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The Geometry of Rewriting Systems: A Proof of the Anick–Groves–Squier Theorem

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Abstract. Let G be a group or monoid which is presented by means of a complete rewriting system. Then one can use the resulting normal forms to collapse the classifying space of G down to a quotient complex (typically "small") of the same homotopy type. If the rewriting system is finite, then the quotient complex has only finitely many cells in each dimension. The proof yields an explicit free resolution of $\mathbf{Z}$ over $\mathbf{Z}G$, similar to resolutions obtained by Anick, Groves, and Squier.

Introduction

Several years ago Ross Geoghegan and I [6] were interested in the homological finiteness properties of a certain group G. We succeeded in constructing a small $K(G,1)$-complex, with only two n-cells for each $n \geq 1$, by an indirect method: We first built a big $K(G,1)$, with infinitely many n-cells for each n, and we then "collapsed away" all but two cells in each dimension. This notion of "collapse" will be explained in §1; for now, one can think of it as analogous to collapsing a maximal tree in a connected complex in order to get rid of all the vertices but one.

The method seemed ad hoc at the time, but it turns out to have much wider applicability than I would have guessed. In particular, it applies to groups and other algebraic objects which come equipped with a complete rewriting system [definition in §2 below]. One simply uses the classical bar construction as the starting point, i.e., as the "big" complex on which to perform the collapse, and one uses the normal forms that come from the rewriting system in order to figure out which cells to collapse.

The result of this process, in the case of a group G, is an explicit $K(G,1)$-complex, typically much smaller than the classical one. One also obtains an explicit free resolution of $\mathbf{Z}$ over $\mathbf{Z}G$, similar to resolutions obtained by Anick [1], Groves [11], and (through dimension 3) Squier [19]. In particular, we recover the Anick–Groves–Squier result that G is of type FP_∞ if the rewriting system is finite.

We begin by explaining in §1 the collapsing method of [6] that we will be using. In §2 we review rewriting systems. The application of the collapsing method to monoids with a rewriting system is then given in §3. We translate the result into algebraic language in §4 and show how it leads to a free resolution. It is easy to describe the basis elements of this resolution, but it is not so clear how to compute the boundary operator. This question is treated in §§5 and 6. Finally, §7 contains some examples.

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1. Collapsing BM: An Overview

Recall that a monoid M gives rise to a semi-simplicial complex BM, whose n-simplices are n-tuples $\sigma = (m_1, \ldots, m_n)$ of elements of M. The face operators are given by

$$d_i\sigma = \begin{cases} (m_2, \ldots, m_n) & i = 0 \\ (m_1, \ldots, m_{i-1}, m_i m_{i+1}, m_{i+2}, \ldots, m_n) & 0 < i < n \\ (m_1, \ldots, m_{n-1}) & i = n, \end{cases}$$

and the degeneracy operators are given by

$$s_i\sigma = (m_1, \ldots, m_i, 1, m_{i+1}, \ldots, m_n) \qquad (0 \le i \le n).$$

The geometric realization $X = |BM|$ is called the *classifying space* of M. It is a CW-complex with one n-cell for every non-degenerate n-simplex of BM, i.e., for every n-tuple $(m_1, \ldots, m_n)$ with $m_i \neq 1$ for all i. We will use the same symbol $(m_1, \ldots, m_n)$ for both a simplex of BM and the corresponding cell of X.

A glance at the 2-skeleton of X shows that $\pi_1(X)$ is the group completion of M, i.e., it is the target of a monoid homomorphism $M \to \pi_1(X)$ which is universal for homomorphisms from M to a group. In particular, $\pi_1(X) = M$ if M is a group. Moreover, X is an Eilenberg–MacLane complex of type $K(M, 1)$ in this case; in fact, it is the "original" $K(M, 1)$ constructed by Eilenberg and MacLane [9] and, independently, by Eckmann [8]. If M is not a group, then X may or may not be an Eilenberg–MacLane complex (cf. [16], [10]).

Our method for analyzing the homotopy type of X is based on the following trivial observation. Suppose we are building X by attaching cells, one at a time, and we are ready to attach a 1-cell $\tau = (m)$. Suppose m admits a non-trivial factorization $m = m'm''$ such that the 1-cells (m') and (m'')

have already been adjoined. Then we can adjoin τ without changing the homotopy type, provided we simultaneously adjoin the 2-cell $\sigma = (m', m'')$. For we have $\tau = d_1\sigma$, and the other two faces of σ are already present; so the adjunction is an elementary expansion, and the resulting inclusion is a homotopy equivalence. It has a canonical homotopy inverse, which "collapses" σ onto the union of the two faces other than the "free face" τ.

In this situation we will say that τ is *redundant*, since we were able to adjoin it without changing the homotopy type. The chosen σ will be called the *collapsible* cell associated to τ and will be denoted $c(\tau)$. The construction has to start somewhere, of course, so we cannot expect all 1-cells to be redundant. Those that we start with will be called *essential*. In practice, they will be the cells (s), where s ranges over some set of generators of M. [Note: We use monoid generators, even if M happens to be a group.]

Turning now to the 2-skeleton, we have already adjoined some of the 2-cells, namely, the collapsible ones. We can expect some of the non-collapsible cells to be "essential", and we must simply adjoin them. In practice, these will correspond to some set of defining relations for M. The remaining 2-cells τ will be declared "redundant", and we will try to find for each such τ a "collapsible" 3-cell $\sigma = c(\tau)$, so that the adjunction of σ and τ can be done as an elementary expansion. Thus τ should be a face of σ, and all other faces should be present already when σ and τ are adjoined.

Continuing this process, we hope to classify the cells of X in all dimensions as "essential", "collapsible", or "redundant". We want to then build X in such a way that the redundant and collapsible cells can be adjoined without changing the homotopy type, so that X will be homotopy equivalent to a complex Y with one cell for each essential cell of X. Let's spell out explicitly what we need in order for this program to work. We will do this in the context of an arbitrary semi-simplicial complex.

Let K be a semi-simplicial complex and let X be its geometric realization. As above, we will identify the set of cells of X with the set of non-degenerate simplices of K. Assume that the cells have been partitioned into three classes, whose elements are called *essential*, *collapsible*, and *redundant*, respectively. The collapsible cells are required to have dimension ≥ 1. Assume further that to each redundant n-cell τ we have associated a collapsible $(n+1)$-cell $\sigma = c(\tau)$ and an integer $i = i(\tau)$ such that $\tau = d_i\sigma$. We will often refer to τ as the *free face* of σ. If τ' is a redundant n-cell such that $\tau' = d_j\sigma$ for some $j \neq i$, then we call τ' an *immediate predecessor* of τ and write $\tau' \prec \tau$. The point of this is that when we adjoin τ and σ, we want any immediate predecessor of τ to be present already.

The given cell partition and functions c and i will be said to constitute a *collapsing scheme* for K if the following two conditions are satisfied:

(C1) *The function c defines, for each $n \geq 0$, a bijection from the set of redundant n-cells to the set of collapsible $(n+1)$-cells.*

(C2) *There is no infinite descending chain $\tau \succ \tau' \succ \tau'' \succ \cdots$ of redundant n-cells.*

Note, as a special case of **(C2)**, that one cannot have $\tau \succ \tau$. In other words, there is a *unique* integer i such that $\tau = d_i c(\tau)$. Another consequence of **(C2)** is that, for any redundant cell τ, there cannot exist arbitrarily long descending chains $\tau = \tau_0 \succ \tau_1 \cdots \succ \tau_k$. This follows from König's lemma (cf. [13], §2.3.4.3), which is applicable since every redundant cell has only finitely many immediate predecessors. The maximal length k of a descending chain as above will be called the *height* of τ.

