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FINITENESS PROPERTIES OF GROUPS

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Recall that a group Γ is said to be of *type* FP_n (resp. FP_∞) if the $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a projective resolution which is finitely generated in dimensions $\leq n$ (resp. in all dimensions), cf. [4] or [8]. For example, Γ is of type FP_1 if and only if it is finitely generated, and Γ is of type FP_2 if it is finitely presented. (The converse of the last assertion is not known.) In this paper we give a necessary and sufficient condition for Γ to be of type FP_n. The condition involves the homological properties of a suitable topological space X on which Γ acts. It is quite easy to use in practice, once one has a suitable X. We will illustrate this by giving several non-trivial examples.

In Section 1 we recall a well-known sufficient condition for the FP_n property. We then give our necessary and sufficient condition in Section 2 (Theorem 2.2). Section 3 contains a similar result about finite presentability. The remainder of the paper deals with examples.

In Section 4 we treat several infinite families of groups, which include some finitely presented simple groups first constructed by R.J. Thompson in 1965 and later generalized by Higman. They also include a certain group F, also constructed by Thompson, which later reappeared in homotopy theory and was eventually shown to be of type FP_∞ [10]. We will give a unified proof that all the groups Γ in these families are finitely presented and of type FP_∞. We will also show that these groups all satisfy $H^*(\Gamma, \mathbb{Z}\Gamma) = 0$. This had previously been proven for some of them by Brown and Geoghegan.

In Section 5 we consider a sequence of groups H_n ($n \geq 1$) introduced by Houghton [18]. Using methods surprisingly similar to those used for the Thompson–Higman groups, we show that H_n is of type FP_{n-1} but not FP_n.

Finally, in Section 6 we look at a sequence Γ_n of solvable S-arithmetic groups first studied by Abels [1]. The work of various people (see [2] for references) has led to the result that Γ_n is of type FP_{n-1} but not FP_n. We give a new proof of this result by applying the criterion of Section 2 to the action of Γ_n on a certain Bruhat–Tits building.

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1. A sufficient condition

We will only consider the case $n<\infty$, since Γ is of type FP_∞ if and only if it is of type FP_n for all $n<\infty$, cf. [8, VIII.4.5]. We will also assume, to avoid trivialities, that $n\geq 1$. By a *Γ-CW-complex* we mean a *CW*-complex X, together with an action of Γ on X by homeomorphisms which permute the cells. We will say that X is *n-good* for Γ if the following two conditions hold:

(a) X is acyclic in dimensions $<n$, i.e., the reduced homology $\tilde{H}_i(X)=0$ for $i<n$.

(b) For $0\leq p\leq n$, the stabilizer Γ_σ of any p-cell σ of X is of type FP_{n-p}.

Such an X always exists. For example, we could take X to be the universal cover of a $K(\Gamma,1)$-complex, in which case (a) and (b) both hold for trivial reasons. Or we could take X to be the n-skeleton of that universal cover.

The following sufficient condition for the FP_n property is well-known:

1.1. Proposition. *Suppose Γ admits an n-good complex X such that X has a finite n-skeleton* mod Γ. *Then Γ is of type* FP_n.

This is in the same spirit as the results in [23, pp. 93ff], but it is not explicitly stated there. For the convenience of the reader we will sketch two proofs; the first is more straightforward, but the second introduces ideas that will be needed later anyway.

Proof 1. Let C be the cellular chain complex of X. For any $p\geq 0$ we have

$$C_p=\bigoplus_\sigma \mathbb{Z}_\sigma\uparrow_{\Gamma_\sigma}^{\Gamma},$$

where σ ranges over a set of representatives for the p-cells of X mod Γ, $\mathbb{Z}_\sigma$ is the orientation module, and the arrow denotes induction from Γ_σ-modules to Γ-modules, cf. [8, p. 68, Example III.5.5(b)], or [23, p. 94, Lemme 3]. In view of the hypotheses, it follows easily that C_p is a Γ-module of type FP_{n-p}, i.e., it admits a projective resolution $(P_{pq})_{q\geq 0}$ such that P_{pq} is finitely generated for $q\leq n-p$. (This is vacuous for $p>n$, in which case we take an arbitrary projective resolution.) One can now find a 'total complex' T, with $T_m=\bigoplus_{p+q=m}P_{pq}$, such that T has the same homology as C [7, Lemma 1.5]. The n-skeleton of T is then a finitely generated partial projective resolution of $\mathbb{Z}$, so Γ is of type FP_n. □

Proof 2. By the Bieri–Eckmann criterion [6], Γ is of type FP_n if and only if the functor $H_i(\Gamma,-)$ preserves direct products for $i<n$. Moreover, it suffices to consider direct products of copies of $\mathbb{Z}\Gamma$, i.e., to show for any index set J that $H_i(\Gamma,\prod_J\mathbb{Z}\Gamma)=0$ for $0<i<$n and that $H_0(\Gamma,\prod_J\mathbb{Z}\Gamma)\xrightarrow{\approx}\prod_J\mathbb{Z}$. Since X is acyclic in dimensions $<n$, we have $H_i(\Gamma,-)=H_i^\Gamma(X,-)$ for $i<n$, the latter being equivariant homology, cf. [8, Chapter VII]. (More concretely, this is simply the homology of

$(C \otimes_{\mathbb{Z}} Q) \otimes_{\Gamma} -$, where C is as in Proof 1 above and Q is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$.) Consider now the equivariant homology spectral sequence [8, VII.7.7], which for our present purposes is most conveniently written in the form

$$E^1_{pq} = \operatorname{Tor}^{\Gamma}_q(C_p, -) \Rightarrow H^{\Gamma}_{p+q}(X, -).$$

(This is simply one of the spectral sequences of the double complex in the definition of equivariant homology above.) Now take the coefficient module to be $\prod \mathbb{Z}\Gamma$, where the index set J is suppressed to simplify the notation. Since C_p is a module of type FP_{n-p}, we have, for $p+q<n$,

$$E^1_{pq} = \begin{cases} 0 & \text{if } q>0, \\ \prod C_p & \text{if } q=0. \end{cases}$$

Moreover, there is a surjection $E^1_{n0} \twoheadrightarrow \prod C_n$; hence $E^2_{p0} = \prod H_p(X)$ for $p<n$. In view of the acyclicity of X in dimensions $<n$, it follows that $E^2_{pq}=0$ for $0<p+q<n$ and that $E^2_{00} = \prod \mathbb{Z}$. Thus $H_i(\Gamma, \prod \mathbb{Z}\Gamma) = H^{\Gamma}_i(X, \prod \mathbb{Z}\Gamma) = 0$ for $0<i<n$, and $H_0(\Gamma, \prod \mathbb{Z}\Gamma) = \prod \mathbb{Z}$. □

2. A necessary and sufficient condition

Let X be an n-good complex for Γ, $1 \le n < \infty$. In case X has an infinite n-skeleton mod Γ, so that 1.1 is inapplicable, we will get an FP_n criterion in terms of the properties of a suitable filtration of X.

By a *filtration* of X we will mean a family $\{X_\alpha\}_{\alpha \in D}$ of Γ-invariant subcomplexes such that D is a directed set, $X_\alpha \subseteq X_\beta$ when $\alpha \le \beta$, and $X = \bigcup_\alpha X_\alpha$. It is always possible to filter X by subcomplexes X_α which have a finite n-skeleton mod Γ. For example, we can simply take D to be the set of *all* such subcomplexes, ordered by inclusion. Such a filtration will be said to be of *finite n-type.* Note that the acyclicity of X in dimensions $<n$ implies, for any filtration, that

$$\varinjlim_{\alpha \in D} \tilde{H}_i(X_\alpha) = 0 \quad \text{for } i<n.$$

We will show that the FP_n property is equivalent to the statement that this limit is 'uniformly' 0 for a filtration of finite n-type.

More precisely, a direct system of groups $\{A_\alpha\}_{\alpha \in D}$ will be called *essentially trivial* if for each $\alpha \in D$ there is a $\beta \ge \alpha$ such that the map $A_\alpha \to A_\beta$ is the trivial map. This is, of course, much stronger than simply requiring the direct limit to be trivial. In fact, we have:

2.1. Lemma. *$\{A_\alpha\}$ is essentially trivial if and only if any direct product of copies of the direct system $\{A_\alpha\}$ has trivial direct limit, i.e.,*

$$\varinjlim_{\alpha \in D} \prod_J A_\alpha = 0$$

for any index set J.

The proof is easy and is left to the reader. We can now state the main result of this section:

2.2. Theorem. *Let X be an n-good Γ-complex with a filtration $\{X_\alpha\}$ of finite n-type. Then Γ is of type* FP_n *if and only if the direct system $\{\tilde{H}_i(X_\alpha)\}$ of reduced homology groups is essentially trivial for each $i<n$.*

Proof. We use the Bieri–Eckmann criterion, as in Proof 2 of 1.1. Note first that, for each $\alpha \in D$ and each $i<n$, the functor $H_i^\Gamma(X_\alpha, -)$ preserves direct products of copies of $\mathbb{Z}\Gamma$, i.e., $H_i^\Gamma(X_\alpha, \prod_J \mathbb{Z}\Gamma) = \prod_J H_i(X_\alpha)$. One can prove this by imitating Proof 2 of 1.1. (Alternatively, show as in Proof 1 of 1.1 that $C(X_\alpha)$ is weakly equivalent to a complex T of projectives with T_i finitely generated for $i \le n$, and deduce the assertion from this.) We now have, for $i<n$,

$$\begin{aligned} H_i(\Gamma, \prod_J \mathbb{Z}\Gamma) &= H_i^\Gamma(X, \prod_J \mathbb{Z}\Gamma) \\ &= \varinjlim_{\alpha \in D} H_i^\Gamma(X_\alpha, \prod_J \mathbb{Z}\Gamma) \\ &= \varinjlim_{\alpha \in D} \prod_J H_i(X_\alpha). \end{aligned}$$

In view of Lemma 2.1, then, $H_i(\Gamma, \prod_J \mathbb{Z}\Gamma) = 0$ for $0<i<n$ and all J if and only if $\{H_i(X_\alpha)\}$ is essentially trivial for $0<i<n$. To deal with the "$i=0$" part of the Bieri–Eckmann criterion, note that we have an exact sequence $0 \to \tilde{H}_0(X_\alpha) \to H_0(X_\alpha) \to \mathbb{Z} \to 0$, and hence an exact sequence

$$0 \to \varinjlim \prod_J \tilde{H}_0(X_\alpha) \to \varinjlim \prod_J H_0(X_\alpha) \to \prod_J \mathbb{Z} \to 0.$$

Thus $H_0(\Gamma, \prod_J \mathbb{Z}\Gamma) \xrightarrow{\approx} \prod_J \mathbb{Z}$ for all J if and only if $\varinjlim \prod_J \tilde{H}_0(X_\alpha) = 0$ for all J, i.e., if and only if $\{\tilde{H}_0(X_\alpha)\}$ is essentially trivial. □

Remarks. (1) If R is an arbitrary ring, then the results of this section and the last remain valid if "FP_n" is replaced by "FP_n over R" and all homology groups are understood to have coefficients in R.

(2) Theorem 2.2 could be formulated as a result about direct systems of Γ-complexes, with no mention of the direct limit X. More precisely, suppose we start with a direct system $\{X_\alpha\}_{\alpha \in D}$ of Γ-complexes. We require the bonding maps $X_\alpha \to X_\beta$ for $\alpha \le \beta$ to be Γ-equivariant and continuous, but they need not respect the cell structure or be inclusions. Assume that each X_α has a finite n-skeleton mod Γ and satisfies condition (b) in the definition of 'n-good', and assume further that $\varinjlim \tilde{H}_i(X_\alpha) = 0$ for $i<n$. Then Γ is of type FP_n if and only if the system $\{\tilde{H}_i(X_\alpha)\}$ is essentially trivial for $i<n$. The proof is similar to that of 2.2 and is left to the reader. [A technicality that arises is that equivariant homology, as we have

defined it in terms of the cellular chain complex, is not obviously functorial with respect to arbitrary continuous Γ-maps. One way to deal with this is to use an alternate definition of equivariant homology; see, for instance, [8, §VII.7.4, Exercise 3].]

