Rewriting Systems and Discrete Morse Theory

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Outline

Review of Discrete Morse Theory

Rewriting Systems and Normal Forms

Collapsing the Classifying Space

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Collapsing the Classifying Space

History

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- (Brown, 1989) Formalized the method ("collapsing scheme"), applied it to groups with a rewriting system.
- (Forman, 1995) Developed discrete Morse theory, motivated by differential topology.
- (Chari, 2000) Formulated discrete Morse theory combinatorially in terms of "Morse matchings"; these are the same as collapsing schemes.

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Given a cell complex X, try to "collapse" it to a homotopy-equivalent quotient complex Y with fewer cells.

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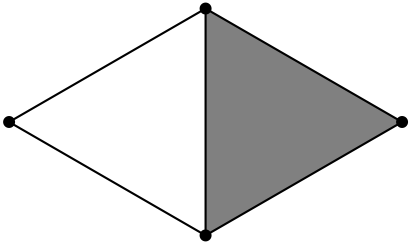
- Homotopy type changes only when we adjoin a critical cell.
- $X \simeq Y$, where Y has one cell for each critical cell of X.

Example 1



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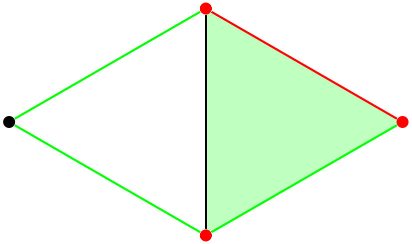

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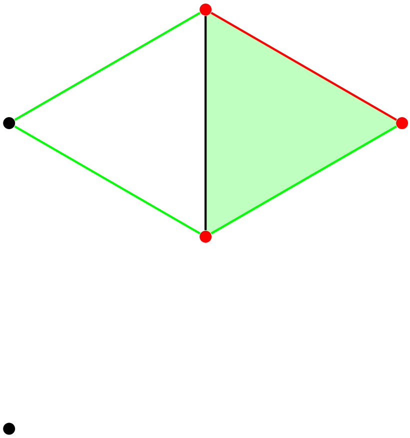

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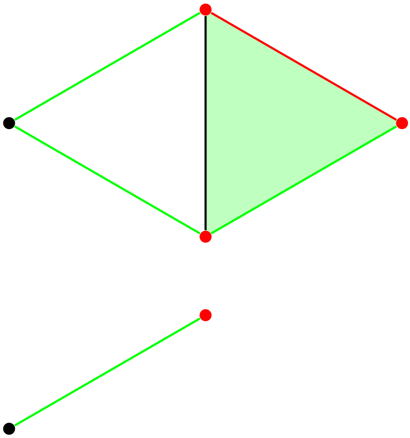

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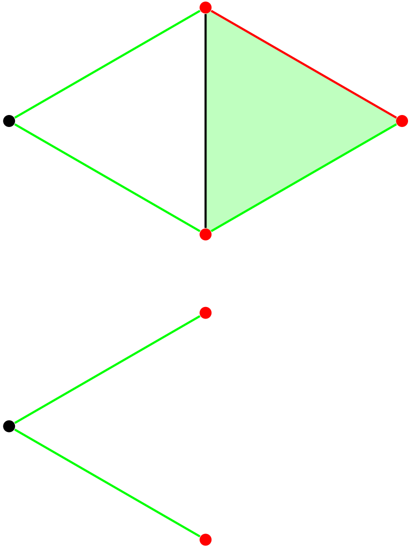

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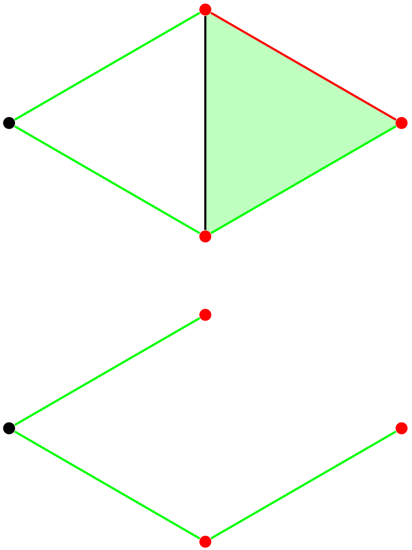