It is now a simple matter to achieve the goal stated above:

PROPOSITION 1. *Let K be a semi-simplicial complex with a collapsing scheme. Then its geometric realization $X = |K|$ admits a canonical quotient CW-complex Y, whose cells are in 1–1 correspondence with the essential cells of X. The quotient map $q : X \to Y$ is a homotopy equivalence. It maps each open essential cell of X homeomorphically onto the corresponding open cell of Y, and it maps each collapsible $(n+1)$-cell into the n-skeleton of Y.*

PROOF: Write X as the union of an increasing sequence of subcomplexes

$$X_0 \subseteq X_0^+ \subseteq X_1 \subseteq X_1^+ \subseteq \cdots,$$

where X_0 consists of the essential vertices, X_n^+ is obtained from X_n by adjoining the redundant n-cells and the collapsible $(n+1)$-cells, and X_{n+1} is obtained from X_n^+ by adjoining the essential $(n+1)$-cells. We can factor the inclusion $X_n \hookrightarrow X_n^+$ as a sequence of adjunctions

$$X_n = X_n^0 \subseteq X_n^1 \subseteq X_n^2 \subseteq \cdots,$$

where we construct X_n^{j+1} from X_n^j by adjoining τ and $c(\tau)$ for every redundant n-cell τ of height j. Note that every face of $c(\tau)$ other than τ is either an immediate predecessor of τ (and hence has height $< j$) or else is essential, collapsible, or degenerate. These faces are therefore already present, and the adjunction of τ and $c(\tau)$ is an elementary expansion.

[Note: The degenerate faces present no problem here since they are identified, when the geometric realization X is constructed, with cells of lower dimension. More precisely, let $\chi : \Delta^{n+1} \to X$ be the characteristic map for $\sigma = c(\tau)$, where Δ^{n+1} is the standard $(n+1)$-simplex. If some face of σ is degenerate, then χ maps the corresponding face of Δ^{n+1} into the $(n-1)$-skeleton of X. So χ maps all but the ith face of Δ^{n+1} into X_n^j, where $i = i(\tau)$, whence our assertion that the adjunction is an elementary expansion.]

The passage from X_n^j to X_n^{j+1} therefore consists of a possibly infinite number of simultaneous elementary expansions. In particular, we have a homotopy equivalence $X_n^j \hookrightarrow X_n^{j+1}$ for each j, whence a homotopy equivalence $X_n \hookrightarrow X_n^+$. Moreover, the collapsing maps associated to the elementary expansions above yield a canonical homotopy inverse $X_n^+ \to X_n$. The rest of the proof is an exercise in elementary homotopy theory. See [6], proof of Theorem 5.3, for more details. □

When we apply these ideas to $K = BM$, we will construct $c(\tau)$ for a redundant cell $\tau = (m_1, \ldots, m_n)$ by factoring one of the m_i, as we indicated above in the case $n = 1$. So we will need some reasonable way of factoring elements of M. This suggests that we need normal forms. It turns out that the normal forms coming from a complete rewriting system do the trick. The next section will be devoted to a review of rewriting systems, and the collapsing scheme will then be constructed in §3.

2. Rewriting Systems and Normal Forms

Let M be a monoid with a fixed set S of generators. Thus M is a quotient of the free monoid F on S. We will call the elements of S "letters" when we are thinking of S as a subset of F, and we will call the elements of F "words"; a word, then, can be viewed as a finite (possibly empty) string of letters. A subset $\mathcal{I} \subseteq F$ will be called a set of *normal forms* for M if $\mathcal{I}$ maps bijectively onto M under the quotient map $\pi : F \to M$. Such a set determines a function $r : F \to \mathcal{I}$ such that $r(w)$ is the unique element of $\mathcal{I}$ with $\pi(r(w)) = \pi(w)$. We are interested in normal forms with the property that $r(w)$ is computable from w by "rewriting", in a sense which we now explain.

Let R be a subset of $F \times F$ such that M admits the presentation $\langle S \,;\, R \rangle$. In other words, M is obtained from F by introducing one relation $w_1 = w_2$ for each $(w_1, w_2) \in R$. The elements of R will be called *rewriting rules.* It is important that they are *ordered* pairs of words. We will often emphasize this by writing $w_1 \to w_2$ if $(w_1, w_2) \in R$. More generally, we write $w \to w'$ whenever $w = uw_1v$ and $w' = uw_2v$ for some $(w_1, w_2) \in R$ and some $u, v \in F$. We then say that w' is obtained from w by *rewriting*, or *reduction.* A word w is called *reducible* if such a reduction is possible, and it is called *irreducible* otherwise.

We say that R is a *complete rewriting system* for M if it satisfies the following two conditions:

(R1) *The set $\mathcal{I}$ of irreducible words is a set of normal forms for M.*

(R2) *There is no infinite chain $w \to w' \to w'' \to \cdots$ of reductions.*

It follows from these axioms that we can compute $r(w)$ for $w \in F$ by starting with w and applying an arbitrary sequence of reductions, until we arrive at an irreducible word. Condition **(R2)** guarantees that this will eventually happen, and clearly the irreducible word we have reached is $r(w)$.

There are various ways of reformulating the axioms; see, for instance, [2], [14], or [19]. See also [12] for further information about rewriting systems.

Here are two simple examples. Several additional examples will be given in §7, and many further examples can be found in the references cited above, especially [14].

EXAMPLES. 1. Let M be the free commutative monoid generated by a set S. If we totally order S, then M admits as normal forms the set of words $s_1 \cdots s_n$ with $s_1 \leq \cdots \leq s_n$ in the chosen ordering. This is the set of irreducible words associated to a complete rewriting system with one rewriting rule $ts \to st$ for every pair $s, t \in S$ with $t > s$.

2. Let G be the one-relator group with two generators a, t and the defining relation $t^{-1}at = a^2$. Let M be the submonoid of G generated by a and t. It is not hard to show that M admits the words $t^i a^j$ $(i, j \geq 0)$ as normal forms. (One can use, for instance, the fact that G is a semi-direct product $\mathbf{Z}[1/2] \rtimes \mathbf{Z}$.) This set of normal forms is the set of irreducible words associated to a complete rewriting system with one rewriting rule $at \to ta^2$.

(Exercise: Verify **(R2)**. Some thought is required here, since application of the rewriting rule increases the length of a word.)

Assume now that we have a complete rewriting system. It will be convenient to introduce an ordering on words which reflects both the rewriting process and the subword relation. Recall first that a *subword* of a word w is any word w' such that $w = uw'v$ for some $u, v \in F$. If u and v are not both empty, then w' is a *proper* subword of w. We now denote by "$\prec$" the smallest transitive relation on F such that $w' \prec w$ if w' is a proper subword of w or if there is a reduction $w \to w'$. Explicitly, then, we have $w' \prec w$ if and only if either (i) w' is a proper subword of w or (ii) w' is a subword (not necessarily proper) of a word w'' such that there is a chain of reductions $w = w_0 \to \cdots \to w_k = w''$ for some $k \geq 1$.

The following assertion is an easy consequence of **(R2)**:

(R3) *There is no infinite chain $w \succ w' \succ w'' \succ \cdots$ of words.*

For the purposes of this paper, it is not important that we have a particular rewriting system; the important thing for us, rather, is the set $\mathcal{I}$ of irreducible words. To emphasize this point of view, we will call a set of normal forms *good* if it is the set $\mathcal{I}$ of irreducible words associated to some complete rewriting system R for M.

REMARK. Squier [19] has noted that there is a canonical choice of complete rewriting system R for a given good set $\mathcal{I}$ of normal forms. Namely, let $\mathcal{L}$ be the set of words $w \notin \mathcal{I}$ such that every proper subword of w is in $\mathcal{I}$; then the canonical R consists of the rules $w \to r(w)$ for $w \in \mathcal{L}$.

Finally, we mention one further condition that we will sometimes assume is satisfied by R (or $\mathcal{I}$).

(R4) *Every $s \in S$, viewed as a word of length 1, is irreducible.*

This condition is harmless. For if it fails, then we can replace S by the subset $S' = S \cap \mathcal{I}$. This is still a set of generators of M, since $\mathcal{I}$ is contained in the submonoid $F' \subset F$ generated by S'. (To see this, note that $\mathcal{I}$ is closed under passage to subwords.) Moreover, $\mathcal{I}$ is a good set of normal forms satisfying **(R4)** with respect to the generating set S'.