We close this section by explicitly stating what 2.2 says for $n \leq 2$.

2.3 Corollary. *Let X be a connected Γ-complex such that the vertex stabilizers are finitely generated. Let $\{X_\alpha\}$ be a filtration of X such that each X_α has a finite 1-skeleton* mod Γ. *Then Γ is finitely generated if and only if the direct system of sets $\{\pi_0(X_\alpha)\}$ is essentially trivial, in the sense that for any α there is a $\beta \geq \alpha$ such that the map $\pi_0(X_\alpha) \to \pi_0(X_\beta)$ is constant.*

This is the case $n=1$. The interested reader can easily prove it directly, without appeal to the machinery used in this section.

For $n=2$ the content of Theorem 2.2 is:

2.4. Corollary. *Let X be a connected Γ-complex such that* (a) $H_1(X)=0$, (b) *the vertex stabilizers are of type* FP_2, *and* (c) *the edge stabilizers are finitely generated. Let $\{X_\alpha\}$ be a filtration of X such that each X_α has a finite 2-skeleton* mod Γ. *If Γ is finitely generated, then Γ is of type* FP_2 *if and only if the direct system $\{H_1\{X_\alpha)\}$ is essentially trivial.*

3. Finite presentation

We begin by recording the analogue of 1.1:

3.1. Proposition [9, Theorem 4]. *Suppose there exists a 1-connected Γ-complex X such that the vertex stabilizers are finitely presented, the edge stabilizers are finitely generated, and X has a finite 2-skeleton* mod Γ. *Then Γ is finitely presented.*

We will use this to prove an analogue of 2.4. To simplify the statement, we assume that our filtration $\{X_\alpha\}_{\alpha \in D}$ has $\bigcap X_\alpha \neq \emptyset$, so that we can choose a basepoint v in this intersection. There is no loss of generality in making this assumption; for we could simply choose an arbitrary $v \in X$ and then replace D by the cofinal subset $\{\alpha \in D \colon v \in X_\alpha\}$.

3.2. Theorem. *Let X be a 1-connected Γ-complex such that the vertex stabilizers are finitely presented and the edge stabilizers are finitely generated. Let $\{X_\alpha\}$ be a filtration of X such that each X_α has a finite 2-skeleton* mod Γ, *and let $v \in \bigcap X_\alpha$ be a basepoint. If Γ is finitely generated, then Γ is finitely presented if and only if the direct system $\{\pi_1(X_\alpha, v)\}$ is essentially trivial.*

Proof. Let D' be the set of α in D such that the connected component X'_α of v in X_α is Γ-invariant. Γ being finitely generated, 2.3 implies that D' is cofinal in D. Replacing D by D' and X_α by X'_α, we reduce to the case where each X_α is connected. Let $\tilde{X}_\alpha$ be the universal cover of X_α, with a chosen basepoint lying over v, and let Γ_α be the group of homeomorphisms of $\tilde{X}_\alpha$ which cover some element of Γ acting on X_α. Then Γ_α is finitely presented by 3.1, and we have a canonical short exact sequence

$$1 \to \pi_1(X_\alpha) \to \Gamma_\alpha \to \Gamma \to 1,$$

cf. [9, §2]. Moreover, the inclusion $X_\alpha \subseteq X_\alpha$ $(\alpha \leq \beta)$ induces a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_\alpha) & \longrightarrow & \Gamma_\alpha & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1(X_\beta) & \longrightarrow & \Gamma_\beta & \longrightarrow & \Gamma & \longrightarrow & 1
\end{array}$$

Suppose now that Γ is finitely presented. Then, for any α, $\pi_1(X_\alpha)$ is finitely generated as a normal subgroup of Γ_α. Since $\varinjlim \pi_1(X_\alpha) = \pi_1(X) = 1$, we may choose $\beta \geq \alpha$ so that $\ker\{\pi_1(X_\alpha) \to \pi_1(X_\beta)\}$ contains a set of normal generators for $\pi_1(X_\alpha)$ in Γ_α and hence is all of $\pi_1(X_\alpha)$. Thus $\{\pi_1(X_\alpha)\}$ is essentially trivial. Conversely, suppose $\{\pi_1(X_\alpha)\}$ is essentially trivial. Let α be arbitrary, and choose $\beta \geq \alpha$ so that the map $\pi_1(X_\alpha) \to \pi_1(X_\beta)$ is trivial. Then the map $\Gamma_\alpha \to \Gamma_\beta$ induces a map $\Gamma \to \Gamma_\beta$, which is a section of the map $\Gamma_\beta \to \Gamma$. Thus Γ is a retract of the finitely presented group Γ_β and hence is finitely presented. $\square$

We close this section by giving, for ease of reference, a special case of the results of Sections 1–3 that will be needed later.

3.3. Corollary. *Let X be a contractible Γ-complex such that the stabilizer of every cell is finitely presented and of type* FP_∞. *Let $\{X_j\}_{j \geq 1}$ be a filtration such that each X_j is finite* mod Γ.

(a) *Suppose that the connectivity of the pair (X_{j+1}, X_j) tends to ∞ as j tends to ∞. Then Γ is finitely presented and of type* FP_∞.

(b) *Fix $n \geq 1$ and suppose that for all sufficiently large j X_{j+1} is obtained from X_j by the adjunction of n-cells, up to homotopy. Then Γ is of type* FP_{n-1} *but not* FP_n. *If $n \geq 3$, then Γ is finitely presented.*

Proof. The hypothesis of (a) implies that for any $i \geq 0$ the system $\{\tilde{H}_i(X_j)\}$ eventually stabilizes to a sequence of isomorphisms. Since the direct limit is 0, these isomorphisms must be zero maps, so the system is essentially trivial. Similarly, $\{\pi_1(X_j)\}$ is essentially trivial. (a) now follows from 2.2 and 3.2.

The positive part of (b) (that Γ is of type FP_{n-1} and is finitely presented if $n \geq 3$) is proved similarly. For the negative part, note that for large j we have a surjection $\tilde{H}_{n-1}(X_j) \twoheadrightarrow \tilde{H}_{n-1}(X_{j+1})$ which, I claim, has a non-trivial kernel. For if the adjunction of n-cells does not kill any $(n-1)$-dimensional homology, then it must introduce some n-dimensional homology. But then the further adjunction of n-cells can never kill off this n-dimensional homology, contradicting the contractibility of X. This proves the claim and shows that $\{\tilde{H}_{n-1}(X_j)\}$ is not essentially trivial. Hence Γ is not of type FP_n. □

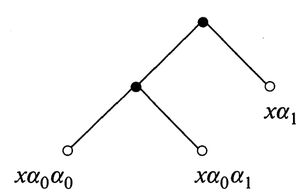
Remark. We do not really need 2.2 and 3.2 to prove (a) and the positive part of (b). In (b), for instance, it follows from the hypotheses that X_j is $(n-2)$-connected for j sufficiently large; so we could simply apply 1.1 and 3.1 to a suitable X_j.

4. Example 1: The Thompson–Higman groups

4A. *Definitions*

Fix an integer $n \geq 2$ and consider the algebraic system consisting of a set V together with a bijection α from V to its n-th Cartesian power V^n. For lack of a better name, we will call (V, α) an *algebra of type n*. Algebras of type 2 were introduced by Jónsson and Tarski [19, Theorem 5] to provide examples of certain phenomena in universal algebra. Higman [17] later considered the general case, for reasons that we will explain in the historical remarks at the end of this section.

Let $\alpha_0, \ldots, \alpha_{n-1}$ be the components of α. Thus each α_i is a unary operator $V \to V$. Following Higman, we write $x\alpha_i$ for the image of x under α_i. It is useful to visualize the action of α by means of trees. For example, suppose we apply α to an element x and then again to $x\alpha_0$. Taking $n=2$, for instance, we represent this by means of the rooted binary tree



where the top node (the root) corresponds to x, and the two descendants of a node represent its images under α_0 and α_1. The nodes with no descendants are called *leaves*; they have been drawn as open circles. For arbitrary n we use n-ary trees instead of binary trees, where an *n-ary tree* is one in which every node that is not a leaf has exactly n descendants.

Let $V_{n,r}$ be the free algebra of type n with r generators x_i, $0 \leq i \leq r-1$. The existence of this free algebra follows from general nonsense, but one can also simply construct it directly [17, §2]. For our purposes it is convenient to describe the construction as follows: For each x_i, construct the *complete* n-ary tree with x_i as the root, where the complete tree, by definition, is the one with no leaves. (It is, of course, infinite.) Let $\mathscr{F}$ be the disjoint union of these trees, and let V_0 be the set of nodes of the 'forest' $\mathscr{F}$. Note that α is defined on V_0 and is injective but not surjective. One now obtains $V_{n,r}$ by formally enlarging V_0 to make α surjective. (We will not need the details of this last step.)

According to [19] and [17], we can construct a new basis from the original basis $X=\{x_i\}$ by replacing any $x \in X$ by its n descendants $x\alpha_0, \ldots, x\alpha_{n-1}$. The new basis (with $r+n-1$ elements) is called a *simple expansion* of X. Iterating this procedure d times, we obtain *d-fold expansions* of X. Expansions of X correspond, in an obvious way, to finite n-ary forests with r roots; the new basis elements are the leaves of the forest. For example, the binary tree pictured above shows a 2-fold expansion of the original 1-element basis $\{x\}$ of $V_{2,1}$; the 3 leaves form a 3-element basis of this algebra.

We will always assume that our forests are drawn with the roots $x_0, x_1, \ldots$ in left-to-right order and with the n descendants $y\alpha_0, \ldots, y\alpha_{n-1}$ of a node y in left-to-right order. In particular, the leaves of the forest (i.e., the elements of the corresponding expansion Y of X) then have a definite left-to-right order.

By a *cyclic-order* on a finite set Y we will mean a free transitive action of $\mathbb{Z}/s\mathbb{Z}$ on Y, where $s=\text{card}(Y)$. Less formally, this simply means that we think of the elements of Y as arranged on a circle, with no preferred starting point. If Y is an expansion of X as above, we will sometimes want to forget the linear order on Y and just remember the underlying cyclic order.

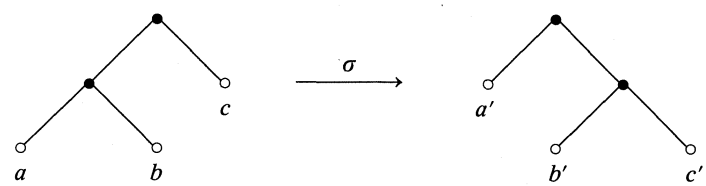
It is shown in [17] that any two bases of $V_{n,r}$ have a common expansion; in particular, any basis can be obtained from X by doing an expansion followed by a contraction, where Z is called a *contraction* of Y if Y is an expansion of Z. Equivalently, Z is obtained from Y by doing finitely many *simple contractions*, where a simple contraction consists of taking an ordered n-tuple $(y_1, \ldots, y_n)$ of distinct elements of Y and replacing them by $z=\alpha^{-1}(y_1, \ldots, y_n)$. If the basis Y is equipped with a linear ordering and the n-tuple consists of consecutive elements (in order), then the new basis has an obvious linear ordering (with z in the position previously occupied by the y's) and is said to be obtained from Y by an *ordered simple contraction*. The notion of *cyclically ordered simple contraction* for bases with a cyclic ordering is defined similarly.

By an *ordered basis* for $V_{n,r}$ we will mean a basis Y which admits a linear ordering such that, for some common expansion Z of X and Y, the order on Z inherited from X is the same as that inherited from Y. One defines *cyclically ordered basis* similarly. In each case the ordering on Y is unique. It is easy to see that a basis Y is ordered (resp. cyclically ordered) if and only if it can be obtained from X by doing an expansion followed by ordered (resp. cyclically ordered) simple contractions.

As in [17], we denote by $G_{n,r}$ the group of automorphisms of $V_{n,r}$, and we adopt the convention that it acts on the right; thus gh denotes g followed by h. For any $g \in G_{n,r}$ there is an expansion Y of X such that $Y \cdot g$ is also an expansion of X.