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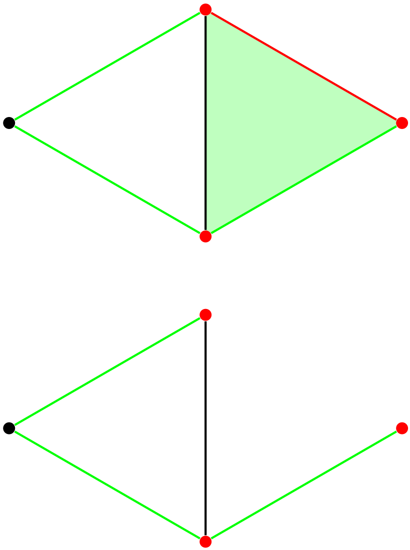

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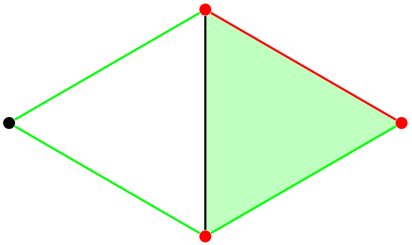

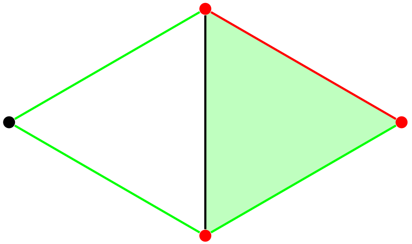

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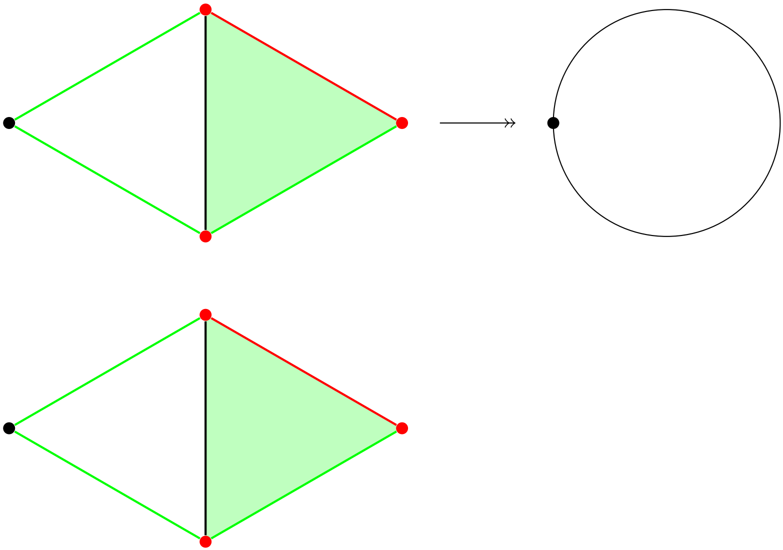

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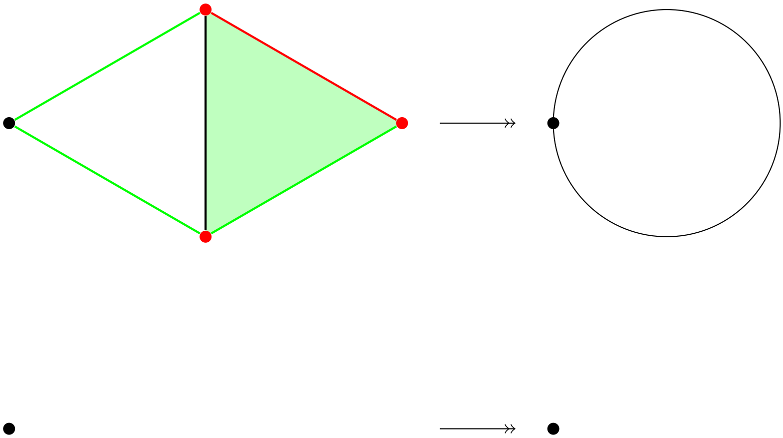