3. Good Normal Forms Yield a Collapsing Scheme

Assume throughout this section that M is a monoid with a generating set S and a good set $\mathcal{I}$ of normal forms. As a reminder of what "good" means, we call the elements of $\mathcal{I}$ *irreducible* and the elements of $F - \mathcal{I}$ *reducible.* We are going to use $\mathcal{I}$ to construct a collapsing scheme for BM. The construction depends only on $\mathcal{I}$, and not on any particular choice of rewriting system R. But the verification of the crucial axiom **(C2)** will require a choice of R. The reader may prefer to simply assume from the outset that we are working with a specific R, e.g., the canonical one.

We will identify M, as a set, with $\mathcal{I}$. We must then be careful to distinguish between multiplication in the free monoid F, denoted by $(w_1, w_2) \mapsto w_1w_2$, and multiplication in M, denoted by $(w_1, w_2) \mapsto w_1 * w_2$. The latter is defined for $w_1, w_2 \in \mathcal{I}$, and it is given by

$$w_1 * w_2 = r(w_1w_2).$$

The simplices of our semi-simplicial complex BM are now viewed as n-tuples $(w_1, \ldots, w_n)$ of irreducible words, and the definition of the face operator d_i for $0 < i < n$ becomes

$$d_i(w_1, \ldots, w_n) = (w_1, \ldots, w_i * w_{i+1}, \ldots, w_n).$$

Finally, the cells of $X = |BM|$ are the n-tuples of *non-empty* irreducible words.

We now wish to carry out the procedure described in §1. Let's look first at low dimensions, for motivation. The unique vertex of X will be essential. A 1-cell $\tau = (w)$ is going to be called essential if $w \in S$ and redundant otherwise. If it is redundant, then the associated collapsible 2-cell will be $c(\tau) = (s, w')$, where $w = sw'$ with $s \in S$. Note that (s, w') is in fact a cell, because s and w' are non-empty subwords of w.

At this point we know that the collapsible 2-cells should be those of the form (s, w) with $s \in S$ and sw irreducible. The other 2-cells τ have the form (i) (w_1, w_2) with $w_1 \notin S$, or (ii) (s, w) with sw reducible. In case (i), τ will be called redundant, and we will set $c(\tau) = (s, w', w_2)$ $(s \in S,\ sw' = w_1)$. In case (ii), we would like to take $c(\tau) = (s, w', w'')$ for some factorization $w = w'w''$. Note, however, that we must choose this factorization so that sw' is reducible; for otherwise (s, w', w'') has already been used as $c(\tau')$, where τ' is the redundant cell (sw', w'') of type (i). Thus we are only going to be able to make (s, w) redundant if sw has a proper initial subword which is reducible. Those (s, w) for which this fails will be the essential 2-cells. Equivalently, (s, w) will be essential if and only if $sw \in \mathcal{L}$, where $\mathcal{L}$ is the set defined in the remark in §2 above.

(This is quite reasonable from the point of view of generators and relations. If **(R4)** holds, for instance, then there will be one essential 1-cell for each element of S and one essential 2-cell for each element of R, where R is the canonical set of rewriting rules for $\mathcal{I}$.)

These considerations lead to the following definitions. An n-cell $\tau = (w_1, \ldots, w_n)$ of X is called *essential* if it satisfies the following three conditions:

(1) $w_1 \in S$.
(2) $w_i w_{i+1}$ is reducible for $1 \leq i < n$.
(3) Every proper initial subword of $w_i w_{i+1}$ $(1 \leq i < n)$ is irreducible.

Note that it suffices, in verifying (3), to check that the subword of $w_i w_{i+1}$ obtained by deleting the last letter is irreducible.

If τ is not essential, let i be the largest integer such that the "$(i-1)$-dimensional front face" $(w_1, \ldots, w_{i-1})$ of τ is an essential $(i-1)$-cell. We have $1 \leq i \leq n$, where the case $i = 1$ occurs if the essential front face is empty, i.e., if w_1 has length $l(w_1) > 1$. If $i = 1$, then we call τ *redundant*, and we set $c(\tau) = (s, w', w_2, \ldots, w_n)$, where $w_1 = sw'$ with $s \in S$. Then $\tau = d_1 c(\tau)$, and we set $i(\tau) = 1$.

Suppose now that $i > 1$. Then either $w_{i-1} w_i$ is irreducible, in which case we say that τ is *collapsible*, or else some proper initial subword of $w_{i-1} w_i$ is reducible, in which case we call τ *redundant*. In the second case, write $w_i = w' w''$, where w' is the smallest initial subword of w_i such that $w_{i-1} w'$ is reducible. The words w' and w'' are necessarily non-empty and irreducible, and we set

$$c(\tau) = (w_1, \ldots, w_{i-1}, w', w'', w_{i+1}, \ldots, w_n).$$

We have $\tau = d_i c(\tau)$, and we set $i(\tau) = i$.

REMARK. To understand what an essential cell $\tau = (w_1, \ldots, w_n)$ looks like, consider the word $w = w_1 w_2 \cdots w_n$ obtained by ignoring the commas. It is not hard to show that τ is determined by w; in other words, there is only one way to re-insert commas into w so as to get an essential cell (cf. [1], Lemma 1.3). Note also that, by (2) and (3), some final subword of $w_i w_{i+1}$ is in $\mathcal{L}$ for $1 \leq i < n$. Consequently, the essential n-cells for $n \geq 3$ correspond to certain words w, consisting of $n-1$ overlapping elements of $\mathcal{L}$. The interested reader can check that the words w that arise here are the same as the "$(n-1)$-chains" of Anick [1]. There is also some similarity with the "critical $(n-1)$-stars" of Groves [11].

Suppose, for example, that M is the free commutative monoid, as in Example 1 of §2. Then the essential n-cells are the cells $(s_1, \dots, s_n)$ with each $s_j \in S$ and $s_1 > \cdots > s_n$ in the chosen ordering on S. The corresponding words $w = s_1 \cdots s_n$ are the words in which every subword of length 2 is in $\mathcal{L}$. The reader is advised to figure out what the collapsible and redundant cells look like in this example before returning to the general theory.

We turn now to the verification of the axioms for a collapsing scheme. First, it is immediate that $c(\tau)$ is collapsible if τ is redundant. It is also quite easy to check that **(C1)** holds. The crux of the matter, then, is **(C2)**. To verify this, choose a complete rewriting system R for which $\mathcal{I}$ is the set of irreducible words. Recall that this determines a relation "$\prec$" on F and that **(R3)** holds. Given an n-cell $\tau = (w_1, \dots, w_n)$, let $w(\tau)$ be the word $w_1 \cdots w_n$.

LEMMA. *Let τ and τ' be redundant n-cells such that τ' is an immediate predecessor of τ. Then one of the following holds:*

(1) $w(\tau') \prec w(\tau)$.
(2) $w(\tau') = w(\tau)$, *and the maximal essential front face of τ' has higher dimension than that of τ.*

PROOF: Let $\sigma = c(\tau)$, and write $\sigma = (w_1, \dots, w_{n+1})$. Thus

$$\tau = (w_1, \dots, w_{i-1}, w_i w_{i+1}, w_{i+2}, \dots, w_{n+1}),$$

where $i = i(\tau)$. [Note that $w_i w_{i+1} = w_i * w_{i+1}$ here.] By definition, then, $\tau' = d_j \sigma$ for some $j \neq i$. If $j = 0$ or $n+1$, then $w(\tau')$ is a proper subword of $w(\sigma) = w(\tau)$, and (1) holds. If $0 < j < n+1$ and $w_j w_{j+1}$ is reducible, then the computation of $d_j \sigma$ involves reducing $w_j w_{j+1}$ to $w_j * w_{j+1}$; so there is a chain of reductions from $w(\tau)$ to $w(\tau')$, and again (1) holds. Note that this case must apply if $0 < j < i$. Suppose, finally, that $i < j < n+1$ and that $w_j w_{j+1}$ is irreducible. Then $w(\tau') = w(\tau)$, and the i-dimensional front face of τ' is essential. But the maximal essential front face of τ has dimension $i-1$, so (2) holds. □

It is now easy to check **(C2)**. For suppose that there is a chain

$$\tau_1 \succ \tau_2 \succ \tau_3 \succ \cdots$$

of redundant n-cells. By **(R3)** we cannot have $w(\tau_j) \succ w(\tau_{j+1})$ for infinitely many j. So τ_{j+1} has a higher-dimensional essential front face than τ_j for all sufficiently large j. But this is absurd, for the dimension of the maximal essential front face of a redundant n-cell is always less than n.