Moreover, there is a unique minimal such Y, and all others are expansions of it [17, Lemma 4.1]. Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be the forest corresponding to Y (resp. Yg) and let σ be the bijection from the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}'$ induced by g. Then g is determined by $(\mathcal{F}, \mathcal{F}', \sigma)$; following Higman, who stated the same result in slightly different language, we call $(\mathcal{F}, \mathcal{F}', \sigma)$ a *symbol* for g. (We say 'a' symbol here because we do not insist on using the minimal Y; we leave it to the reader to figure out how a general symbol can be obtained by 'expansion' of the minimal one.)

Conversely, given a pair $(\mathcal{F}, \mathcal{F}')$ of finite n-ary forests with r roots and the same number of leaves, and given a bijection σ from the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}'$, there is an associated element of $G_{n,r}$. For example, there is an element $g \in G_{2,1}$ given by the symbol

(4.1)


where σ is given by $a \mapsto a'$, etc. (Note that the picture 4.1 only indicates where g sends the leaves; to figure out what g does to other elements, such as the root, we have to use the fact that g commutes with α.)

We will also be interested in two subgroups of $G_{n,r}$, defined in terms of order-preservation properties. Given $g \in G_{n,r}$, choose a symbol $(\mathcal{F}, \mathcal{F}', \sigma)$ representing it; we call g *order-preserving* (resp. *cyclic-order-preserving*) if σ preserves the order (resp. cyclic order) of the leaves. This is independent of the choice of symbol. For example, the element of $G_{2,1}$ pictured in 4.1 is order-preserving. If instead of the given σ we had used $a \mapsto b'$, $b \mapsto c'$, $c \mapsto a'$, then the resulting element would be cyclic-order-preserving but not order-preserving. And if we had used $a \mapsto b'$, $b \mapsto a'$, $c \mapsto c'$, then the resulting element would be neither order-preserving nor cyclic-order-preserving.

If g is order-preserving, then the image Yg of any ordered basis Y is again ordered, and the induced bijection $\sigma : Y \to Yg$ is order-preserving as a map of ordered sets. Conversely, if there exists an ordered basis Y such that Yg is ordered and g induces an order-preserving bijection $Y \to Yg$, then g is order-preserving. Similar remarks apply to cyclic-order-preserving automorphisms.

The order-preserving (resp. cyclic-order-preserving) elements of $G_{n,r}$ form a subgroup which we denote by $F_{n,r}$ (resp. $T_{n,r}$). We have

$$F_{n,r} \subset T_{n,r} \subset G_{n,r}.$$

Since X can be replaced by an expansion in the definitions of these groups, the groups depend only on $r \bmod n-1$. In the case of $F_{n,r}$, there is even less dependence on r:

4.1. Proposition. *$F_{n,r}$ is independent of r, up to isomorphism.*

Proof. We will introduce a new group $F_{n,\infty}$ and show that $F_{n,r} \approx F_{n,\infty}$ for any r. Let $V_{n,\infty}$ be a free algebra of type n on an infinite basis $X = \{x_0, x_1, \ldots\}$. We will say that an automorphism of $V_{n,\infty}$ is of *finite type* if there is an expansion Y of X such that $Z = Yg$ is also an expansion of X. [As when we were dealing with finite bases, we call Y an *expansion* of X it is obtained from X by doing *finitely many* simple expansions. Thus Y corresponds to a forest $\mathscr{F}$ with roots x_i, such that every tree in $\mathscr{F}$ is finite and almost every tree consists of the root alone.] As above, Y and Z are ordered in a natural way, and we call g *order-preserving* if the induced bijection $\sigma : Y \to Z$ is order-preserving. Let $F_{n,\infty}$ be the group of order-preserving automorphisms of $V_{n,\infty}$ of finite type. For any $r < \infty$ the algebra $V_{n,r}$ contains an isomorphic copy of $V_{n,\infty}$, defined as follows: Consider the infinite n-ary forest obtained by starting with the set of roots $\{x_0, \ldots, x_{r-1}\}$ and repeating infinitely often the step of letting the right-most leaf sprout n descendants. Let W be the set of leaves of this forest. Then W freely generates a subalgebra of $V_{n,r}$ isomorphic to $V_{n,\infty}$. It is easy to check that this subforest is invariant under $F_{n,r}$ and that the resulting restriction map $F_{n,r} \to F_{n,\infty}$ is an isomorphism. □

Having introduced $V_{n,\infty}$ and $F_{n,\infty}$, we go one step further and consider the free algebra V_n on a 'doubly infinite' set of generators $X = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$, indexed by the integers. As an algebra, of course, this is isomorphic to $V_{n,\infty}$; but the order type of the basis X is different, and this affects the corresponding 'F-group'. We denote by F_n the group of order-preserving isomorphisms of V_n of finite type, this being defined exactly as above.

The group F_n is closely related to $F_{n,\infty}$. On the one hand, F_n is easily seen to be an ascending HNN extension of $F_{n,\infty}$; here $F_{n,\infty}$ is embedded in F_n as the subgroup fixing the x_i with $i < 0$, and the stable letter for the HNN extension is the shift automorphism $s \in F_n$, $x_i \cdot s = x_{i+1}$. On the other hand, we can recover an isomorphic copy of $F_{n,\infty}$ as a subgroup of F_n of finite index:

4.3. Proposition. *There is an embedding of $F_{n,\infty}$ as a subgroup of F_n of index $n-1$.*

Proof. For any $g \in F_n$ there are integers m, m' such that $x_i \cdot g = x_{i+m}$ for $i \gg 0$ and $x_i \cdot g = x_{i-m'}$ for $i \ll 0$. It is easy to see that $m + m' \equiv 0 \pmod{n-1}$. Set $\theta(g) = m \pmod{n-1} = -m' \pmod{n-1}$. Then $\theta : F_n \to \mathbb{Z}/(n-1)\mathbb{Z}$ is a surjective homomorphism. I claim $\ker\theta \approx F_{n,\infty}$. Consider the infinite n-ary forest inside $V_{n,\infty}$ obtained by starting with the set of roots $\{x_0, x_1, \ldots\}$ and repeatedly letting the left-most leaf sprout n descendants. The set of leaves of this forest generates a subalgebra which can be identified with V_n and which is invariant under $F_{n,\infty}$. Restriction to this subalgebra defines an injective homomorphism $F_{n,\infty} \to F_n$, whose image is $\ker\theta$. □

Historical remarks. In 1965 R.J. Thompson constructed a certain group G, which he thought of as a group of rules for rearranging formal expressions, cf. [20, pp.

475ff]. As an aid in the study of G, Thompson introduced groups F and T, with $F \subset T \subset G$. (G is the group called $\mathfrak{C}'$ in [20] and $Ft({}^{\omega}2)$ in [24]; F is denoted $\mathfrak{P}'$ in [20] and $\mathbb{P}$ in [24]; T seems to appear only in some unpublished handwritten notes.) Thompson showed that all three groups were finitely presented, and he showed that T and G were simple. This attracted the attention of group theorists, because there were no previously known examples of finitely presented infinite simple groups.

Shortly thereafter, F. Galvin and Thompson [unpublished] observed that G was isomorphic to the automorphism group of the Jónsson–Tarski algebra $V_{2,1}$, i.e., $G \approx G_{2,1}$. This is very easy to explain from our present point of view; for Thompson's formal expressions can be represented by binary trees, and his rearrangement rules are precisely those which are given by symbols $(\mathscr{F}, \mathscr{F}', \sigma)$ as above. The element of $G_{2,1}$ corresponding to the symbol 4.1, for example, corresponds to the 'rearrangement' $(AB)C \to A(BC)$.

Higman, upon hearing about Thompson's G (as the automorphism group of $V_{2,1}$), introduced algebras of type n and the generalizations $G_{n,r}$ of G. The groups $F_{n,r}$ and $T_{n,r}$ that we have defined here were not considered by Higman, but they are simply the obvious generalizations of Thompson's F and T; in particular, $F = F_{2,1}$ and $T = T_{2,1}$.

The group F later appeared independently in homotopy theory, in connection with the study of homotopy idempotents ([11], [12], [15]). Using this point of view, Geoghegan and I [10] proved that F was of type FP_∞. [*Note*: It is not immediately obvious that the group F of [10] is the same as the group we are calling F here; we will explain why they are the same in Remark 1 after Proposition 4.8 below.] We knew about Thompson's T and G (but not about the interpretation of G as $G_{2,1}$), and we were able to deduce the FP_∞ property for T from that for F, cf. Remark 2 in Section 4B below. But G remained a mystery.

It turns out, however, that the theory of algebras of type n makes it quite easy to give a unified proof that all of the groups $F_{n,r}$, $T_{n,r}$, and $G_{n,r}$ are finitely presented and of type FP_∞. That is what we will do in Section 4E below, after making a few preliminary observations about the groups.

4B. *Interpretation as homeomorphism groups*

Thompson found it useful to use representations of his groups F, T, and G as homeomorphism groups. In the case of G, this is made explicit in [20] and [24]: G is a certain group of homeomorphisms of the Cantor set. In the case of F and T, the unpublished notes referred to above contain representations of them as groups of PL homeomorphisms of the unit interval and the circle, respectively. The unpublished paper of Freyd and Heller [15] contains two similar representations of F, one as homeomorphisms of the half-line $[0, \infty)$ and the other as homeomorphisms of the entire line $\mathbb{R}$. In this section we will generalize all these representations.

Let I_r be the interval $[0, r] \subset \mathbb{R}$. The n-ary forests that we used in Section 4A to represent expansions of X can be read as recipes for subdividing the interval I_r.

Namely, the r roots correspond to the r intervals $[i, i+1]$, $0 \le i \le r-1$, and a node with n descendants corresponds to an interval which is subdivided into n equal parts. Given a symbol $(\mathcal{F}, \mathcal{F}', \sigma)$ representing an element $f \in F_{n,r}$, we get a homeomorphism $\tilde{f}$ of I_r by forming the subdivisions of I_r associated to $\mathcal{F}$ and $\mathcal{F}'$ and mapping each subinterval of the first subdivision linearly to the corresponding subinterval of the second. This makes sense because σ is order-preserving, and it is easily seen to be independent of the choice of symbol. We therefore have a well-defined homomorphism $f \mapsto \tilde{f}$ from $F_{n,r}$ to the group of PL homeomorphisms of I_r.

4.4. Proposition. *The homomorphism $f \mapsto \tilde{f}$ is injective. Its image consists of all PL homeomorphisms h of I_r with the following two properties:*

(a) *All singularities of h are in $\mathbb{Z}[1/n]$.*

(b) *The derivative of h at any non-singular point is n^k for some $k \in \mathbb{Z}$.*

Proof. The only non-trivial assertion is that any h satisfying (a) and (b) is in the image. The following proof is due to M. Brin; it is much shorter than my original proof. By an *admissible* subdivision of I_r we will mean a subdivision corresponding to a forest as above. Choose an admissible subdivision $\mathcal{P}$ such that h is linear on each subinterval; for example, we could take $\mathcal{P}$ to be a uniform subdivision into subintervals of length n^{-k} for k sufficiently large. The image subdivision $\mathcal{P}' = h(\mathcal{P})$ has its subdivision points in $\mathbb{Z}[1/n]$, so it can be refined to a uniform admissible subdivision $\mathcal{Q}'$. Let $\mathcal{Q}$ be the refinement $h^{-1}(\mathcal{Q}')$ of $\mathcal{P}$. Since h is linear with slope a power of n on each subinterval J in $\mathcal{P}$, this refinement uniformly subdivides J into subintervals of length n^{-j} for some j. It follows that $\mathcal{Q}$ is admissible and hence $h = \tilde{f}$, where $f \in F_{n,r}$ is defined by the forests corresponding to $\mathcal{Q}$ and $\mathcal{Q}'$. $\square$

In a similar way, $F_{n,\infty}$ can be represented as a group of PL homeomorphisms of the half-line $[0,\infty)$, F_n can be represented as a group of homeomorphisms of the whole line $\mathbb{R}$, and $T_{n,r}$ can be represented as a group of homeomorphisms of the circle obtained by identifying the endpoints 0 and r of $[0,r]$.