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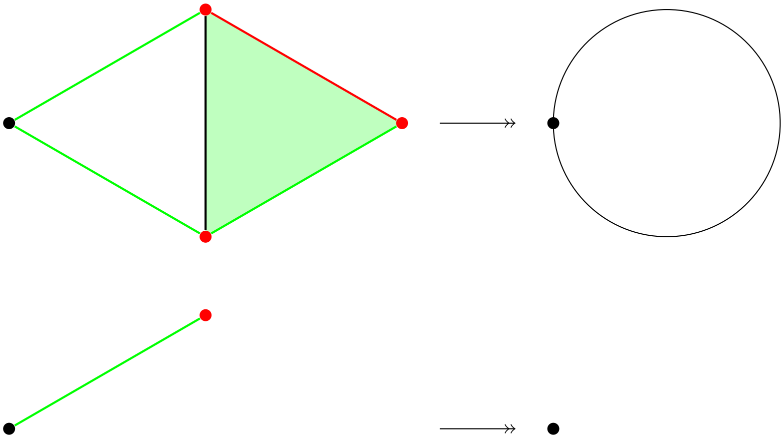

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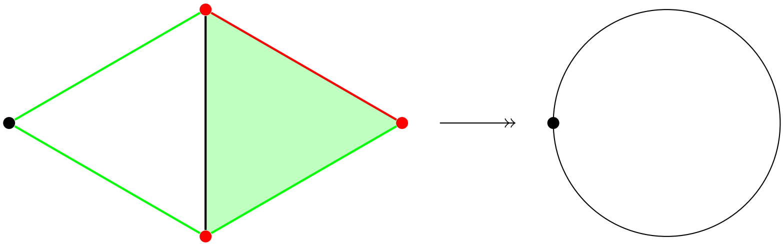

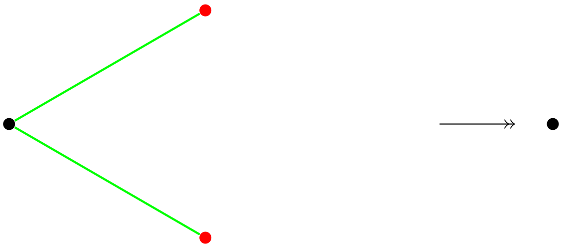

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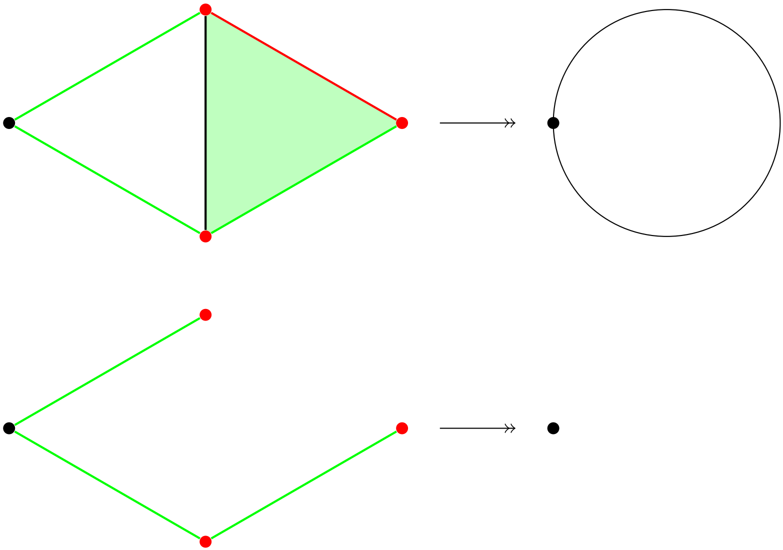