Thus we have indeed constructed a collapsing scheme. Proposition 1 of §1 now yields:

THEOREM 1. *Let M be a monoid with a good set of normal forms, and let $X = |BM|$ be its classifying space. Then X admits a canonical quotient CW-complex Y, whose cells are in 1–1 correspondence with the essential cells of X. The quotient map $q : X \to Y$ is a homotopy equivalence. It maps each open essential cell of X homeomorphically onto the corresponding open cell of Y, and it maps each collapsible $(n+1)$-cell into the n-skeleton of Y.* □

A good set $\mathcal{I}$ of normal forms will be said to have *finite type* if S is finite and $\mathcal{I}$ is the set of irreducible words associated to a complete rewriting system with only finitely many rewriting rules. It is easy to see that there are then only finitely many essential cells in each dimension. Consequently:

COROLLARY. *If M admits a good set of normal forms of finite type, then its classifying space has the homotopy type of a complex with only finitely many cells in each dimension.* □

REMARKS. 1. In applying the results of this section to a given example, it is very likely that we will see ways to do further collapsing and thereby reduce Y to an even smaller complex. Here is one such situation that comes up fairly often: Suppose that our rewriting system contains rules $s\bar{s} \to 1$ and $\bar{s}s \to 1$ for some pair $s, \bar{s}$ of distinct generators. Assuming that s and $\bar{s}$ are irreducible, we will then have for each $n \geq 1$ a pair of essential n-cells $\sigma_n = (s, \bar{s}, s, \bar{s}, \dots)$ and $\bar{\sigma}_n = (\bar{s}, s, \bar{s}, s, \dots)$. The only non-degenerate faces of σ_n for $n \geq 2$ are $d_0\sigma_n = \bar{\sigma}_{n-1}$ and $d_n\sigma_n = \sigma_{n-1}$. We now modify our previous definitions by declaring that $\bar{\sigma}_n$ is redundant for $n \geq 1$ and that σ_n is collapsible for $n \geq 2$ (with free face $\bar{\sigma}_{n-1}$). Thus the only cell from these two infinite families that remains essential is σ_1.

2. Recall that a monoid is the same thing as a category with one object and that the classifying space construction extends to categories

(cf. [18] or [17]). For simplicity, we have confined ourselves in this section to monoids, but one can equally well treat more general categories. We will give an example of this in §7.

4. Algebraic Interpretation

In case the monoid M of §3 is a group, the cellular chain complex of the universal cover of X is the standard (or "bar") resolution of $\mathbf{Z}$ over $\mathbf{Z}M$. By taking the universal cover of the complex Y of Theorem 1, we then get a quotient complex of the bar resolution, which is a free resolution with one basis element for each essential cell of X.

We would like, more generally, to construct a "small" resolution of this type for any monoid M with a good set of normal forms, not just for groups. I do not know any way to formally deduce such a resolution from the existence of the homotopy equivalence $X \to Y$ above. What one can do, however, is simply work directly with the bar resolution and imitate the method used in §1. In other words, we do our homotopy theory in the category of chain complexes of free $\mathbf{Z}M$-modules instead of the category of CW-complexes. Here are the details.

For any monoid M, let EM be the following semi-simplicial complex: The n-simplices are $(n+1)$-tuples of elements of M, a typical such $(n+1)$-tuple being written in the form $(m_1, \ldots, m_n)m$. If $m = 1$, we suppress it from the notation and simply write $(m_1, \ldots, m_n)$. The face operators in EM are given by

$$d_i(m_1, \ldots, m_n)m = \begin{cases} (m_2, \ldots, m_n)m & i = 0 \\ (m_1, \ldots, m_i m_{i+1}, \ldots, m_n)m & 0 < i < n \\ (m_1, \ldots, m_{n-1})m_n m & i = n, \end{cases}$$

and the degeneracy operators are given by

$$s_i(m_1, \ldots, m_n)m = (m_1, \ldots, m_i, 1, m_{i+1}, \ldots, m_n)m \qquad (0 \leq i \leq n).$$

There is an obvious right action of M on EM by simplicial maps, where the action of m is given by $(m_1, \ldots, m_n)m' \mapsto (m_1, \ldots, m_n)m'm$. This action makes the normalized chain complex $C = C_*(EM)$ a complex of free right $\mathbf{Z}M$-modules, and, in fact, it is precisely the normalized bar resolution of $\mathbf{Z}$ over $\mathbf{Z}M$ (cf. [7], §X.2). The module C_n of n-chains has a

$\mathbf{Z}M$-basis consisting of the n-tuples $(m_1, \ldots, m_n)$ with each $m_i \neq 1$. Thus the basis elements of C correspond to the cells of $X = |BM|$. But when we compute the boundary operator $d : C_n \to C_{n-1}$ on these basis elements, we must remember to use the face operators in EM, as defined above, not those in BM. This makes a difference only for the face operator d_n.

Returning now to the situation of §3, we have a classification of the $\mathbf{Z}M$-basis elements of C as essential, collapsible, or redundant. If σ is a collapsible $(n+1)$-cell and τ is its free face, then the boundary of σ in C can be written in the form

$$d\sigma = \pm\tau - x, \tag{1}$$

where x is a $\mathbf{Z}M$-linear combination of collapsible n-cells, essential n-cells, and redundant n-cells that are immediate predecessors of τ. [Note that σ might have degenerate faces in EM, but these are 0 in C and hence do not appear in (1).]

Suppose now that we are trying to build C by successive adjunctions, as we did for X in the proof of Theorem 1. If we proceed as in that proof, then each adjunction of a redundant basis element τ along with its associated collapsible σ yields a chain homotopy equivalence. It has a canonical homotopy inverse, which maps σ to 0 (and hence τ to $\pm x$, where x is as in (1) above).

The following analogue of Theorem 1 is now immediate:

THEOREM 2. *Let M be a monoid with a good set of normal forms, and let $C = C_*(EM)$ be its normalized bar resolution. Then C admits a canonical quotient complex D, which is a free resolution of $\mathbf{Z}$ over $\mathbf{Z}M$ with one basis element for each essential cell. The quotient map $q : C \to D$ maps each essential cell of C to the corresponding basis element of D, and it maps each collapsible cell to* 0. □

COROLLARY (ANICK–GROVES–SQUIER). *If M admits a good set of normal forms of finite type, then M is of type FP_∞, i.e., $\mathbf{Z}$ admits a free resolution D over $\mathbf{Z}M$ with D_n finitely generated for each n.* □

REMARK. The method used in this section works, with no essential change, if the ring $\mathbf{Z}M$ is replaced by an arbitrary augmented k-algebra A which comes equipped with a presentation satisfying the conditions of Bergman's diamond lemma ([2], Theorem 1.2). Here k can be any commutative ring. One starts with the normalized bar resolution C of k over A, and one obtains a quotient resolution D, with one generator for each "essential" generator of the bar resolution. In particular, we recover Anick's theorem ([1], Theorem 1.4). It seems likely that one can similarly study the homological algebra of other kinds of algebraic rewriting systems (i.e., other than monoids and associative algebras), but I have not tried to do this.

