THE BOLZANO-WEIERSTRASS THEOREM

MATH 1220

The Bolzano-Weierstrass Theorem: Every sequence $\{x_n\}_{n=1}^{\infty}$ in a closed interval $[a, b]$ has a convergent subsequence.

Proof I. Indication of Proof:

(1) Divide $[a, b]$ in half (at its midpoint) into two closed subintervals. At least one subinterval (call it I_1) must contain infinitely many elements of the sequence. Let x_{n_1}, the first element in our subsequence, be a term in the sequence which also lies in I_1.

(2) Now repeat. Specifically, suppose we have (inductively) defined closed intervals $I_1, \ldots, I_k$ and $x_{n_1}, \ldots, x_{n_k}$ (for all $2 \leq m \leq k$) so that
   (a) I_m is a subinterval of I_{m-1} having half its length.
   (b) $x_{n_m} \in I_m$.
   (c) $n_m > n_{m-1}$.
   (d) I_m contains infinitely many elements of the sequence.

   Then divide I_k again in half at its midpoint and choose I_{k+1} to be a half containing infinitely many terms of the sequence. And pick $n_{k+1} > n_k$ with $x_{n_{k+1}}$ a point in I_{k+1}.. Then

$$\bigcap_{k=1}^{\infty} I_k$$

   is a single point by the nested intervals theorem and the subsequence converges to this point.

Proof II. The Bolzano-Weierstrass Theorem follows from the next Theorem and Lemma.

Theorem: An increasing sequence that is bounded converges to a limit.

We proved this theorem in class. Here is the proof.

Proof: Let (a_n) be such a sequence. By assumption, (a_n) is non-empty and bounded above. By the least-upper-bound property of the real numbers, $s = \sup_n(a_n)$ exists. Now, for every $\epsilon > 0$, there exists a natural number N such that $a_N > s - \epsilon$, since otherwise $s - \epsilon$ is an upper bound of (a_n), which contradicts the fact that s is a least upper bound of (a_n). Since (a_n) is increasing snd s is an upper bound, for every $n > N$, $|a_n - s| = s - a_n \leq s - a_N < \epsilon$. Hence s is the limit of (a_n).

Lemma: Every sequence (a_n) has a monotone subsequence: We proved the lemma in class. You can also look at Lemma 1.1, p.38.