Remarks. (1) When $n=2$, we have $F_{2,1} \approx F_{2,\infty} \approx F_2$, cf. 4.2 and 4.3. So there is really one abstract group F represented as homeomorphisms of $[0,1]$, $[0,\infty)$, and $(-\infty, -\infty)$. These are the Thompson and Freyd–Heller representations mentioned at the beginning of this section.

(2) It is now easy to explain how Geoghegan and I deduced finiteness properties of $T = T_{2,1}$ from those of $F = F_{2,1}$. View F (resp. T) as a group of homeomorphisms of the unit interval I (resp. the circle $S = I/\{0,1\}$). Let S_0 be the image in S of $I \cap \mathbb{Z}[\frac{1}{2}]$, and let K be the simplicial complex whose simplices are the finite subsets of S_0. Then K is contractible, T acts simplically on K, and the action is transitive on the simplices in any given dimension. Moreover, the stabilizer of a simplex is the direct product of a finite number of copies of F. So once F is known to be finitely presented and of type FP_∞, the same follows for T (cf. 1.1 and 3.1).

Finally, to represent $G_{n,r}$ as a group of homeomorphisms, let C_n be the Cantor set of infinite sequences $a=(a_i)_{i\geq 1}$, where $a_i \in \{0, \dots, n-1\}$. ($C_n$ is topologized as the product of copies of a discrete n-point space.) Note that there is an obvious way to "subdivide C_n into n equal parts", these parts being the clopen subsets $\{a \in C_n: a_1 = k\}$, $0 \leq k \leq n-1$, each canonically homeomorphic to C_n itself. Now let $C_{n,r}$ be the disjoint union of r copies of C_n. We can then imitate what we did for $F_{n,r}$ and $T_{n,r}$ to get a faithful representation of $G_{n,r}$ as a group of homeomorphisms of $C_{n,r}$. The essential point here is that we are now subdividing our space into *disjoint* subsets, so we get a homeomorphism associated to a symbol $(\mathscr{F}, \mathscr{F}', \sigma)$ no matter how badly σ fails to be order-preserving.

4C. *Generators and relations*

As we have already indicated, we are eventually going to prove that all of the groups under discussion are finitely presented. But in this section we give a very simple *infinite* presentation of $F_{n,\infty}$ which is useful for some purposes. We also make a few brief remarks about generators and relations for some of the other groups.

Let X be the basis $\{x_0, x_1, \dots\}$ of $V_{n,\infty}$ and let Y_i be the simple expansion of X at x_i, $i \geq 0$. Let $g_i \in F_{n,\infty}$ be the element such that $Y_i g = X$.

4.5. Proposition. *$F_{n,\infty}$ is generated by the elements g_i, $i \geq 0$.*

Proof. Call an element $p \in F_{n,\infty}$ *positive* if there is an expansion Y of X such that $Yp = X$. It is obvious that every element of $F_{n,\infty}$ has the form pq^{-1}, where p and q are positive. So the proposition will follow if we show that every positive element is a product of g_i's.

If Y is a d-fold expansion of x, then we set $d(Y) = d$. We will prove by induction on $d(Y)$ that the positive element p_Y which takes Y to X is a product of $d(Y)$ g_i's. This is trivial if $d(Y) = 0$, so assume $d(Y) > 0$ and choose $i \geq 0$ such that Y is an expansion of Y_i. Then $Y' = Yg_i$ is an expansion of $Y_i g_i = X$ with $d(Y') = d(Y) - 1$, so $p_{Y'}$ is a product of $d(Y) - 1$ g_j's by induction. Thus $p_Y = g_i p_{Y'}$ is a product of $d(Y)$ g_j's, as required. □

Note that, in view of the choice of i in this proof, there is usually more than one way to write a given positive element as a product of g_j's. This leads to relations among the generators g_j. For example, suppose Y is obtained from X by expanding both x_i and x_j, where $i < j$. Then there are two ways to write p_Y as a product of g_k's, and one finds

(4.6) $$g_j g_i = g_i g_{j+n-1} \quad \text{for } i<j,$$

or, equivalently,

(4.7) $$g_i^{-1} g_j g_i = g_{j+n-1} \quad \text{for } i<j.$$

4.8. Proposition. $F_{n,\infty}$ *admits a presentation with generators* $g_0, g_1, \dots$ *and relations* 4.6 (*or* 4.7).

Proof. We will think of $F_{n,\infty}$ as a group of PL homeomorphisms of the half-line as in Section 4B. Note that g_i is then supported in $[i,\infty)$ and has right-hand derivative n at i. Consequently, if w is any word in the generators g_i, and if j is the smallest integer such that $g_j^{\pm 1}$ occurs in w, then the homeomorphism corresponding to w has right-hand derivative n^e at j, where e is the exponent sum of g_j in w.

Let $\tilde{F}$ be the abstract group with generators g_i subject to the relations above. We have a surjection $\tilde{F} \twoheadrightarrow F_{n,\infty}$, which we must prove is injective. Suppose not, and choose a word w of minimal length representing a non-trivial element of the kernel. Let j be as in the previous paragraph. Then the exponent sum of g_j in w must be 0. On the other hand, we may use the defining relations for $\tilde{F}$ to rewrite w (without changing its length) so that all occurrences of g_j^{+1} are at the beginning and all occurrences of g_j^{-1} are at the end. So w is conjugate to a shorter word, contradicting its minimality. □

Remarks. (1) The group called F in [10] was *defined* by the presentation above, with $n=2$; hence that F is the same as $F_{2,\infty}$. The presentation was significant in the context of [10] because it showed that F supported the universal example of an endomorphism which was idempotent up to conjugacy. In the same way, $F_{n,\infty}$ supports the universal example of an endomorphism φ such that φ^n is conjugate to φ. φ is simply the shift map, $g_i \mapsto g_{i+1}$.

(2) With a little more effort, one can use the method of proof of 4.8 to prove a *normal form theorem* for $F_{n,\infty}$ analogous to that of F, cf. [10, 1.3].

(3) In view of 4.7, $F_{n,\infty}$ is obviously generated by the g_i with $0 \le i \le n-1$.

We now make a few remarks about generators and relations for some of the other groups we have been discussing.

4.9. Let $X = \{\dots, x_{-1}, x_0, x_1, \dots\}$ be the given basis for V_n, let Y_i ($i \in \mathbb{Z}$) be the simple expansion of X at x_i, and let $g_i \in F_n$ be the element which takes Y_i to X and fixes the x_j for $j<i$. Let $s \in F_n$ be the shift automorphism of V_n, $x_i \mapsto x_{i+1}$. Then F_n is generated by s and the g_i, subject to the relations 4.6 and

$$s^{-1} g_i s = g_{i+1}.$$

(So it is actually generated by s and g_0.) If we identify $F_{n,\infty}$ with the subgroup of F_n fixing the x_i for $i<0$, then the generator g_i of $F_{n,\infty}$ ($i \ge 0$) corresponds to the generator of F_n with the same name. On the other hand, we have an embedding of $F_{n,\infty}$ as a subgroup of F_n of index $n-1$ (4.3); this is given by $g_0 \mapsto s^{n-1}$ and $g_i \mapsto g_i$ for $i>0$.

4.10. Let $X = \{x_0, \dots, x_{r-1}\}$ be the given basis for $V_{n,r}$, let Y_i be the simple expansion of X at x_i, and let γ_{ij} for $i<j$ be the element of $F_{n,r}$ which takes Y_i to Y_j. We call an element of this form (with X possibly replaced by a different ordered basis) a *glide*. Set $\gamma_i=\gamma_{i,i+1}$, $0\le i\le r-2$. If we identify $F_{n,r}$ with the subgroup of $F_{n,\infty}$ fixing the basis elements x_i of $V_{n,\infty}$ with $i>r-1$, then $\gamma_{ij}=g_i g_j^{-1}$. On the other hand, we have an isomorphism $F_{n,\infty}\approx F_{n,r}$ (cf. proof of 4.2), under which $g_i\in F_{n,\infty}$ corresponds to $\gamma_{i,r-1}$ for $i<r-1$. In particular, $F_{n,r}$ is generated by the $\gamma_{i,r-1}$, hence also by the γ_i, provided $r\ge n+1$. Since X can always be replaced by an expansion with at least $n+1$ elements, it follows that $F_{n,r}$ is generated by glides for any $r<\infty$.

4.11. Let $X=\{x_i\}$ as in 4.10, but view the indices i as integers mod r. Let γ_{ij} $(i,j\in\mathbb{Z}/r\mathbb{Z})$ be the cyclic-order-preserving automorphism which takes Y_i to Y_j, with the first descendant of x_i going to x_j. This notation is consistent with that above if $0\le i<j\le r-2$. Again we call elements of this form (relative to some cyclically ordered basis) *glides*, and we set $\gamma_i=\gamma_{i,i+1}$, $i\in\mathbb{Z}/r\mathbb{Z}$. Replace X by an expansion, if necessary, to assure $r\ge 2$, and let $\varrho\in T_{n,r}$ be the 'rotation' $x_i\mapsto x_{i+1}$ of order r. Let ϱ' be the rotation of order $r+n-1$ defined in the same way but with respect to the basis Y_{r-1}. The following relations hold:

(i) $$\varrho^{-1}\gamma_i\varrho=\gamma_{i+1} \qquad (i\in\mathbb{Z}/r\mathbb{Z}),$$

(ii) $$\gamma_{r-2}^{-1}\varrho=\varrho',$$

(iii) $$\gamma_{r-1}\gamma_0\gamma_1\cdots\gamma_{r-2}=(\varrho')^{n-1}.$$

I claim that $T_{n,r}$ is generated by $F_{n,r}$ and ϱ, hence by glides and ϱ. For it is obvious that $T_{n,r}$ is generated by $F_{n,r}$ and the rotations $\varrho,\varrho',\varrho'',\dots$ with respect to the successive right-most expansions of X; we can now use (ii) to eliminate ϱ', and we can similarly use analogues of (ii) (with X replaced by its expansions) to eliminate ϱ'', etc. This proves the claim.

Note, finally, that ϱ^{n-1} is a product of glides. In fact, if we use (ii) to eliminate ϱ' from (iii) and then use (i) to lump together the resulting $n-1$ occurrences of ϱ, we obtain

(iv) $$\varrho^{n-1}=\gamma_{r-n}\gamma_{r-n+1}\cdots\gamma_{r-1}\gamma_0\cdots\gamma_{r-2},$$

where there are $r+n-1$ factors on the right.

4D. Normal subgroups

Recall that group theorists became interested in Thompson's G and T because of the combination of finiteness properties and simplicity. Although the present paper is about finiteness properties, we would like to state, with sketches of the proofs, the simplicity properties of all of the groups we have been discussing. To put these results in perspective, the reader should recall that homeomorphism groups are often simple, or, at least, have a simple commutator subgroup, cf. Epstein [13]. So the simplicity properties stated in this section are not at all surprising. What is perhaps surprising is that a homeomorphism group 'big' enough to fit into Epstein's framework also has good finiteness properties; indeed, the examples that usually come to mind are not even countable.

We begin with $F_{n,r}$. It is convenient to introduce some terminology motivated by the homeomorphism representation of $F_{n,r}$. Suppose V' is a subalgebra of $V=V_{n,r}$ generated by a set Y' of consecutive elements in some ordered basis Y of V. We will say that an element $g\in F_{n,r}$ has *support* in V' if g fixes all the basis elements of $Y-Y'$. One can check by considering symbols that this notion depends only on V' and not on the choice of Y and Y'. Moreover, V' is necessarily invariant under g.

Two subalgebras V' and V'' as above will be called *disjoint* if they are generated by disjoint consecutive subsets of a single ordered basis for V. In particular, it makes sense now to say that two elements of $F_{n,r}$ have *disjoint supports.*

Finally, we say that g has *interior support* if it has support in a V' as above such that Y' contains neither the first nor the last element of Y. This is equivalent to saying that the homeomorphism of $[0,r]$ corresponding to g has compact support in $(0,r)$.

Examples. The glide γ_{ij} defined in 4.10 is supported in the algebra generated by $x_i,\ldots,x_j$. It has interior support if and only if $i>0$ and $j<r-1$. The two glides γ_i and γ_j of 4.10 have disjoint supports if and only if i and j differ by at least 2.