5. Computational Techniques

The statement of Theorem 2 does not contain an explicit formula for the boundary operator in D. To obtain such a formula, we need to compute the boundary $d\sigma$ of an arbitrary essential cell $\sigma \in C$ and then find the image $q(d\sigma) \in D$. The only difficulty here is that $d\sigma$ might involve some redundant cells τ. Now for each such τ, we have an element $x \in C$ such that $q(\tau) = \pm q(x)$. So the problem of computing $q(\tau)$ is reduced to the simpler problem of computing $q(x)$. This is simpler because the redundant cells that occur in x are immediate predecessors of τ. We are therefore led to the following "rewriting" procedure for explicitly describing D:

Identify D, as a graded $\mathbf{Z}M$-module, with the submodule of C generated by the essential cells. For each collapsible cell σ, write down the rule

$$\sigma \to 0.$$

For each redundant cell τ, let $\sigma = c(\tau)$, and write down the rule

$$\tau \to \pm x,$$

where x and the ambiguous sign are taken from (1). We can, of course, delete from x any terms involving collapsible cells. The remaining terms involve either essential cells or redundant cells that are immediate predecessors of τ and hence have height less than that of τ. It follows that we can use the rewriting rules above to "reduce" an element $u \in C$ to an element $\bar{u} \in D$, and this element is precisely the image $q(u)$. In particular, we can now compute the boundary operator ∂ in D; it is given by

$$\partial e = \overline{de},$$

where e is an essential cell and de is its boundary in C.

Let's make this rewriting process more explicit. Let $\tau = (w_1, \ldots, w_n)$ be redundant, and let $i = i(\tau)$. Then $\sigma = c(\tau)$ is given by

$$\sigma = (w_1, \ldots, w_{i-1}, w', w'', w_{i+1}, \ldots, w_n)$$

for some factorization $w'w''$ of w_i. The faces $d_j\sigma$ for $j > i+1$ are either degenerate or collapsible; they can therefore be ignored. There are always at least two other faces (aside from the free face $d_i\sigma = \tau$), namely, $d_{i-1}\sigma$ and $d_{i+1}\sigma$. Set

$$\lambda = d_{i-1}\sigma = \begin{cases} (w'', w_2, \ldots, w_n) & \text{if } i = 1 \\ (w_1, \ldots, w_{i-2}, w_{i-1} * w', w'', w_{i+1}, \ldots, w_n) & \text{if } i > 1. \end{cases}$$

Roughly speaking, it is obtained from τ by pushing the left half of w_i to the left. Set

$$\rho = d_{i+1}\sigma = \begin{cases} (w_1, \ldots, w_{n-1}, w')w'' & \text{if } i = n \\ (w_1, \ldots, w_{i-1}, w', w'' * w_{i+1}, w_{i+2}, \ldots, w_n) & \text{if } i < n. \end{cases}$$

It is obtained from τ by pushing the right half of w_i to the right. The rewriting rule $\tau \to \pm x$ above now becomes

$$\tau \to \lambda + \rho + \sum_{j=0}^{i-2} (-1)^{i-j-1} d_j\sigma. \tag{2}$$

In specific examples, one often finds that the right side of (2) contains a large number of redundant cells τ', many of which are ultimately seen to satisfy $q(\tau') = 0$ after further rewriting rules are applied. If we could recognize such cells τ' in advance, then we could simply delete them from (2) and save a lot of work. With this goal in mind, we prove the following modest result:

PROPOSITION 2. *Let $\tau = (w_1, \ldots, w_n)$ be an n-cell for some $n \geq 2$. If w_1w_2 is irreducible, then $q(\tau) = 0$.*

PROOF: We argue by induction on the length $l(w_1)$. If $l(w_1) = 1$, then τ is collapsible and there is nothing to prove. So assume that $w_1 = sw$ with $s \in S$ and $w \neq 1$. Using the notation above, we have $i = 1$, $\lambda = (w, w_2, \ldots)$, and $\rho = (s, ww_2, \ldots)$. Since ww_2 and sww_2 are irreducible, we can apply the induction hypothesis to conclude that $q(\lambda) = q(\rho) = 0$. The proposition now follows from (2). □

Even with the aid of Proposition 2, the task of computing q (and hence ∂) can be quite tedious if one simply uses (2). A better strategy is to try to guess a formula for q and then use (2) to prove inductively that the guess is correct. The induction here is with respect to the "immediate predecessor" relation, i.e., one proves the desired formula for a redundant cell τ under the assumption that the result is already known for all redundant $\tau' \prec \tau$. In other words, the result can be assumed known for all redundant cells that appear on the right side of (2). Condition **(C2)** justifies this sort of induction.

Here is a simple example. Let G be a cyclic group of finite order $m \geq 2$. It is presented (as a monoid) by the complete rewriting system with one generator s and one rule $s^m \to 1$. There is one essential n-cell $e_n = (s, s^{m-1}, s, s^{m-1}, \dots)$ in each dimension n. Moreover, one can check that the maximal essential front face of every redundant cell τ has even dimension. A straightforward induction now yields the following formula for q: Let $\tau = (w_1, \dots, w_n)$ be an n-cell. If $w_i w_{i+1}$ is irreducible for some odd $i < n$, then $q(\tau) = 0$. Otherwise $q(\tau) = e_n a$, where $a \in \mathbf{Z}G$ is given by

$$a = \begin{cases} 1 & \text{if } n \text{ is even} \\ 1 + s + \cdots + s^{k-1} & \text{if } n \text{ is odd and } w_n = s^k. \end{cases}$$

The inductive proof, which is left to the reader, is simplified by the fact that, in the notation of (2), $d_j \sigma$ is degenerate for $0 < j < i - 1$.

It now follows at once that $\partial e_n = e_{n-1} a$, where

$$a = \begin{cases} 1 - s & \text{if } n \text{ is odd} \\ 1 + s + \cdots + s^{m-1} & \text{if } n \text{ is even.} \end{cases}$$

This is the formula that one would expect (cf. [7], §XII.7, or [4], §I.6).

We close this section by deriving, for arbitrary M, the expected formula for $\partial : D_2 \to D_1$ in terms of free derivatives (cf. [4], Exercise 3 of §II.5 or Exercise 3 of §IV.2). To simplify the statement, we assume that **(R4)** holds. Let F, as before, be the free monoid generated by S. There is a unique function $\delta : F \to D_1$ such that $\delta(s) = (s)$ for all $s \in S$ and $\delta(uv) = \delta(u)r(v) + \delta(v)$ for all $u, v \in F$. For any 1-cell $\tau = (w)$, I claim that $q(\tau) = \delta(w) \in D_1$. To prove this, we may assume that $w = sw'$ with $s \in S$ and $w' \neq 1$, and we may assume inductively that $q(w') = \delta(w')$.

Then

$$\begin{aligned} q(\tau) &= q(w') + (s)w' \qquad \text{by (2)} \\ &= \delta(w') + \delta(s)w' \\ &= \delta(w), \end{aligned}$$

as claimed.

Recall that, since we have assumed **(R4)**, there is one essential 2-cell for each $(w_1, w_2) \in R$, where R is the canonical rewriting system for $\mathcal{I}$.

PROPOSITION 3. *Let e be the essential 2-cell corresponding to a rewriting rule $(w_1, w_2) \in R$. Then $\partial e = \delta(w_1) - \delta(w_2)$.*

PROOF: We have $e = (s, w)$, with $w_1 = sw$ and $w_2 = r(sw) = s * w$. Hence $\partial e = q(de) = q(w) - q(w_2) + (s)w$. Now $q(w_2) = \delta(w_2)$, and $q(w) + (s)w = \delta(sw) = \delta(w_1)$, whence the proposition. □

6. The Case of Conjugation Relations

This section is motivated by [6] and by some unpublished work of Craig Squier, in which "cubical" resolutions were constructed for a number of monoids.

We continue to assume that M is a monoid with a good set $\mathcal{I}$ of normal forms. Suppose that the generating set S comes equipped with a total order and that the elements of $\mathcal{I}$ are the non-decreasing words $s_1 \cdots s_n$, as in the case of the free commutative monoid (Example 1 of §2). The canonical rewriting system R for $\mathcal{I}$ then has one rule $ts \to r(ts)$ for each pair of generators t, s with $t > s$. Suppose further that s occurs as the first letter in $r(ts)$, so that the rule has the form

$$ts \longrightarrow su,$$

where u is an irreducible word whose first letter is greater than or equal to s. Following the customary notation for conjugates in group theory, we will write $u = t^s$. In the case of the free commutative monoid, for instance, we have $t^s = t$. The reader may wish to concentrate on this case, at least on first reading.