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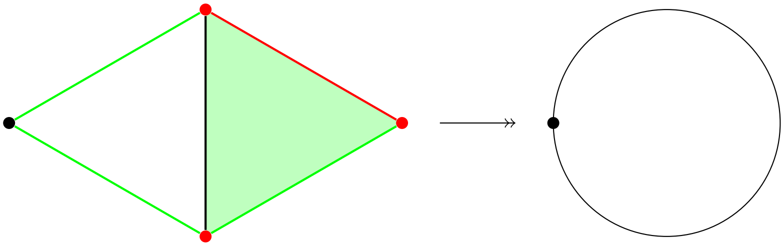

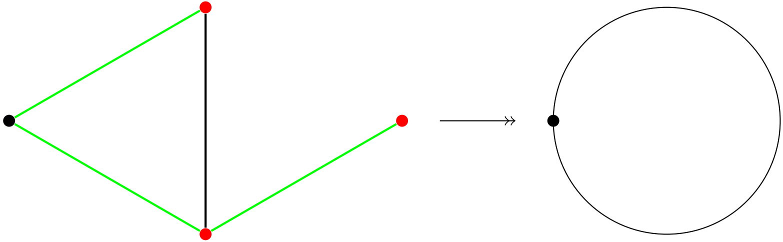

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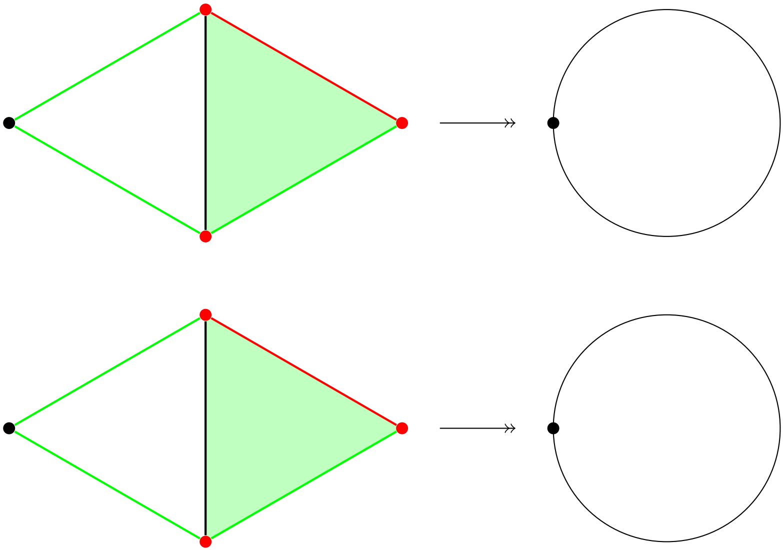

Example 2

- X: boundary of 3-simplex
- Vertices: 1, 2, 3, 4
- Simplices: nonempty proper subsets
- Match by inserting/deleting vertex 1 when possible.

$$
\begin{array}{rcl}
1 & & \\
2 & \leftrightarrow & 12 \\
3 & \leftrightarrow & 13 \\
4 & \leftrightarrow & 14 \\
23 & \leftrightarrow & 123 \\
24 & \leftrightarrow & 124 \\
34 & \leftrightarrow & 134 \\
234 & &
\end{array}
$$

X collapses to a 2-sphere with one vertex and one 2-cell.

Morse Matchings: Summary

Given X as before (classification of cells, matching), want to build X by adjoining, for $n = 0, 1, 2, \ldots$

- **Critical** n-cells.
- Redundant n-cells τ, along with associated collapsible $(n+1)$-cells σ.

Want all (redundant) faces of σ other than τ to be there already.

Definition

Given $\sigma \leftrightarrow \tau$ and another redundant face $\tau' < \sigma$, write $\tau \succ \tau'$. The data above define a **Morse matching** if there is no infinite descending chain $\tau \succ \tau' \succ \tau'' \succ \cdots$ of redundant cells.

Proposition

A Morse matching yields a canonical homotopy equivalence $X \twoheadrightarrow Y$, where Y has one cell for each critical cell of X.

Outline

Notation and Terminology

- M: A monoid
- S: A set of generators
- F: The free monoid on S
- $q \colon F \twoheadrightarrow M$: The quotient map

F consists of words on the alphabet S, and q takes a word w to the element of M represented by w.