Let F_{nr}^0 be the subgroup of $F_{n,r}$ consisting of elements with interior support. By applying 4.10 to the group of order-preserving automorphisms of subalgebras V' as above, we see that F_{nr}^0 is generated by glides.

We are going to show by the methods of [13] that the commutator subgroup of F_{nr}^0 is simple. Before stating the precise result, we determine the abelianizations of F_{nr}^0 and $F_{n,r}$.

Let A be the abelian group with generators e_-, e_+, and e_i $(i\in\mathbb{Z})$ subject to the relations $e_i=e_j$ if $i\equiv j \pmod{n-1}$. Thus A is free abelian of rank $n+1$. We will construct a homomorphism $a: F_{n,r}\to A$ which records the positions at which expansions occur in the construction of a symbol for an element of $F_{n,r}$.

If Y is an ordered basis of $V_{n,r}$, write the elements of Y as

$$y_- < y_1 < \cdots < y_s < y_+,$$

where $\operatorname{card}(Y)=s+2$. In case $\operatorname{card}(Y)=1$, we agree that the unique element of Y is y_-; this arbitrary choice will have no effect on our definition of the map a. By the *position* of an element $y\in Y$ we will mean the index i such that $y=y_i$; thus $i\in\mathbb{Z}$ or i is one of the symbols $-$, $+$.

If a new basis is formed from Y by doing a simple expansion at y, then the position of y in Y will be called the position of the expansion. Similarly, if a new (ordered) basis is formed by contracting n elements starting at y, then the position of y will be called the position of the contraction.

4.12. Lemma. *One can associate to any two ordered bases Y, Z an element*

$\delta(Z, Y) \in A$, in such a way that the following hold:

(i) *For any three ordered bases Y, Z, W, we have*

$$\delta(W, Y) = \delta(W, Z) + \delta(Z, Y).$$

(ii) *Suppose Z is obtained from Y by doing a sequence of simple expansions and contractions, at positions $i, j, \ldots$. Then*

$$\delta(Z, Y) = \pm e_i \pm e_j \pm \cdots,$$

where we take the plus sign in case of an expansion and the minus sign in case of a contraction.

Proof. Suppose first that Z is an expansion of Y, and let $\mathscr{F}$ be the corresponding forest, with Y as the set of roots. For each node v of $\mathscr{F}$ which is not a leaf, choose an n-ary subforest $\mathscr{F}_v$ of $\mathscr{F}$ containing all the roots and having v as a leaf. The leaves of $\mathscr{F}_v$ are the elements of an expansion Z_v of Y, and we denote by $i(v)$ the position of v in Z_v. Set $\delta(Z, Y) = \sum e_{i(v)}$, where v ranges over all the nodes which are not leaves; this is independent of the choice of the $\mathscr{F}_v$. Note that we can also describe $\delta(Z, Y)$ as $e_i + e_j + \cdots$, where $i, .j, \ldots$ are the positions of the expansions in any sequence of simple expansions leading from Y to Z. It follows that (i) holds if W is an expansion of Z.

Now suppose Y and Z are arbitrary ordered bases and let W be a common expansion. Using the special case of (i) just verified, one checks easily that $\delta(W, Y) - \delta(W, Z)$ is independent of the choice of W. We may therefore set $\delta(Z, Y) = \delta(W, Y) - \delta(W, Z)$. It is now easy to verify (i) for any three ordered bases. Finally, (ii) is clear in the case of a single simple expansion or contraction, and the general case follows from (i). □

We now define $a: F_{n,r} \to A$ by

$$a(g) = \delta(Y, Yg),$$

where Y is any ordered basis. This is independent of the choice of Y, and the resulting map a is a homomorphism. If, for example, we take Y so that Y and Yg are both expansions of X (as in the definition of 'symbol'), then

$$a(g) = \delta(Y, X) - \delta(Yg, X).$$

By computing $a(\gamma)$ for a glide γ, one sees that $\operatorname{im} a = \{\sum \lambda_i e_i \in A: \sum \lambda_i = 0\}$. In particular, $\operatorname{im} a \approx \mathbb{Z}^n$. Similarly, $a(F_{n,r}^0) \approx \mathbb{Z}^{n-2}$. Let $F_{n,r}^{\mathrm{s}} = \ker a = \ker(a \mid F_{n,r}^0)$. [The 's' here stands for 'simple'.] I claim that $F_{n,r}^{\mathrm{s}}$ is the commutator subgroup of both $F_{n,r}$ and $F_{n,r}^0$. It suffices to prove this for $F_{n,r}^0$.

For this purpose it is convenient to use the embedding $F_{n,r} \xrightarrow{\approx} F_{n,\infty} \hookrightarrow F_n$ (4.2 and 4.3), under which $F_{n,r}^0$ is carried to the subgroup F_n^{c} of F_n consisting of elements with support in a finitely generated subalgebra. (Equivalently, F_n^{c} consists of those $g \in F_n$ such that the corresponding homeomorphism of $\mathbb{R}$ has compact support.) It is easy to see that F_n^{c} is generated by the glides $\gamma_{ij}=g_i g_j^{-1}$, where the g_i are as in 4.9; this follows, for instance, from what we already know about generators for the subgroups $F_{n,r}$ of F_n^{c}. Setting $\gamma_i=\gamma_{i,i+1}$, it follows that F_n^{c} is generated by the γ_i, $i\in\mathbb{Z}$. Under the map a, transported to F_n^{c}, we have $\gamma_i\mapsto e_i-e_{i+1}\in A$. The claim will now follow if we prove:

(a) $\gamma_i=\gamma_{i+n-1}$ in the abelianization $(F_n^{\mathrm{c}})_{\mathrm{ab}}$. In particular, $(F_n^{\mathrm{c}})_{\mathrm{ab}}$ is generated by the images of any $n-1$ consecutive γ's.

(b) $\gamma_1\cdots\gamma_{n-1}$ is in the commutator subgroup of F_n^{c}.

For (a) note first that any g_j with $j\ll 0$ conjugates γ_i to γ_{i+n-1} by 4.7. Now take $k\gg 0$. Then g_k has support disjoint from that of γ_{i+n-1}, so $\gamma_{jk}=g_j g_k^{-1}$ also conjugates γ_i to γ_{i+n-1}. This proves (a). Similarly,

$$\begin{aligned}\gamma_1\cdots\gamma_{n-1}&=g_1g_n^{-1}=g_1g_0^{-1}g_1^{-1}g_0=[g_1,g_0^{-1}]\\&=[g_1g_0^{-1},g_0^{-1}]=[\gamma_0^{-1},g_0^{-1}]\\&=[\gamma_0^{-1},g_0^{-1}g_k]\end{aligned}$$

for $k\gg 0$, since g_k has disjoint support from γ_0. This proves (b) and hence the claim.

4.13. Theorem. *Any non-trivial subgroup of $F_{n,r}$ normalized by the commutator subgroup $F_{n,r}^{\mathrm{s}}$ contains it. In particular, $F_{n,r}^{\mathrm{s}}$ is simple, and every non-trivial normal subgroup of $F_{n,r}$ contains it.*

Sketch of proof. As above, it is convenient to work in F_n. We will use, in this sketch of the proof, informal language motivated by the homeomorphism group interpretation of F_n.

The first step is to observe that we can freely move parts of bases around by means of elements of $H=[F_n^{\mathrm{c}},F_n^{\mathrm{c}}]$: Suppose Y is a finite set of consecutive elements in some expansion of our basis $X=\{\dots,x_{-1},x_0,x_1,\dots\}$ for V_n. Choose $g\in F_n$ which is a product of the g_i's and which takes Y to a set of consecutive elements of X. Note that we have a lot of freedom as to where in X the set Yg starts. Now replace g by its product with enough elements g_k^{-1} ($k\gg 0$) to make $g\in F_n^{\mathrm{c}}$ without changing Yg. Repeating this argument, we can find $h\in F_n^{\mathrm{c}}$ which moves the support of g far away from Y. Then the commutator $(g^{-1})^h\cdot g$ (where the exponent denotes conjugation) is in H and agrees with g on Y.

Now let N be a non-trivial subgroup of F_n normalized by H, and choose $1\neq g\in N$. Then we can find a subalgebra V' of V_n (generated by one element, for instance) such that $V'g$ is disjoint from V'. If h is a commutator of automorphisms supported in V', the commutator $(h^{-1})^g\cdot h=g^{-1}\cdot g^h$ is in N and agrees with h on V'. So we now have lots of non-trivial elements of N, enough to freely move things around in V'. Repeating this argument, but with these new elements of N instead of the original g, we can show (cf. [13, 1.4.6]) that there is a subalgebra V'' of V' such that N contains all commutators of automorphisms supported in V''. By the previous paragraph, we can move V'' by an element of H to a subalgebra containing an arbitrarily big chunk X' of X. Hence N contains the commutator of any two elements supported in the algebra generated by X', so $N \supseteq H$. □

Remark. The group H is not the commutator subgroup of F_n. In fact, it is easy to see that F_n^c is the commutator subgroup of F_n. So it is the *second* commutator of F_n that is simple.

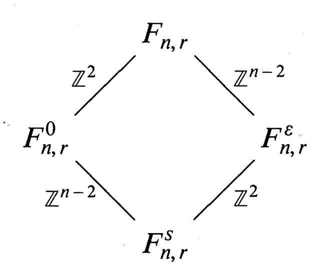
To complete our discussion of normal subgroups of $F_{n,r}$, we wish to define one more subgroup. Let $\bar{A}$ be the quotient of A obtained by introducing the additional relations $e_- = e_0$ and $e_+ = e_{r-1}$. It is free abelian of rank $n-1$, with basis (e_i) $(i \in \mathbb{Z}/(n-1)\mathbb{Z})$. Let $\bar{a}: F_{n,r} \to \bar{A}$ be the composite

$$F_{n,r} \xrightarrow{a} A \twoheadrightarrow \bar{A}.$$

In other words, $\bar{a}$ is defined in the same way as a, but with no special treatment for first and last basis elements. Let $F_{n,r}^{\varepsilon} = \ker \bar{a}$; its abelianization is given by a map $F_{n,r}^{\varepsilon} \to \mathbb{Z} \oplus \mathbb{Z}$ which records the 'endpoint' behavior of an automorphism.

Recall that we have an inclusion $F_{n,r} \subset F_n^c$, obtained by regarding $F_{n,r}$ as the group of order-preserving automorphisms of V_n with support in the subalgebra generated by $x_0, \dots, x_{r-1}$. It follows easily from our computation of the abelianization of F_n^c above that $F_{n,r}^{\varepsilon} = F_{n,r} \cap [F_n^c, F_n^c]$. This fact will be used in the next section.

The following diagram summarizes the subgroups of $F_{n,r}$ that have been defined; the labels indicate the corresponding quotients:



Now consider $T_{n,r}$. Let $d = \gcd(r, n-1)$. We define a homomorphism $\theta: T_{n,r} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$ as follows. Given $g \in T_{n,r}$, choose a symbol $(\mathscr{F}, \mathscr{F}', \sigma)$, and let $y_0, \dots, y_{s-1}$ (resp. $y'_0, \dots, y'_{s-1}$) be the leaves of $\mathscr{F}$ (resp. $\mathscr{F}'$), in order. By definition of $T_{n,r}$, there exists $k \in \mathbb{Z}/s\mathbb{Z}$ such that $y_i \cdot \sigma = y'_{i+k}$, where the subscripts are considered as integers mod s. We set $\theta(g) = k \pmod d$. It is easy to check that θ is well-defined and a homomorphism.

Note that $\theta(\varrho) = 1$ if ϱ is a rotation as in 4.11, and $\theta(\gamma) = 0$ if γ is a glide. It follows that the kernel $T_{n,r}^0$ of θ is generated by ϱ^d and glides; in fact, one easily deduces from 4.11 that $T_{n,r}^0$ is generated by ϱ^d and the γ_i $(i \in \mathbb{Z}/r\mathbb{Z})$ if r is large enough. [Use relation (i).] On the other hand, relation (iv) of 4.11 implies that ϱ^d is a product of glides. So $T_{n,r}^0$ is the subgroup of $T_{n,r}$ generated by glides. In particular, $T_{n,r}$ itself is generated by glides if and only if r is relatively prime to $n-1$.