For simplicity, we will assume that $l(t^s) = 1$, i.e., that $t^s \in S$. Some of what we do is valid more generally, but the statements become more complicated. We will also assume that the "conjugation function" satisfies

$$s < t_1 < t_2 \implies t_1^s < t_2^s.$$

It is convenient to extend the conjugation notation slightly: Given $s \in S$ and a word $w = t_1 \cdots t_n$ such that $t_i > s$ for all i, set

$$w^s = t_1^s \cdots t_n^s.$$

Note, then, that $r(ws) = sw^s$ if w is irreducible. Next, given $t \in S$ we wish to define $t^w \in S$ for certain words $w = s_1 \cdots s_n$. Proceeding by induction on n, let $u = s_1 \cdots s_{n-1}$. We then set

$$t^w = (t^u)^{s_n},$$

provided t^u is defined and $t^u > s_n$.

We now apply the method of §§4 and 5 to M. The essential cells of $X = |BM|$ are the cells $(s_1, \ldots, s_n)$ with $s_i \in S$ and $s_1 > \cdots > s_n$. We therefore obtain a free resolution D with one generator for each such cell. In order to describe the boundary operator ∂ in D, we introduce some operators A_i and B_i which map D_n to D_{n-1}. They are to be $\mathbf{Z}M$-linear, so we need only specify them on the essential n-cells. Given an essential cell $\sigma = (s_1, \ldots, s_n)$ and an integer i with $1 \leq i \leq n$, set

$$A_i\sigma = (s_1^{s_i}, \ldots, s_{i-1}^{s_i}, s_{i+1}, \ldots, s_n)$$
$$B_i\sigma = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)s_i^u,$$

where $u = s_{i+1} \cdots s_n$. (Note that, in view of our assumptions, the right-hand sides are well-defined and in D_{n-1}.) In terms of the word $s_1 \cdots s_n$ associated to σ, we compute $A_i\sigma$ by moving s_i to the left and then deleting it, and we compute $B_i\sigma$ by moving s_i to the right and then retaining the resulting conjugate of s_i as an operator.

PROPOSITION 4. *The boundary operator $\partial : D_n \to D_{n-1}$ is given by*

$$\partial = \sum_{i=1}^{n} (-1)^{i-1}(A_i - B_i).$$

To prove this, it is convenient to extend the operators A_i and B_i to arbitrary cells $\sigma = (w_1, \ldots, w_n)$. The definitions are slightly more complicated, because it might not be possible to move w_i all the way to the left and right. What we do, roughly speaking, is move them as far as possible. The precise definitions follow.

We define $A_i\sigma$ for $1 \le i \le n$ by induction on i. Consider the face

$$d_{i-1}\sigma = \begin{cases} (w_2, \ldots, w_n) & \text{if } i = 1 \\ (w_1, \ldots, w_{i-1} * w_i, \ldots, w_n) & \text{if } i > 1. \end{cases}$$

We set $A_i\sigma = d_{i-1}\sigma$ if any of the following conditions hold: (a) $i = 1$; (b) $l(w_i) > 1$; (c) $i > 1$, $l(w_i) = 1$, and there is a letter t occurring in w_{i-1} such $t \le w_i$ in the given ordering on S. If (a)–(c) all fail, then $i > 1$ and

$$\sigma = (w_1, \ldots, w_{i-1}, s, w_{i+1}, \ldots, w_n)$$

with $s \in S$ and $t > s$ for all t occurring in w_{i-1}. Hence

$$d_{i-1}\sigma = (w_1, \ldots, w_{i-2}, sw_{i-1}^s, w_{i+1}, \ldots, w_n).$$

We now set $A_i\sigma = A_{i-1}\sigma'$, where

$$\sigma' = (w_1, \ldots, w_{i-2}, s, w_{i-1}^s, w_{i+1}, \ldots, w_n).$$

Similarly, B_i is defined for $1 \le i \le n$ by descending induction on i. Consider the face

$$d_i\sigma = \begin{cases} (w_1, \ldots, w_{n-1})w_n & \text{if } i = n \\ (w_1, \ldots, w_i * w_{i+1}, \ldots, w_n) & \text{if } i < n. \end{cases}$$

We set $B_i\sigma = d_i\sigma$ if any of the following conditions hold: (a) $i = n$; (b) $l(w_{i+1}) > 1$; (c) $i < n$, $l(w_{i+1}) = 1$, and $t \le w_{i+1}$ for some t occurring in w_i. If (a)–(c) all fail, then $i < n$ and

$$\sigma = (w_1, \ldots, w_i, s, w_{i+2}, \ldots, w_n)$$

with $s \in S$ and $t > s$ for all t occurring in w_i. We now have

$$d_i\sigma = (w_1, \ldots, w_{i-1}, sw_i^s, w_{i+2}, \ldots, w_n),$$

and we set $B_i\sigma = B_{i+1}\sigma'$, where

$$\sigma' = (w_1, \ldots, w_{i-1}, s, w_i^s, w_{i+2}, \ldots, w_n).$$

The proof of the proposition is based on the following lemma:

Lemma. *Let τ be a redundant n-cell, let $i = i(\tau)$, and let $\sigma = c(\tau)$. Then $q(\tau) = q(A_i\sigma + B_{i+1}\sigma)$.*

This shows that the rewriting rule (2) of §5 can be replaced by the much simpler rule

$$\tau \longrightarrow A_i\sigma + B_{i+1}\sigma.$$

Accepting the lemma for the moment, we can easily prove the proposition. For suppose $e = (s_1, \dots, s_n)$ is essential, and consider $d_j e$ for $0 < j < n$. We have

$$\begin{aligned} d_j e &= (s_1, \dots, s_j * s_{j+1}, \dots, s_n) \\ &= (s_1, \dots, s_{j-1}, s_{j+1} s_j^{s_{j+1}}, s_{j+2}, \dots, s_n), \end{aligned}$$

and the lemma yields $q(d_j e) = A_{j+1}e + B_j e$. Since $d_0 e = A_1 e$ and $d_n e = B_n e$, we conclude that

$$\begin{aligned} \partial e &= A_1 e + (-1)^n B_n e + \sum_{j=1}^{n-1} (-1)^j q(d_j e) \\ &= A_1 e + (-1)^n B_n e + \sum_{j=1}^{n-1} (-1)^j (A_{j+1} e + B_j e) \\ &= \sum_{j=1}^{n} (-1)^{j-1} (A_j e - B_j e), \end{aligned}$$

as required.

Proof of the Lemma: We argue by induction with respect to the "immediate predecessor" relation, as in §5. Thus we may assume that the lemma is already known for all redundant τ' that appear on the right side of (2). Using this induction hypothesis, one can check that, in the notation of (2), $q(\rho) = q(B_{i+1}\sigma)$, $q(\lambda) = q(A_i\sigma)$, and $q(d_j\sigma) = 0$ for $j < i - 1$, whence the lemma. Here, for instance, is the proof of the assertion about $\rho = d_{i+1}\sigma$; the proofs of the other two assertions are similar and are left to the reader.

Write $\tau = (s_1, \dots, s_{i-1}, s_i w, w_{i+1}, \dots, w_n)$, where $s_1 > \dots > s_i$ in S and w is a non-trivial irreducible word whose first letter is greater than or equal to s_i. Then $\sigma = (s_1, \dots, s_i, w, w_{i+1}, \dots, w_n)$. The result to be proved is trivial in the cases where $B_{i+1}\sigma$ was defined to be $d_{i+1}\sigma$. So we may assume that $i < n$ and that $\sigma = (s_1, \dots, s_i, w, s, \dots)$, with $s \in S$ and $t > s$ for all t which occur in w. Then $\rho = (s_1, \dots, s_i, sw^s, \dots)$. This is collapsible if $s_i \leq s$, in which case $q(\rho) = 0 = q(B_{i+1}\sigma)$. If $s_i > s$, on the other hand, then ρ is redundant, with $i(\rho) = i + 1$. Let

$$\sigma' = c(\rho) = (s_1, \dots, s_i, s, w^s, \dots).$$

We have $A_{i+1}\sigma' = (s_1^s, \dots, s_i^s, w^s, \dots)$, which is collapsible. So the induction hypothesis yields

$$\begin{aligned} q(\rho) &= q(A_{i+1}\sigma' + B_{i+2}\sigma') \\ &= q(B_{i+2}\sigma') \\ &= q(B_{i+1}\sigma), \end{aligned}$$

as required. □

7. Examples

We have already treated the case of a finite cyclic group in §5. We present here a few more examples to illustrate how the methods of this paper can be used to obtain explicit Eilenberg–MacLane complexes, or explicit free resolutions, for various groups or monoids. When we are interested in a group G, however, we will often get results about G by studying a suitable submonoid $M \subset G$. (This idea is suggested by the work of Craig Squier.) And in one example, instead of replacing G by a submonoid, we will replace G by a category $\mathcal{M}$ and apply our methods to the classifying space $|B\mathcal{M}|$.