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- Given $(w_1, w_2) \in R$, write $w_1 \to w_2$ ("rewriting rule").
- More generally, write $uw_1v \to uw_2v$ for $u, v \in F$.

We say that uw_1v reduces to uw_2v.

Want to use rewriting to reduce every element to a normal form.

Complete Rewriting Systems

Definition

R is a complete rewriting system for M if:

- The set of irreducible words is a set of normal forms for M.
- There is no infinite chain $w \to w' \to w'' \to \cdots$ of reductions.

The first condition is equivalent to the diamond property (M. H. A. Newman, 1942).

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Example (Free commutative monoid on 2 generators)

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Example (Free group on 2 generators)

Four monoid generators $a, \bar{a}, b, \bar{b}$, four rewriting rules

$$a\bar{a} \to 1 \qquad \bar{a}a \to 1 \qquad b\bar{b} \to 1 \qquad \bar{b}b \to 1$$

leading to the standard normal forms (reduced words in the sense of group theory).

Example (Thompson's Group and Monoid)

- Group presentation: $\langle\, x_0, x_1, \ldots ; x_i^{-1} x_n x_i = x_{n+1} \text{ for } i < n \,\rangle$
- This is MM^{-1}, where M is defined by the rewriting rules

$$x_n x_i \to x_i x_{n+1} \quad (i < n)$$

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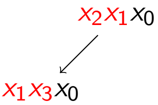

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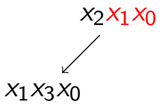

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$$\swarrow \qquad \searrow$$

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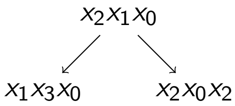
Completing the Diamond



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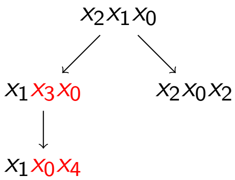

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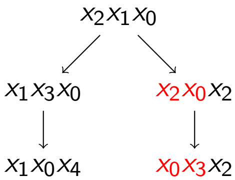

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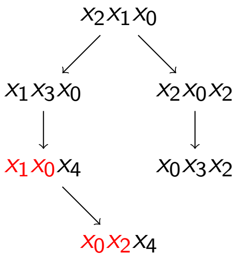

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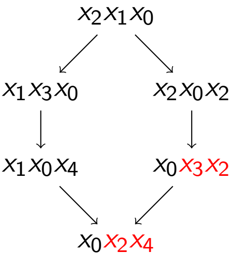

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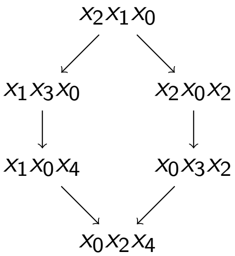

- That's all there is to it! M has a complete rewriting system.

Outline

Review of Discrete Morse Theory

Rewriting Systems and Normal Forms

Collapsing the Classifying Space

The Classifying Space of a Monoid

Associated to a monoid M is a CW-complex $X = BM$.

- Cells are simplices with face identifications.
- One n-cell for each n-tuple $(m_1 \mid m_2 \mid \cdots \mid m_n)$.
- Face operators delete m_1, delete a bar, delete m_n.

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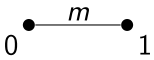
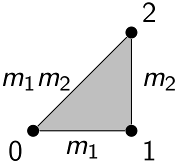
$(\,)$

(m)



$(m_1 \mid m_2)$

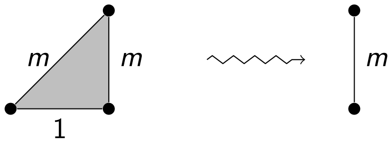


Normalization

If some $m_i = 1$, the cell $(m_1 \mid m_2 \mid \cdots \mid m_n)$ is degenerate; squash it to a suitable face.

(1)

$(1 \mid m)$



So X has one n-cell for each n-tuple of nontrivial elements of M.

What is BM?

- If M is a group, then $BM = K(M, 1)$, the (original) Eilenberg–MacLane space with $\pi_1 = M$ and $\pi_i = 0$ for $i > 0$.
- Its cellular chain complex is the standard complex for defining $H_*(M)$ algebraically.
- More generally, if M admits a group of fractions $G = MM^{-1}$, then $BM \simeq K(G, 1)$.
- It's always true that $\pi_1(BM)$ is the group completion of M.