We will show by the methods of [13] that the commutator subgroup of $T^0_{n,r}$ is simple. Before stating the precise result, we compute the abelianization of $T^0_{n,r}$.

Let A be a free abelian group of rank d with basis $\{e_i\}$, $i\in\mathbb{Z}/d\mathbb{Z}$. There is a homomorphism $a: T^0_{n,r}\to A$, defined in exactly the same way as the homomorphism $\bar{a}: F_{n,r}\to\bar{A}$ above. As before, $\operatorname{im} a\approx\mathbb{Z}^{d-1}$; in fact, $a(\gamma_i)=e_i-e_{i+1}$. I claim that the kernel $T^s_{n,r}$ is the commutator subgroup of $T^0_{n,r}$. It suffices to show:

(a) $\gamma_i=\gamma_{i+d}$ in $(T^0_{n,r})_{\mathrm{ab}}$; in particular, $(T^0_{n,r})_{\mathrm{ab}}$ is generated by the image of any d consecutive γ's.

(b) $\gamma_1\cdots\gamma_d$ is in the commutator subgroup of $T^0_{n,r}$.

(a) follows from relation (i) of 4.11, since $\varrho^d\in T^0_{n,r}$. To prove (b), note first that $\gamma_1\cdots\gamma_{n-1}$ is trivial mod commutators. In fact, it is in the kernel of the abelianization map $F_{n,r}\twoheadrightarrow\mathbb{Z}^n$, so it is already in the commutator subgroup of $F_{n,r}\subset T^0_{n,r}$. Thus if we denote by t the product of d consecutive γ's mod commutators, we have $t^{(n-1)/d}=1$. Now apply relation (iv) of 4.11 to conclude that $t^{r/d}\equiv\varrho^{n-1}$ mod commutators. Since the left side is already known to have order dividing $(n-1)/d$ and the right side has order dividing r/d, it follows that $t^{r/d}=1$ and hence that $t=1$, as required. This proves the claim.

We can summarize the situation by the diagram

$$
\begin{array}{lll}
 & T_{n,r} & \twoheadrightarrow \mathbb{Z}/d\mathbb{Z} \\
 & \cup & \\
(4.14) & T^0_{n,r} & \twoheadrightarrow \mathbb{Z}^{d-1} \\
 & \cup & \\
 & T^s_{n,r} &
\end{array}
$$

where the lower arrow is an abelianization map.

Remark. $T^0_{n,r}$ is not the commutator subgroup of $T_{n,r}$, nor is $T^s_{n,r}$, unless $d=1$. In fact, one can show that the abelianization of $T_{n,r}$ is $\mathbb{Z}/d\mathbb{Z}\oplus\mathbb{Z}/d\mathbb{Z}$.

4.15. Theorem. *Any non-trivial subgroup of $T_{n,r}$ normalized by $T^s_{n,r}$ contains it. In particular, $T^s_{n,r}$ is a simple group and is the second commutator subgroup of $T_{n,r}$, and every non-trivial normal subgroup of $T_{n,r}$ contains it.*

If $d=1$, for instance, as in Thompson's original situation where $n=2$, then $T^s_{n,r}=T_{n,r}$, so the latter is simple.

Sketch of proof of 4.15. We could repeat the arguments of 4.13, but it is easier to deduce 4.15 from 4.14. If N is a non-trivial subgroup normalized by $T^s_{n,r}$, then $N\cap F_{n,r}$ is normalized by $F^s_{n,r}$. It is easy to see that this intersection is non-trivial, so N contains $F^s_{n,r}$ by 4.13. In particular, N contains the commutator $[\gamma_i,\gamma_j]$ of any two of our generating glides for $F_{n,r}$. We may repeat this argument with X reordered (but with the same underlying cyclic order), so N in fact contains the commutator $[\gamma_i,\gamma_j]$ of any two of our generating glides for $T^0_{n,r}$. Now any conjugate of this commutator is a commutator of a similar pair of glides, defined with respect to a cyclically ordered basis other than X, so N also contains it. Hence N contains a set of generators for $T^{\mathrm{s}}_{n,r}$. $\square$

For the sake of completeness, we finish the simplicity discussion by recalling the result of [17] analogous to 4.13 and 4.15. Let d now denote $\gcd(2, n-1)$. Given a symbol $(\mathscr{F}, \mathscr{F}', \sigma)$ for an element $g \in G_{n,r}$, let $\sigma' = \sigma\tau$, where τ is the order-preserving bijection from the leaves of $\mathscr{F}'$ to those of $\mathscr{F}$. σ' is a permutation of the leaves of $\mathscr{F}$, and we set $\theta(g) = 0 \pmod d$ (resp. $1 \pmod d$) if σ' is even (resp. odd). Then θ is a well-defined surjection $G_{n,r} \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}$, and we set $G^{\mathrm{s}}_{n,r} = \ker\theta$.

4.16. Theorem. *$G^{\mathrm{s}}_{n,r}$ is the commutator subgroup of $G_{n,r}$, and every non-trivial subgroup normalized by $G^{\mathrm{s}}_{n,r}$ contains it. In particular, $G_{n,r}$ is simple if n is even and contains a simple subgroup of index* 2 *if n is odd.*

It is instructive to read Higman's proof of this while keeping in mind the homeomorphism group interpretation of $G_{n,r}$. One sees the same 'small support' idea as in the proofs of 4.13 and 4.15, but stated in purely algebraic language.

4E. Finiteness properties

4.17. Theorem. *The groups $F_{n,r}$, $F^{\varepsilon}_{n,r}$, $T_{n,r}$, $T^{0}_{n,r}$, $T^{\mathrm{s}}_{n,r}$, $G_{n,r}$, and $G^{\mathrm{s}}_{n,r}$ are all finitely presented and of type* FP_∞. *The groups $F^{0}_{n,r}$ and $F^{\mathrm{s}}_{n,r}$ can each be expressed as an increasing union of finitely presented* FP_∞ *groups.*

We will give the proof in detail for $G_{n,r}$ and then indicate the modifications needed to handle the other groups.

Let $\mathscr{B}$ be the set of all bases of $V = V_{n,r}$. For $Y, Z \in \mathscr{B}$ we write $Y < Z$ if Z is an expansion of Y. This makes $\mathscr{B}$ a poset. For any $Y \in \mathscr{B}$ let $\mathscr{B}_{<Y} = \{Z \in \mathscr{B} : Z < Y\}$, and define $\mathscr{B}_{\leq Y}$ similarly. We denote by $|\mathscr{B}|$ the simplicial complex whose simplices are the finite linearly ordered subsets of $\mathscr{B}$. As usual, we will use this construction of a simplicial complex associated to a poset to assign topological concepts to posets. In particular, we can talk about connectivity, contractibility, dimension, etc., for $\mathscr{B}$ and various subsets of $\mathscr{B}$.

The first observation is that $\mathscr{B}$ is *contractible*, being a directed set. (Any two bases have a common expansion.) Now consider the action of $G = G_{n,r}$ on $\mathscr{B}$ induced by its action on V. The stabilizer of a basis $Y \in \mathscr{B}$ is the symmetric group of permutations of Y. In particular, it is finite, and it follows that the stabilizer of every simplex of $\mathscr{B}$ is finite. Thus $|\mathscr{B}|$ is m-good for G in the sense of Section 1 for every m. But $|\mathscr{B}|$ has infinite skeleta mod G because bases can have arbitrarily large cardinality. We will therefore filter $|\mathscr{B}|$ and apply Sections 2 and 3.

For $Y \in \mathscr{B}$ let $h(Y)$ be the largest integer h such that there is a chain $Y = Y_h > \cdots > Y_0$ in $\mathscr{B}$. Equivalently, $h(Y) = \dim \mathscr{B}_{\leq Y}$. We call $h(Y)$ the *height* of Y. For any $h \geq 0$ set

$$\mathscr{B}_h = \{Y \in \mathscr{B} : h(Y) \leq h\}.$$

Then $\mathscr{B}_h$ is G-invariant and is finite mod G. We wish to understand the homotopy properties of the inclusion $\mathscr{B}_h \subset \mathscr{B}_{h+1}$. We will need the following three lemmas.

4.18. Lemma. *Suppose $Y_1, \dots, Y_k$ are distinct simple contractions of a basis Y. Then $Y_1, \dots, Y_k$ have a lower bound in $\mathscr{B}$ if and only if the k contracted n-tuples are pairwise disjoint in Y. In this case the Y_i have a greatest lower bound, namely, the basis Z obtained by contracting all k of the n-tuples.*

Proof. If the n-tuples are disjoint, then the Y_i obviously admit Z as a lower bound. Conversely, suppose W is a lower bound for the Y_i, i.e., each Y_i is an expansion of W. Recall that bases which are expansions of W are in 1–1 correspondence with finite n-ary forests with W as the set of roots. Let $\mathscr{F}$ be the forest corresponding to Y and let $\mathscr{F}_i$ be the subforest corresponding to Y_i. Then $\mathscr{F}_i$ is gotten from $\mathscr{F}$ by removing n siblings, i.e., n leaves which are the n descendants of some node. Clearly k distinct sets of siblings are disjoint, so the contracted n-tuples are indeed disjoint. The basis Z corresponds to the forest gotten by removing all k sets of siblings, so $W \leq Z$. Since W was an arbitrary lower bound, this shows that Z is the greatest lower bound. □

4.19. Lemma. *For any $Y \in \mathscr{B}$, the complex $|\mathscr{B}_{<Y}|$ is homotopy equivalent to the following simplicial complex $\Sigma = \Sigma(Y)$: The vertices of Σ are the ordered n-tuples of distinct elements of Y, and a collection of such n-tuples is a simplex of Σ if and only if the underlying sets of the n-tuples are pairwise disjoint.*

Proof. We will use a standard argument, which essentially goes back to Folkman [14]. Let $K = |\mathscr{B}_{<Y}|$. For any $Y' < Y$ let $K_{Y'}$ be the subcomplex $|\mathscr{B}_{\leq Y'}|$ of K. It is a cone, hence it is contractible. Note that $K = \bigcup K_{Y'}$, where Y' ranges over the simple contractions of Y. Given a collection $\{Y_i\}$ of simple contractions of Y, the intersection $L = \bigcap K_{Y_i}$ is non-empty if and only if the Y_i have a lower bound, and in this case L is the contractible complex K_Z, Z being the greatest lower bound. Hence K is homotopy equivalent to the nerve of the cover $\{K_{Y'}\}$. (See, for instance, [16, 1.9].) But this nerve is precisely Σ by 4.18. □

4.20. Lemma. *For any integer $k \geq 0$ there is an integer $\mu(k)$ such that $\Sigma(Y)$ is k-connected whenever* $\text{card}(Y) \geq \mu(k)$.

Proof. If $\text{card}(Y) \geq 3n-1$, then any two vertices of $\Sigma(Y)$ can be connected by an edge path of length ≤ 2. For either they are disjoint, in which case they are connected by an edge, or else they involve at most $2n-1$ elements of Y, in which case they can both be connected to a third vertex. So we can take $\mu(0) = 3n-1$. More precisely, if we also set $\nu(0) = 3$, we have proven the following assertion for $k = 0$:

(A_k) Whenever $\mathrm{card}(Y) \geq \mu(k)$, the k-skeleton $\Sigma(Y)^k$ is null-homotopic in $\Sigma(Y)$ by a homotopy under which any k-simplex remains in a subcomplex having at most $\nu(k)$ vertices.