The examples are only intended to illustrate the method; in all cases except possibly the last, the results obtained were already known.

Example 1: Free groups. Let F be the free group generated by a set S. The usual normal forms for the elements of F are obtained from a monoid rewriting system with generators $s, \bar{s}$ $(s \in S)$ and rules $s\bar{s} \to 1$ and $\bar{s}s \to 1$. The only essential cells in positive dimensions are those of the form $(s, \bar{s}, s, \bar{s}, \dots)$ or $(\bar{s}, s, \bar{s}, s, \dots)$, which we discussed in Remark 1 at the end of §3. As we explained there, it is possible to modify the essential/collapsible/redundant classification in order to "collapse away" most of these cells. The result, then, is the usual $K(F, 1)$-complex Y, with one 1-cell for each $s \in S$ and no higher-dimensional cells.

Example 2: Free abelian groups. Let G be the free abelian group generated by a set S. The easiest way to proceed here is to work not with G directly, but rather with the submonoid M generated by S. Note that $G =$

MM^{-1}, i.e., every element of G can be written as $m_1 m_2^{-1}$ with $m_i \in M$. It is not hard to show, in this situation, that $|BM|$ is homotopy equivalent to $|BG|$ (cf. [10], Proposition 4.4). So there is no harm in replacing G by M. From the point of view of homological algebra, the content of this statement is that $\mathbf{Z}G$ is flat as a left $\mathbf{Z}M$-module (cf. [7], Chapter X, proof of Proposition 4.1), so if we have a resolution D of $\mathbf{Z}$ by free right $\mathbf{Z}M$-modules, then $D \otimes_{\mathbf{Z}M} \mathbf{Z}G$ is a resolution of $\mathbf{Z}$ by free right $\mathbf{Z}G$-modules.

Choose a total order on S, and use the normal forms of Example 1 of §2. We have already noted that this fits into the framework of §6, so we obtain a free resolution D of $\mathbf{Z}$ over $\mathbf{Z}M$ with one basis element for every decreasing sequence $s_1 > \cdots > s_n$ of elements of S. The boundary operator $\partial : D_n \to D_{n-1}$ is given by

$$\partial = \sum_{i=1}^{n} (-1)^{i-1}(A_i - B_i),$$

where

$$A_i(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$
$$B_i(s_1, \ldots, s_n) = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)s_i.$$

This resolution is well-known. See, for instance, [7], §X.5.

Example 3: A one-relator group. Let G be the group with two generators a, t and the single relation $a^t = a^2$, where $a^t = t^{-1}at$. Let M be the submonoid generated by a and t. As in Example 2, we have $G = MM^{-1}$, so we can obtain a $K(G, 1)$ by working with M instead of G. We have already given a good set of normal forms for M (Example 2 of §2). One checks that the only essential cells are $()$, (a), (t), and (a, t), so Theorem 1 yields a $K(G, 1)$ with one vertex, two 1-cells, and one 2-cell. The corresponding resolution $D \otimes_{\mathbf{Z}M} \mathbf{Z}G$ is the famous Lyndon resolution [15].

Example 4: Thompson's group. Let G be the group with infinitely many generators $x_0, x_1, x_2, \ldots$ and infinitely many relations $x_j^{x_i} = x_{j+1}$ for $i < j$. This group was first introduced by R. Thompson, and it was the main object of study in [6], where it was called F. See [6] and [5] for further references and a discussion of the history of this group. As in Examples 2 and 3, we will apply our methods to the submonoid $M \subset G$ generated by the x_i. Once again, we have $G = MM^{-1}$, so there is no harm in doing this.

The elements of M have normal forms $x_{i_1} \cdots x_{i_n}$ with $i_1 \leq \cdots \leq i_n$ (cf. [6], 1.3), and these normal forms arise from the complete rewriting system with rules $x_j x_i \to x_i x_{j+1}$ for $i < j$. As in Example 2, then, we are in the situation of §6. To simplify the notation, let's write $(i_1, \ldots, i_n)$ for the generator of D corresponding to the essential cell $(x_{i_1}, \ldots, x_{i_n})$. The result, then, is that D has a basis consisting of decreasing sequences $i_1 > \cdots > i_n$ of non-negative integers, and $\partial : D_n \to D_{n-1}$ is given by

$$\partial = \sum_{j=1}^{n} (-1)^{j-1}(A_j - B_j),$$

where

$$A_j(i_1, \ldots, i_n) = (i_1 + 1, \ldots, i_{j-1} + 1, i_{j+1}, \ldots, i_n)$$
$$B_j(i_1, \ldots, i_n) = (i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_n) x_{i_j+n-j}.$$

Note that the irreducible words $r(w(\sigma))$ corresponding to the essential cells σ are precisely the words $x_{q_1} \cdots x_{q_n}$ with $q_{j+1} \geq q_j + 2$ for $j = 1, \ldots, n-1$. If we rewrite the A and B operators in terms of the sequences $(q_1, \ldots, q_n)$, we see that our resolution $D \otimes_{\mathbf{Z}M} \mathbf{Z}G$ is isomorphic to the "big" resolution constructed in [6]. In particular, we have obtained a new proof of Theorem 4.1 of [6]. Geoghegan and I went on in [6] to collapse this big resolution to one with only two generators in each positive dimension. This further collapse was based on the combinatorics of the "cubical face operators" A_j and B_j; I do not know how to explain it in terms of rewriting systems.

Example 5: Thompson's group again. Let G be the group of piecewise linear homeomorphisms g of the unit interval $[0, 1]$ with the following two properties: (a) the singularities of g occur at dyadic rational numbers, i.e., at points in $\mathbf{Z}[1/2]$; and (b) for any non-singular x, one has $g'(x) = 2^n$ for some $n \in \mathbf{Z}$. It is known that G is isomorphic to the group of Example 4 (see, for instance, [5], Propositions 4.1, 4.4, and 4.8). Our approach this time, however, will be to construct, directly from the definition of G as a homeomorphism group, a category $\mathcal{M}$ whose classifying space is a $K(G, 1)$-complex. We will then apply the methods of §3 to $\mathcal{M}$ (cf. Remark 2 at the end of §3). Our discussion will be sketchy and will assume familiarity with the definition and basic properties of the classifying space of a category, as given in [18] or [17].

Let $\mathcal{G}$ be the groupoid whose objects are the intervals I of the form $[0, l]$, where l is a positive integer, and whose morphisms $I \to I'$ are the piecewise linear homeomorphisms satisfying conditions (a) and (b) above. Then $\mathcal{G}$ is connected (i.e., all objects are isomorphic), and our group G is the group of maps from $[0,1]$ to itself in $\mathcal{G}$. Hence the classifying space of $\mathcal{G}$ is a $K(G,1)$-complex.

By an *admissible subdivision* of $[0,l]$ we will mean a subdivision obtained by starting with the standard partition into l subintervals $[i-1,i]$ $(1 \le i \le l)$ and then repeating 0 or more times the operation of inserting a midpoint into a subinterval. One can show that every map $g : I \to I'$ in $\mathcal{G}$ can be described in terms of admissible subdivisions, i.e., there are admissible subdivisions of I and I' (into the same number of subintervals) such that g maps the intervals of the first subdivision linearly to the intervals of the second (cf. [5], proof of Proposition 4.4).