Matching in Low Dimensions

Assume M has a complete rewriting system. View n-simplices as n-tuples of (irreducible) words $(w_1 \mid w_2 \mid \cdots \mid w_n)$.

1-cells

- A 1-cell (w) is **critical** if and only if $w \in S$.
- If $l(w) > 1$, write $w = su$ and make (w) redundant via $(w) \leftrightarrow (s \mid u)$. [Faces (u), (w), (s).]

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2-cells

- $(s \mid u)$ is collapsible if su is irreducible.
- $(su \mid v) \leftrightarrow (s \mid u \mid v)$.
- $(s \mid uv) \leftrightarrow (s \mid u \mid v)$ if suv is reducible? OK if su still reducible; in this case use smallest prefix u.
- $(s \mid w)$ is **critical** if sw is reducible but every proper prefix is irreducible.

The Morse Matching

Given a cell $(w_1 \mid w_2 \mid \cdots \mid w_n)$, read from left to right and try to insert or delete a bar. A cell is redundant if we insert a bar, collapsible if we delete a bar, and **critical** otherwise.

Restrictions

- $(\cdots \mid u \mid v \mid \ldots) \mapsto (\cdots \mid uv \mid \ldots)$ is OK only if uv is irreducible.
- $(\cdots \mid u \mid vw \mid \ldots) \mapsto (\cdots \mid u \mid v \mid w \mid \ldots)$ is OK only if uv is reducible.

The Morse Matching

Given a cell $(w_1 \mid w_2 \mid \cdots \mid w_n)$, read from left to right and try to insert or delete a bar. A cell is redundant if we insert a bar, collapsible if we delete a bar, and **critical** otherwise.

Restrictions

- $(\cdots \mid u \mid v \mid \ldots) \mapsto (\cdots \mid uv \mid \ldots)$ is OK only if uv is irreducible.
- $(\cdots \mid u \mid vw \mid \ldots) \mapsto (\cdots \mid u \mid v \mid w \mid \ldots)$ is OK only if uv is reducible.

Theorem

This works.

The Morse Matching

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Theorem

If M is a monoid with a set of normal forms that comes from a complete rewriting system, then the procedure above is a Morse matching. Thus $X = BM$ has a canonical quotient Y with one cell for each critical cell of X, and the quotient map is a homotopy equivalence.

The Morse Matching

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Remarks

- *The Morse matching depends only on the normal forms, not on the rewriting rules.*
- *But the fact that we have a complete rewriting system is used in the proof.*
- *And the rules are needed to figure out what Y looks like.*

Example (free commutative monoid, normal forms $s^i t^j$)

$$
\begin{aligned}
()& \\
(s)& \\
(t)& \\
(sw)& \leftrightarrow (s \mid w) \\
(tw)& \leftrightarrow (t \mid w)
\end{aligned}
$$

Example (free commutative monoid, normal forms $s^i t^j$)

$$
\begin{array}{r}
(\,) \\
(s) \\
(t) \\
(sw) \leftrightarrow (s \mid w) \\
(tw) \leftrightarrow (t \mid w) \\
(su \mid v) \leftrightarrow (s \mid u \mid v) \\
(tu \mid v) \leftrightarrow (t \mid u \mid v) \\
(t \mid su) \leftrightarrow (t \mid s \mid u) \\
(t \mid s)
\end{array}
$$

Example (free commutative monoid, normal forms $s^i t^j$)

$$(\,)$$
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$$(t \mid su) \leftrightarrow (t \mid s \mid u)$$
$$(t \mid s)$$

No more critical cells. For example, consider dimension 3:

$$(u \mid v \mid w)$$

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$$()$$
$$(s)$$
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$$(sw) \leftrightarrow (s \mid w)$$
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$$(tu \mid v) \leftrightarrow (t \mid u \mid v)$$
$$(t \mid su) \leftrightarrow (t \mid s \mid u)$$
$$(t \mid s)$$