Assume inductively that $\mu(k-1)$ and $\nu(k-1)$ have been defined and that (A_{k-1}) holds. Suppose $\mathrm{card}(Y) \geq \mu(k-1)$ and choose a null-homotopy of Σ^{k-1} as in (A_{k-1}), where $\Sigma = \Sigma(Y)$. For any k-simplex σ of Σ, each of its codimension 1 faces τ remains in a subcomplex having at most $\nu(k-1)$ vertices, hence $\partial\sigma$ remains in a subcomplex Σ' having at most $m = (k+1)\nu(k-1)$ vertices. If $\mathrm{card}(Y) \geq c = nm + n$, we can find another vertex υ such that Σ' is contained in the link of υ. We can therefore extend the null-homotopy to σ by coning, with σ remaining in the full subcomplex generated by Σ' and υ. Hence (A_k) holds with $\mu(k) = \max(\mu(k-1), c)$ and $\nu(k) = m + 1$. □

Remark. This proof yields a value of $\mu(k)$ that is very far from the best possible. When $n = 2$, for instance, K. Vogtmann has shown [private communication] that one can take $\mu(k) = 3k + 5$, whereas the proof above yields a value of $\mu(k)$ bigger than $k!$.

Proof of 4.17 (for $G = G_{n,r}$). The passage from $|\mathscr{B}_h|$ to $|\mathscr{B}_{h+1}|$ consists of adjoining, for each Y with $h(Y) = h + 1$, a cone over $|\mathscr{B}_{<Y}|$. In view of 4.19 and 4.20, it follows that the connectivity of $(|\mathscr{B}_{h+1}|, |\mathscr{B}_h|)$ tends to ∞ as h tends to ∞. Hence G is finitely presented and of type FP_∞ by 3.3(a). □

We now take up the other groups listed in the statement of 4.17.

(i) $F_{n,r}$ and $T_{n,r}$: Instead of $\mathscr{B}$, use the subposet consisting of ordered bases (or cyclically ordered bases). The proof goes through with minor changes. For example, the complex $\Sigma(Y)$ is replaced by the subcomplex consisting of disjoint consecutive (or cyclically consecutive) n-tuples.

(ii) $T^0_{n,r}$ and $G^{\mathrm{s}}_{n,r}$: These have finite index in $T_{n,r}$ and $G_{n,r}$.

(iii) $F^\varepsilon_{n,r}$ and $T^{\mathrm{s}}_{n,r}$: Recall the abelian group $A \approx \mathbb{Z}^{n+1}$ *and the function* $\delta(\cdot,\cdot)$ of 4.12. Let $\bar{A}$ be as in the definition of $F^\varepsilon_{n,r}$ and let $\bar{\delta}$ be δ followed by the quotient map $A \twoheadrightarrow \bar{A}$. Then we can prove 4.17 for $F^\varepsilon_{n,r}$ by using the poset of ordered bases Y such that the coefficient of e_i in $\bar{\delta}(Y, X)$ is non-negative for all $i \in \mathbb{Z}/(n-1)\mathbb{Z}$. A similar poset works for $T^{\mathrm{s}}_{n,r}$.

(iv) $F^0_{n,r}$ and $F^{\mathrm{s}}_{n,r}$: We use the isomorphisms $F^0_{n,r} \approx F^{\mathrm{c}}_n$, $F^{\mathrm{s}}_{n,r} \approx [F^{\mathrm{c}}_n, F^{\mathrm{c}}_n]$ (cf. Section 4D). For any set Y of consecutive elements of our basis X for V_n, consider the subgroup of F^{c}_n consisting of automorphisms supported in the subalgebra generated by Y. This subgroup is isomorphic to $F_{n,r}$ ($r = \mathrm{card}(Y)$), hence is known to be finitely presented and of type FP_∞. Clearly F^{c}_n is the increasing union of such subgroups. Similarly, $[F^{\mathrm{c}}_n, F^{\mathrm{c}}_n]$ is the increasing union of subgroups isomorphic to $F^\varepsilon_{n,r}$ by the remarks following the definition of $F^\varepsilon_{n,r}$.

4F. Vanishing cohomology

It is a remarkable fact that almost all of the known examples of groups Γ with good finiteness properties also have $H^i(\Gamma, \mathbb{Z}\Gamma)=0$ for all i where this is theoretically possible. Indeed, all examples of non-vanishing $H^i(\Gamma, \mathbb{Z}\Gamma)$ that I know of (for Γ of type FP_∞, say) can be explained by one of the following two facts:

(a) If Γ is of type FP_∞ and Γ has virtual cohomological dimension $m<\infty$, then $H^m(\Gamma, \mathbb{Z}\Gamma)\neq 0$, cf. [8, VIII.6.7 and III.6.5].

(b) One can construct new examples of non-vanishing $H^*(\Gamma, \mathbb{Z}\Gamma)$ from given ones by forming amalgamated free products or HNN extensions.

We will see here that this vanishing cohomology phenomenon continues for the groups Γ we have been discussing.

4.21. Theorem. *All of the* FP_∞ *groups* Γ *mentioned in the statement of* 4.17 *satisfy* $H^*(\Gamma, \mathbb{Z}\Gamma)=0$.

(For $F_{2,1}$ and $T_{2,1}$ this is due to Brown and Geoghegan, cf. [10] and Remark 2 of Section 4B above.)

Proof of 4.21. We will give the proof for $G=G_{n,r}$ and leave it to the reader to check that a similar proof works for the other groups.

For any basis Y of $V_{n,r}$ let $\lambda(Y)$ be the number of simple expansions required to get from Y to the minimal common expansion of X and Y. For any $p\geq 0$ set

$$\mathscr{B}^p=\{Y\in\mathscr{B}: \lambda(Y)+h(Y)\geq p\}.$$

Then we have

$$\mathscr{B}=\mathscr{B}^0\supset\mathscr{B}^1\supset\cdots,$$

and $\bigcap_{p\geq 0}\mathscr{B}^p=\emptyset$. Note that $\mathscr{B}^p$ is closed under expansion, hence it is still a directed set; in particular, it is contractible. Note also that $\mathscr{B}-\mathscr{B}^p$ is finite; for if $Y\notin\mathscr{B}^p$, then Y is a contraction of an expansion Z of X with $h(Z)<p$, and there are only finitely many of these. One should think of the $\mathscr{B}^p$ as a decreasing sequence of neighborhoods of ∞ in $\mathscr{B}$.

We now use the connection between $H^*(\Gamma, \mathbb{Z}\Gamma)$ and cohomology with compact supports, cf. [8, §VIII.7, Exercise 4]. Since $\mathscr{B}_h$ is highly connected for large h, we have, for any i,

$$\begin{aligned} H^i(G,\mathbb{Z}G) &= H^i_c(\mathscr{B}_h) \quad \text{for } h\gg 0 \\ &= \varinjlim_p H^i(\mathscr{B}_h, \mathscr{B}_h^p), \end{aligned}$$

where $\mathscr{B}_h^p=\mathscr{B}_h\cap\mathscr{B}^p$. To complete the proof it suffices to show that, for fixed i and large enough h, $\mathscr{B}_h^p$ is i-connected for all p. But this is easy. One need only go back to 4.19 and 4.20, prove analogues for the poset $\mathscr{B}^p$, and observe that there is a single μ in 4.20 that works for all p. $\square$

5. Example 2: Houghton's groups

The groups to be discussed in this section were introduced by Houghton [18]. See also [22] and [26].

Fix an integer $n \geq 1$, let $\mathbb{N}$ be the set of positive integers, and let $S = \mathbb{N} \times \{1, \ldots, n\}$. We think of S as the disjoint union of n copies of $\mathbb{N}$, each arranged along a ray emanating from the origin in the plane. Let H be the group of all permutations g of S such that on each ray g is eventually a translation. More precisely, we require:

(∗) There is an n-tuple $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ such that for each $i \in \{1, \ldots, n\}$ one has $(x, i) \cdot g = (x + m_i, i)$ for all sufficiently large $x \in \mathbb{N}$.

The assignment $g \mapsto (m_1, \ldots, m_n)$ defines a homomorphism $a : H \to \mathbb{Z}^n$ whose image is the subgroup $\{(m_1, \ldots, m_n) \in \mathbb{Z}^n \colon \sum m_i = 0\}$, of rank $n-1$. The kernel of a is the infinite symmetric group, consisting of all permutations of S with finite support. It is the commutator subgroup of H if $n \geq 3$. For $n = 1$ and 2, one checks that the commutator subgroup of H is the infinite alternating group. In all cases the second commutator subgroup of H is a locally finite infinite simple group.

If $n = 1$, then H is the infinite symmetric group and hence is not finitely generated. But H is easily seen to be finitely generated if $n = 2$, and it was shown by R.G. Burns and D. Solitar [unpublished] to be finitely presented if $n = 3$. We complete the picture by proving:

5.1. Theorem. *The group H is of type* FP_{n-1} *but not* FP_n. *For $n \geq 3$ it is finitely presented.*

The proof is an imitation of what we did for the Thompson–Higman groups. The first and crucial step is to find a poset to play the role of the poset $\mathscr{B}$ used in Section 4E. Let M be the monoid of 1–1 maps $S \to S$ satisfying (∗). [As in Section 4 our groups and monoids act on the right; thus $\alpha\beta$ denotes α followed by β for $\alpha, \beta \in M$.] Let $T \subset M$ be the commutative submonoid consisting of *translations*, i.e., elements $t \in M$ satisfying $(x, i) \cdot t = (x + m_i, i)$ for *all* $x \in \mathbb{N}$. Necessarily, then, $m_i \geq 0$. T is a free commutative monoid generated by elements $t_1, \ldots, t_n$, where t_i translates by 1 on the i-th ray and is the identity on the others.

Given $\alpha, \beta \in M$ we write $\alpha \leq \beta$ if $\beta = t\alpha$ for some $t \in T$. We denote by $\mathscr{M}$ the underlying set of M, equipped with this partial order. It is a directed set, and the underlying set of the submonoid T is a cofinal subset. [The latter should be thought of as the analogue of the subset of $\mathscr{B}$ consisting of the expansions of the given basis X of $V_{n,r}$.] The group H is a subgroup of M, hence it acts by right multiplication. This commutes with the action of T by left multiplication and induces an action of H on the poset $\mathscr{M}$.

Note that the stabilizer of a simplex of $|\mathscr{M}|$ is finite. Moreover, if we filter $\mathscr{M}$ by height, then each $|\mathscr{M}_h|$ is finite mod H. We are therefore in a position to apply the results of Sections 2 and 3.

5.2. Lemma. *Given $\alpha \in \mathcal{M}$, let Y be the finite set $S - S \cdot \alpha$. Then $|\mathcal{M}_{<\alpha}|$ is homotopy equivalent to the following simplicial complex $\Sigma = \Sigma(Y)$: The set of vertices of Σ is $\{1, \ldots, n\} \times Y$, and a collection of these vertices forms a simplex if the first coordinates are all distinct and the second coordinates are all distinct.*

(Thus Σ is a certain subcomplex of the complex of disjoint pairs of elements of $\{1, \ldots, n\} \amalg Y$.)

Proof. The maximal elements of $\mathcal{M}_{<\alpha}$ are those β such that $\alpha = t_i \beta$ for some i. Such a β agrees with α except on the i-th ray, and on the i-th ray it satisfies $(x+1, i) \cdot \beta = (x, i) \cdot \alpha$ for all $x \in \mathbb{N}$. Thus β is determined by specifying i and specifying $y = (1, i) \cdot \beta \in Y$. It is easy to check that a collection of such β's has a lower bound in $\mathcal{M}$ if and only if the corresponding collection of pairs (i, y) is a simplex of Σ. Moreover, in this case there is a greatest lower bound. The result now follows as in the proof of 4.19. □

Let $\Sigma_{n,h}$ denote the complex $\Sigma(Y)$ for a set Y of cardinality h. [*Note*: If $Y = S - S\alpha$ as above, then the cardinality h is equal to the height of α in the poset $\mathcal{M}$.]