Call g *positive* if it admits a description of this form in which the subdivision of I' is the standard one with l' subintervals, where $I' = [0,l']$. Note, then, that a general map $g : I \to I'$ in $\mathcal{G}$ is a composite $p^{-1}q$, where $q : I \to I''$ and $p : I' \to I''$ are positive maps with the same target; indeed, this is just a restatement of the fact that g is describable in terms of admissible subdivisions of I and I'.

Identity maps are positive, and a composite of positive maps is positive; so $\mathcal{G}$ has a subcategory $\mathcal{M}$ whose maps are the positive maps. We can now express the result of the previous paragraph by writing $\mathcal{G} = \mathcal{M}^{-1}\mathcal{M}$. This implies that $|B\mathcal{M}|$ is a $K(G,1)$-complex.

[Sketch of proof: Let $\mathcal{E}$ be the following category: An object of $\mathcal{E}$ is a map $g : [0,1] \to I$ in $\mathcal{G}$, where I is an arbitrary object of $\mathcal{G}$; given $g : [0,1] \to I$ and $g' : [0,1] \to I'$, a map from g to g' in $\mathcal{E}$ is a map $m : I \to I'$ in $\mathcal{M}$ such that $g' = mg$. Using the equation $\mathcal{G} = \mathcal{M}^{-1}\mathcal{M}$, one shows that for any two objects g, g' of $\mathcal{E}$ there is a third object g'' to which they both map. It follows that $\mathcal{E}$ is a filtering category and hence that $|B\mathcal{E}|$ is contractible. Our assertion that $|B\mathcal{M}|$ is a $K(G,1)$-complex is now immediate, since G acts freely on $|B\mathcal{E}|$ with $|B\mathcal{M}|$ as quotient. Alternatively, one can deduce the assertion from Quillen's Theorem A, applied to the inclusion $\mathcal{M} \hookrightarrow \mathcal{G}$.]

For any $l \ge 1$ and any i with $1 \le i \le l$, let $\delta_i^l : [0,l] \to [0,l+1]$ be the map in $\mathcal{M}$ which has slope 2 on $[i-1,i]$ and slope 1 elsewhere. To simplify the notation, we will often indicate this map by writing $[0,l] \xrightarrow{i} [0,l+1]$. The maps δ_i^l generate $\mathcal{M}$. In fact, every map $[0,l] \to [0,l+n]$ in $\mathcal{M}$ is uniquely expressible as a composite

$$[0,l] \xrightarrow{i_1} [0,l+1] \xrightarrow{i_2} \cdots \xrightarrow{i_n} [0,l+n]$$

with $i_1 \leq \cdots \leq i_n$. Moreover, these normal forms for the maps in $\mathcal{M}$ are produced by the rewriting rules

$$\delta_i^{l+1}\delta_j^l \to \delta_{j+1}^{l+1}\delta_i^l \qquad (1 \leq i < j \leq l).$$

Recall now that the classifying space X of $\mathcal{M}$ is the geometric realization of a semi-simplicial complex $B\mathcal{M}$ with one n-simplex for every diagram $I_0 \to \cdots \to I_n$ consisting of n composable maps. (When $n = 0$, such a "diagram" is simply an object of $\mathcal{M}$.) Using the methods of §3, one constructs a collapsing scheme for $B\mathcal{M}$ whose essential cells are those of the form

$$[0, l] \xrightarrow{i_1} \cdots \xrightarrow{i_n} [0, l+n]$$

with $i_1 > \cdots > i_n$. Thus we can collapse X to a $K(G,1)$-complex Y with one n-cell for every sequence $l \geq i_1 > \cdots > i_n$ of positive integers. The corresponding resolution D of $\mathbf{Z}$ over $\mathbf{Z}G$ is a quotient of $C = C_*(B\mathcal{E})$, where $\mathcal{E}$ is the category introduced above.

REMARKS. 1. This $K(G,1)$-complex Y is essentially the same as one that was obtained by Melanie Stein [unpublished], using different methods. As in Example 4, it is easy to collapse Y further, to a complex with one vertex and exactly two n-cells for each $n \geq 1$. The complex Y and analogous complexes for other homeomorphism groups have proved to be quite useful, for reasons that will be explained elsewhere.

2. The category $\mathcal{E}$ which arose above is actually a poset viewed as a category. It is isomorphic to the poset of ordered bases that was used in [5].

Example 6: A group of Brin and Squier. Our last example is intended to illustrate the advantage of using the "right" set of generators, or the "right" submonoid, for a group G. Let G be the group of piecewise linear homeomorphisms g of the half-line $[0, \infty)$ such that g has only finitely many singularities and satisfies the slope and singularity conditions (a) and (b) of Example 5. This is the group called $G(2)$ in [3]. It admits a presentation with infinitely many generators $x_0, x_1, x_2, \ldots$ and infinitely many relations $x_j^{x_i} = x_{2j-i}$ for $i < j$, cf. [3], (2.11). We can now proceed exactly as in Example 4 to obtain a "cubical" free resolution D with one generator for each decreasing sequence $i_1 > \cdots > i_n$ of non-negative integers.

Now it is known that G is of type FP_∞; in fact, G is an ascending HNN extension of Thompson's group (Example 4). So one might expect to be able to use the cubical face operators A_i and B_i to collapse D to a small resolution, as in [6]. But I have not been able to do this. The situation changes drastically, however, if we replace the generators x_i by new generators $t, y_0, y_1, \ldots$, where $t = x_0$ and $y_i = x_{i+1}^{-1} x_i = x_i x_{i+2}^{-1}$. With these new generators, the defining relations become

$$\begin{aligned} y_i^t &= y_{2i} y_{2i+2} \\ y_j^{y_i} &= y_{j+1} \qquad \text{for } i < j. \end{aligned}$$

This presentation reflects the fact that G is an HNN extension of Thompson's group.

It is not hard to show that the submonoid M of G generated by t and the y_i admits a presentation by a complete rewriting system with rules $y_i t \to t y_{2i} y_{2i+2}$ and $y_j y_i \to y_i y_{j+1}$ for $i < j$. And, again, we have $G = MM^{-1}$. So we obtain a resolution with basis elements $(y_{i_1}, \ldots, y_{i_n})$ and $(y_{i_1}, \ldots, y_{i_{n-1}}, t)$, where the subscripts are strictly decreasing. Our rewriting system does not quite fit the framework of §6, because of the presence of two y's on the right side of the first rule. But one can still use the methods of §6 to obtain a cubical boundary formula $\partial = \sum_{i=1}^n (-1)^{i-1}(A_i - B_i)$, where the A_i and B_i are defined as before by moving generators to the right or left. The only difference is that if σ is an essential cell of the form $(y_{i_1}, \ldots, y_{i_{n-1}}, t)$, then the result of moving t to the left is $\tau = (y_{2i_1} y_{2i_1+2}, \ldots, y_{2i_{n-1}} y_{2i_{n-1}+2})$, which is not essential. Thus $A_n \sigma$ has to be interpreted as $q(\tau)$ rather than τ. Since t does not occur in τ, we can compute $q(\tau)$ by repeated applications of the lemma in §6. This will yield a linear combination of essential cells involving y's only, and not t. It is not necessary, for our present purposes, to know any more about $q(\tau)$.

It is now a simple matter to imitate [6] and collapse the resolution D to one with only finitely many generators in each dimension. Here are some hints as to how to do this: As essential cells in dimensions 1 and 2, take (t), (y_0), (y_1), (y_2, y_0), (y_3, y_1), (y_0, t), and (y_1, t). To collapse the 1-cell $\tau = (y_i)$, which is now considered redundant if $i \geq 2$, set $c(\tau) = (y_{i-1}, y_{i-2})$; note that $A_2 c(\tau) = \tau$. Similarly, collapse the 2-cell $\tau = (y_i, t)$ with $i \geq 2$ by setting $c(\tau) = (y_{i-1}, y_{i-2}, t)$. Suppose, finally, that $\tau = (y_j, y_i)$ with either (a) $j \geq i + 3$ or (b) $j = i + 2$ and $i \geq 2$. We then take $c(\tau)$ to be (y_{j-1}, y_{j-2}, y_i) in case (a) and $(y_{j-1}, y_{i-1}, y_{i-2})$ in case (b). Further details are omitted.

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