WHAT IS . . .

# A Building?

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Buildings were introduced by Jacques Tits to provide a geometric framework for understanding certain classes of groups. The definition evolved gradually during the 1950s and 1960s and reached a mature form about 1965. My treatment will be based on Tits's 1965 definition, in which a building is a simplicial complex with certain properties. It is possible to give a more modern answer to "What is a building?", which is equivalent but looks very different; see [2].

Buildings are made up of apartments, also called *thin buildings* or *Coxeter complexes*; these correspond to Coxeter groups. A *Coxeter group* is a group generated by elements of order 2, subject to relations that give the orders of the pairwise products of the generators. The simplest example is the dihedral group $D_{2m}$ of order $2m$, with presentation

$$D_{2m} = \left\langle s, t \mid s^2 = t^2 = (st)^m = 1 \right\rangle .$$

The infinite dihedral group

$$D_\infty = \left\langle s, t \mid s^2 = t^2 = 1 \right\rangle$$

is also a Coxeter group; there is no relation for the product $st$ because it has infinite order. Readers who have studied Lie theory have seen Weyl groups, which are the classical examples of Coxeter groups. The symmetric group $S_4$ on four letters, for instance, is the Weyl group of type $A_3$, with presentation

$$S_4 = \left\langle s, t, u \mid s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^2 = 1 \right\rangle .$$

Every finite Coxeter group can be realized in a canonical way as a group of orthogonal transformations of Euclidean space, with the generators of order 2 acting as reflections with respect to hyperplanes. Thus $D_{2m}$ acts on the plane, with $s$ and $t$ acting as reflections through lines that meet at an angle of $\pi/m$. And $S_4$ admits a reflection representation on 3-space, obtained by starting with the obvious action of $S_4$ on $\mathbb{R}^4$ and restricting to the subspace $x_1 + x_2 + x_3 + x_4 = 0$. More geometrically, we get this action by viewing $S_4$ as the group of symmetries of a regular tetrahedron.

Given a finite Coxeter group $W$ and its reflection representation on Euclidean space, consider the set of hyperplanes whose reflections belong to $W$. If we cut the unit sphere by these hyperplanes, we get a cell decomposition of the sphere. The cells turn out to be (spherical) simplices, and we obtain a simplicial complex $\Sigma = \Sigma(W)$ triangulating the sphere. This is the Coxeter complex associated with $W$.

For $D_{2m}$ acting on the plane, $\Sigma$ is a circle decomposed into $2m$ arcs. For the action of $S_4$ on 3-space, $\Sigma$ is the triangulated 2-sphere shown in Figure 1. There are six reflecting hyperplanes, which cut the sphere into twenty-four triangular regions. Combinatorially, $\Sigma$ is the barycentric subdivision of the boundary of a tetrahedron, as indicated in the picture. (One face of an inscribed tetrahedron is visible.) The vertex labels will be explained below.

A similar but more complicated construction yields a Coxeter complex associated with an arbitrary Coxeter group $W$. For example, $\Sigma(D_\infty)$ is a triangulated line, with the generators $s$ and $t$ acting as affine reflections with respect to the endpoints of an edge.

In general, a simplicial complex $\Sigma$ is said to be a *Coxeter complex* if it is isomorphic to $\Sigma(W)$ for some Coxeter group $W$. Such complexes are glued together to make buildings. Here is the canonical example of a building: Let $k$ be a field and let $\Delta = \Delta(k^n)$ be the abstract simplicial complex whose vertices are the nonzero proper subspaces of the vector space $k^n$ and whose simplices are the chains

$$V_1 < V_2 < \cdots < V_r$$

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