5.3. Lemma. *$\Sigma_{n,h}$ is $(n-2)$-connected for $h \geq 2n-1$. More precisely, it is homotopy equivalent to a bouquet of $p_n(h)$ spheres of dimension $n-1$, where p_n is the monic polynomial of degree n defined inductively by $p_0 = 1$ and*

$$p_n(u) = (u-1)p_{n-1}(u-2) + (n-1)p_{n-1}(u-1).$$

Proof. We argue by induction on n, using a method due to K. Vogtmann [private communication]. If $n=1$, then $\Sigma_{n,h}$ is a bouquet of $h-1$ copies of S^0, provided $h \geq 1 = 2 \cdot 1 - 1$. Now suppose $n \geq 2$ and assume that $\Sigma_{n-1,k}$ is homotopy equivalent to a bouquet of $p_{n-1}(k)$ copies of S^{n-2} for $k \geq 2(n-1)-1$. We build up $\Sigma_{n,h}$ in several steps. Start with the contractible complex $K = \mathrm{st}(v)$ [= the star of v] for some vertex v of the form $(1, y_0)$. Now let K' be the full subcomplex generated by K and the vertices $(1, y)$ for $y \neq y_0$. Any simplex of K' that is not in K involves exactly one of the new vertices, so K' is obtained from K by adjoining, for each $y \neq y_0$, a cone over $\mathrm{lk}_K(1, y)$ [= the link of $(1, y)$ in K]. Now $\mathrm{lk}_K(1, y) \approx \Sigma_{n-1,h-2}$ (with vertex set $\{2, \ldots, n\} \times (Y - \{y_0, y\})$), so K' is homotopy equivalent to a bouquet of $h-1$ copies of the suspension of $\Sigma_{n-1,h-2}$. In view of the inductive hypothesis, then, K' is equivalent to a bouquet of $(h-1)p_{n-1}(h-2)$ copies of S^{n-1}.

Now adjoin the remaining vertices (i, y_0), $i \neq 1$. We have $\mathrm{lk}_{K'}(i, y_0) \approx \Sigma_{n-1,h-1}$, so $\Sigma_{n,h}$ is obtained from K' by adjoining cones over $n-1$ copies of $\Sigma_{n-1,h-1}$. Using the inductive hypothesis again, we see that, up to homotopy, we are adjoining $(n-1)p_{n-1}(h-1)$ cells of dimension $n-1$ to our bouquet of $(n-1)$-spheres, whence the lemma. □

Proof of 5.1. Arguing as in the proof of 4.17, we deduce from 5.2 and 5.3 that, for large h, the passage from $\mathcal{M}_h$ to $\mathcal{M}_{h+1}$ consists of the adjunction of n-cells, up to homotopy. The theorem therefore follows from 3.3(b). □

Remark. The proof suggests that, in some sense, $\mathcal{M}$ is essentially n-dimensional. One can make this intuition precise by constructing an n-dimensional cubical complex X on which H acts, with $\mathcal{M}$ as the set of vertices. For each $\alpha \in \mathcal{M}$, the vertices $t_1^{\varepsilon_1} \cdots t_n^{\varepsilon_n}\alpha$ $(\varepsilon_i = 0, 1)$ span an n-cube of X.

6. Example 3: The groups of Abels

For $n \geq 1$ and p a prime number, let $\Gamma_n \subset \mathrm{GL}_{n+1}(\mathbb{Z}[1/p])$ be the group of upper triangular matrices g with $g_{11} = g_{n+1,n+1} = 1$. For example,

$$\Gamma_1 = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that Γ_1 is not finitely generated. But Γ_2 is easily seen to be finitely generated, and Γ_3 was shown by Abels [1] to be finitely presented. Further work by various people eventually led to the following result (see [2] for references):

6.1. Theorem. *Γ_n is of type* FP_{n-1} but not FP_n; *for $n \geq 3$ it is finitely presented.*

We will give a new proof of 6.1 by applying the results of Sections 2 and 3 to the Bruhat–Tits building X associated to the group GL_{n+1} and the p-adic valuation of $\mathbb{Q}$. We recall the relevant definitions.

Notation

K = a field with discrete valuation v.
A = the valuation ring.
π = a generator of the maximal ideal of A.
k = the residue field $A/\pi A$.
$V = K^{n+1}$, $n \geq 1$.

A *lattice* L in V is a finitely generated A-submodule of V spanning V as a vector space over K. Equivalently, L is a free A-submodule of V with an A-basis which is also a vector space basis of V. The set $\mathcal{L}$ of lattices in V is a poset under inclusion. We denote by W the n-dimensional subcomplex of $|\mathcal{L}|$ consisting of the simplices $\{L_0, \ldots, L_q\}$ such that

$$L_0 < L_1 < \cdots < L_q < \pi^{-1} L_0.$$

Note that W is locally finite if the residue field k is finite.

Let $i: K \hookrightarrow V$ be the inclusion of the first factor, and let $\mathrm{pr}: V \twoheadrightarrow K$ be the projection onto the last factor. For any $L \in \mathscr{L}$ the image $\mathrm{pr}(L)$ is a lattice in K, hence

$$\mathrm{pr}(L) = \pi^{-\varepsilon(L)} A$$

for some integer $\varepsilon(L)$. The resulting function $\varepsilon: \mathscr{L} \to \mathbb{Z}$ is an example of an *augmentation* in the sense of [16]. (The minus sign in the definition is used so that ε will be order-preserving.) Similarly, we define $\eta: \mathscr{L} \to \mathbb{Z}$ by

$$i^{-1}(L) = \pi^{-\eta(L)} A.$$

It is also an augmentation.

In case $K = \mathbb{Q}$ and v is the p-adic valuation, there is an obvious action of $\Gamma = \Gamma_n$ on $\mathscr{L}$, and both ε and η are Γ-invariant. We will use one of our augmentations, as in [16], to construct a model X for the Bruhat–Tits building as a subcomplex of W, and we will use the other one to filter X.

Let R be the abstract ordered simplicial complex which triangulates the real line $\mathbb{R}$, with $\mathbb{Z}$ as vertex set and the usual ordering. For $r, s \in \mathbb{Z}$ with $r \le s$, we denote by $[r, s]$ the subcomplex of R which triangulates the interval from r to s. The functions ε and η extend to simplicial maps $W \to R$, still called ε and η. We set $X = \varepsilon^{-1}(0)$, $X_r = X \cap \eta^{-1}(r)$ $(r \in \mathbb{Z})$, and $X_{rs} = X \cap \eta^{-1}([r, s])$ $(r, s \in \mathbb{Z}, r \le s)$. X is isomorphic to the usual Bruhat–Tits building and hence is contractible [16].

Example. Suppose $n = 1$. Then X is a tree, and a vertex L of X is a lattice in K^2 with a basis of the form $\{(\pi^{-r}, 0), (x, 1)\}$ for some $x \in K$, where $r = \eta(L)$. The class of $x \bmod \pi^{-r} A$ is uniquely determined by L. Thus the 'slice' X_r of X is a 0-dimensional complex whose vertex set can be identified with $K/\pi^{-r}A$. It is easy to see that a vertex L of X_r as above is connected by an edge to a unique vertex of X_{r+1}, namely, the lattice with basis $\{(\pi^{-r-1}, 0), (x, 1)\}$. So X is the mapping telescope of the sequence

$$\cdots \twoheadrightarrow K/\pi^2 A \twoheadrightarrow K/\pi A \twoheadrightarrow K/A \twoheadrightarrow K/\pi^{-1} A \twoheadrightarrow \cdots$$

of 0-dimensional complexes. It follows, in particular, that X_{rs} is homotopy equivalent to X_s and that $X_{r,s+1}$ is obtained from X_{rs} by the adjunction of cones over bouquets of 0-spheres.

For general n we will show that the system of subcomplexes X_{rs} has similar homotopy theoretic properties:

6.2. Lemma. (a) *X_s is a deformation retract of X_{rs}.*

(b) *Up to homotopy, $X_{r,s+1}$ is obtained from X_{rs} by the adjunction of n-cells.*

Proof. The subcomplex $[r, s]$ of R is an ordered simplicial complex, so we may form its simplicial product with X_{rs}. There is a simplicial map

$$X_{rs} \times [r, s] \to X_{rs}$$

given on vertices by $(L,t)\mapsto L+A\pi^{-t}e_1$, where e_1 is the first standard basis vector of V. This map yields a deformation retraction of X_{rs} onto X_s, whence (a).

Now consider the inclusion $X_{rs}\subset X_{r,s+1}$. A simplex σ of $X_{r,s+1}$ has the form

$$(*)\qquad L_0<\cdots<L_p<M_0<\cdots<M_q$$

$(p\geq -1, q\geq -1)$, where $\eta(L_j)\leq s$ and $\eta(M_j)=s+1$. We will filter $X_{r,s+1}$ according to the dimension of M_q/M_0 over the residue field k. (By convention, this dimension is -1 if $q=-1$, i.e., if there are no M's.) Thus we have

$$X_{rs}=F_{-1}\subset F_0\subset\cdots\subset F_n=X_{r,s+1},$$

where F_d consists of the simplices with $\dim M_q/M_0\leq d$. I claim that for each $d\geq 0$ the passage from F_{d-1} to F_d is, up to homotopy, the adjunction of n-cells. This implies (b), so it remains to prove the claim.

For each simplex σ of F_d not in F_{d-1}, write σ as in $(*)$ and let τ be the simplex $\{M_0, M_q\}$ of X_{s+1}. (τ is a 1-simplex if $d>0$ and a vertex if $d=0$.) Then σ is in $\mathrm{st}(\tau)$ [= the star of τ in F_d], and σ is not in the star of any other simplex $\{M', M\}$ of X_{s+1} with $\dim M/M'=d$. It follows that F_d is obtained from F_{d-1} by the adjunction, for each such $\tau=\{M',M\}$ with $\dim M/M'=d$, of $\mathrm{st}(\tau)$ relative to $\mathrm{st}(\tau)\cap F_{d-1}$. To prove the claim, then, it suffices to show that this intersection has the homotopy type of a bouquet of $(n-1)$-spheres.

We will use the standard notation for 'intervals' in the poset $\mathscr{L}$; for instance, $[L_1, L_2]=\{L\in\mathscr{L}: L_1\subseteq L\subseteq L_2\}$. Let $\mathscr{S}=\{L\in[\pi M, M']: \eta(L)=s \text{ and } \varepsilon(L)=0\}$. Then $\mathrm{st}(\tau)$ is the join $|\mathscr{S}|*|[M',M]|$, and $\mathrm{st}(\tau)\cap F_{d-1}$ is the subjoin

$$|\mathscr{S}|*(|[M',M)|\cup|(M',M]|).$$

Now the second factor of this join is the suspension of $|(M',M)|$, and (M',M) is isomorphic to the poset $\mathscr{T}(M/M')$ of proper non-zero subspaces of the d-dimensional vector space M/M' over k. Hence this second factor is homotopy equivalent to a bouquet of $(d-1)$-spheres by the Solomon–Tits theorem (cf. [21]). [In case $d=0$, this simply means that the second factor is empty.]

Turning now to the first factor $|\mathscr{S}|$, note that $\mathscr{S}$ is isomorphic to a certain subposet $\mathscr{S}'$ of $\mathscr{T}(M'/\pi M)$. Namely, $\mathscr{S}'$ consists of the subspaces of $M'/\pi M$ which do not contain the line $(\pi M+A\pi^{-s-1}e_1)/\pi M$ and which are not contained in the kernel of the linear map $M'/\pi M\twoheadrightarrow k$ induced by $\mathrm{pr}: V\to K$. By Vogtmann's generalization of the Solomon–Tits theorem (cf. [25, Proposition 1.4]), $|\mathscr{S}'|$ is homotopy equivalent to a bouquet of $(n-1-d)$-spheres. The join is therefore homotopy equivalent to a bouquet of $(n-1)$-spheres, as required. □

Proof of 6.1. Let $\Gamma=\Gamma_n$, and consider the contractible Γ-complex X we have just been discussing, where $K=\mathbb{Q}$ with the p-adic valuation. Filter X by the subcomplexes $X_{(r)}=X_{-r,r}$ $(r\geq 0)$. It is easy to see that each $X_{(r)}$ is finite mod Γ (cf. [2, proof of Theorem B(c)]) and that the stabilizer of every simplex is finitely presented and of type FP_∞ (cf. [2, proof of Theorem B(b)]). And Lemma 6.2 implies that, up to homotopy, $X_{(r+1)}$ is obtained from $X_{(r)}$ by the adjunction of n-cells. The theorem now follows from 3.3(b). □

Remark. It follows from the proof that X_0 [$=X_{(0)}$] is $(n-2)$-connected. This was proved in [2] by different methods and was used there to prove the positive part of 6.1.

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