No more critical cells. For example, consider dimension 3:

$$(t \mid v \mid w)$$

Example (free commutative monoid, normal forms $s^i t^j$)

$$
\begin{aligned}
&()\\
&(s)\\
&(t)\\
(sw) &\leftrightarrow (s \mid w)\\
(tw) &\leftrightarrow (t \mid w)\\
(su \mid v) &\leftrightarrow (s \mid u \mid v)\\
(tu \mid v) &\leftrightarrow (t \mid u \mid v)\\
(t \mid su) &\leftrightarrow (t \mid s \mid u)\\
(t \mid s)&
\end{aligned}
$$

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&(t \mid s)
\end{aligned}
$$

No more critical cells. For example, consider dimension 3:

$$(t \mid s \mid w)$$

Note

The collapsed complex Y is a torus.

Example (free group on a, b, usual normal forms)

Four critical cells in each positive dimension:

$$(a) \quad (\bar{a}) \quad (b) \quad (\bar{b})$$
$$(a \mid \bar{a}) \quad (\bar{a} \mid a) \quad (b \mid \bar{b}) \quad (\bar{b} \mid b)$$
$$\vdots$$

Example (free group on a, b, usual normal forms)

Four critical cells in each positive dimension:

$$\begin{array}{cccc} (a) & (\bar{a}) & (b) & (\bar{b}) \\ (a \mid \bar{a}) & (\bar{a} \mid a) & (b \mid \bar{b}) & (\bar{b} \mid b) \\ & & \vdots & \end{array}$$

Extend Morse matching to get rid of most of them...

$$\begin{gathered} (\bar{a}) \leftrightarrow (a \mid \bar{a}) \\ (\bar{a} \mid a) \leftrightarrow (a \mid \bar{a} \mid a) \\ \vdots \end{gathered}$$

Example (free group on a, b, usual normal forms)

Four critical cells in each positive dimension:

$$(a) \quad (\bar{a}) \quad (b) \quad (\bar{b})$$
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Extend Morse matching to get rid of most of them...

$$(\bar{a}) \leftrightarrow (a \mid \bar{a})$$
$$(\bar{a} \mid a) \leftrightarrow (a \mid \bar{a} \mid a)$$
$$\vdots$$

...leaving three critical cells $(\,)$, (a), (b); Y is a figure 8.

Thompson's Monoid

Generators $x_0, x_1, \ldots$, normal forms

$$x_{i_1} x_{i_2} \cdots x_{i_n} \text{ with } i_1 \leq i_2 \leq \cdots \leq i_n$$

Critical cells

$$(x_{i_1} \mid x_{i_2} \mid \cdots \mid x_{i_n}) \text{ with } i_1 > i_2 > \cdots > i_n$$

The resulting collapsed complex Y is the "big" cubical complex found by Brown–Geoghegan. This can be further collapsed.

Variants

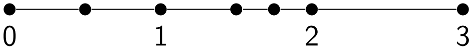
- Chain complexes
- Algebras with rewriting system
- Small categories

Thompson's Group via a Category

Let M be the following category of PL-homeomorphisms:

- Objects: The intervals $[0, l] \subset \mathbb{R}$, $l = 1, 2, \ldots$.
- Morphisms: PL-maps $[0, l] \to [0, m]$ obtained by dyadically subdividing $[0, l]$ into m subintervals and mapping them linearly to the standard unit intervals in $[0, m]$.

Dyadic subdivision: Start with standard subdivision into unit intervals, repeatedly insert midpoints.



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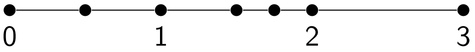

- BM is an Eilenberg–MacLane space for Thompson's group.
- Generators: $i^{(l)} \colon [0, l] \to [0, l+1]$, $i = 1, \ldots, l$.
- Normal forms: $[0, l] \to [0, l+1] \to \cdots \to [0, l+n]$ with weakly increasing i's; these come from rewriting rules.
- Result is a space constructed by Melanie Stein for the study of PL-homeomorphism groups; it can be collapsed further.

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