GROUP ACTIONS ON SYMPLECTIC MANIFOLDS: LECTURE 1

LIAT KESSLER

1. Basic definitions

A *Lie group* G is a manifold equipped with a group structure, where the group operations: $G \times G \to G$ sending $(a, b) \mapsto ab$, and $G \to G$, sending $a \mapsto a^{-1}$ are smooth maps.

Examples:

- $\mathbb{R}$ (with addition).
- S^1 regarded as unit complex numbers with multiplication, represents rotations of the plane: $S^1 = \mathrm{U}(1) = \mathrm{SO}(2)$.
- $T^n = \mathbb{R}^n / \mathbb{Z}^n$, the compact torus.
- $(\mathbb{C}^*)^n$, the complex torus.
- $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$.
- $\mathrm{U}(n)$, unitary linear transformations of $\mathbb{C}^n$.
- $\mathrm{SU}(n)$, unitary linear transformations of $\mathbb{C}^n$ with $\det = 1$.
- $\emptyset(n)$, orthogonal linear transformations of $\mathbb{R}^n$; $\mathrm{SO}(n)$.
- $\mathrm{GL}(V)$, invertible linear transformations of a vector space V.

A *linear representation* of a Lie group G on a vector space V is a group homomorphism $G \to \mathrm{GL}(V)$.

An action of a Lie group G on a manifold M is a group homomorphism

$$\tau \colon G \to \mathrm{Diff}(M)$$
$$g \mapsto \tau_g.$$

We will write g instead of τ_g, and $g \cdot m$ instead of $\tau_g(m)$. An action of G on M is smooth if the associated map

$$G \times M \to M, \quad (g, m) \mapsto g \cdot m$$

is smooth.

Example. If M is a linear vector space and τ_a are linear transformations, this is a *linear representation* of G.

We will usually, but not always, consider compact Lie groups. Sometimes the compactness requirement can be replaced by the following condition.

An action of G on M is *proper* if the map

$$G \times M \to M \times M, \quad (a, m) \mapsto (a \cdot m, m),$$

is proper. i.e., the preimage of any compact set is compact. Recall that a proper map between locally compact Hausdorff spaces (as are manifolds) is closed (exercise). If G is compact, the action is proper. If G is not compact, but M is compact, the action is not proper.

If the group G acts on two manifolds M and N, and if $\phi\colon M \to N$ is a smooth map, it is said that ϕ is an *equivariant map* if

$$\forall m \in M, \quad \forall g \in G, \quad \phi(g \cdot m) = g \cdot \phi(m).$$

Clearly, if an equivariant map ϕ is a diffeomorphism, the inverse map ϕ^{-1} is also equivariant.

2. Orbits and stabilizers

Let a Lie group G act on a manifold M. The orbit of a point $m \in M$ is

$$G \cdot m = \{g \cdot m | g \in G\} \subseteq M,$$

and its stabilizer G_m,

$$G_m = \{g \in G | g \cdot m = m\} \subseteq G.$$

Being a closed subgroup of the Lie group G, the stabilizer G_m is a Lie group. (See Theorem 2.27 in J.F. Adams, *Lectures on Lie Groups*, Benjamin Inc., New York, 1969.). If the action is proper the stabilizer G_m is compact. Notice that stabilizers of points in the same orbit are conjugate to each other, as $G_{g\cdot m} = g^{-1}G_m g$ (this equality also shows that all possible conjugates appear). Also, if $\phi\colon M \to N$ is an equivariant map, $G_m \subseteq G_{\phi(m)}$.

The action is said to be *free* if $G_m = \{e\}$ for all $m \in M$.

The action is said to be *effective* if the homomorphism $G \to \mathrm{Diff}(M)$ that defines the action is one-to-one, or, equivalently, if $\cap_{m\in M} G_m = \{e\}$.

The evaluation map $ev_m\colon g \to g \cdot m$ induces a bijection from the quotient G/G_m to the orbit $G \cdot m$. The manifold M decomposes into a disjoint union of orbits. The quotient M/G is the set of orbits with the quotient topology.

Example. Let S^1 act on the unit sphere $S^2 \subset \mathbb{R}^3$ by rotations around the $z - axis$:

$$\theta \mapsto \text{ rotation in angle } \theta \text{ around the } z\text{-axis}.$$

Then the quotient S^2/S^1 is the closed interval $[-1, 1]$.

Example. The circle S^1 acts on $\mathbb{C}^n$ by complex multiplication

$$t \cdot (z_1, \dots, z_n) = (tz_1, \dots, tz_n).$$

The point 0 is fixed; all the other orbits are circles.

Exercise. The circle S^1 also acts on (say) $S^3 \subset \mathbb{C}^2$ by

$$t \cdot (z_1, z_2) = (t^{m_1} z_1, t^{m_2} z_2),$$

where $m_1, m_2 \in \mathbb{Z}$. Show that this action is effective if and only if m_1 and m_2 are relatively prime.

Exercise. The complex torus $(\mathbb{C}^*)^n$ act on $\mathbb{C}^n$ by coordinate-wise multiplication

$$(\lambda_1, \dots, \lambda_n)(z_1, \dots, z_n) = (\lambda_1 z_1, \dots, \lambda_n z_n).$$

(The action is not proper.) What are the orbits? What are the stabilizers? Describe the quotient $\mathbb{C}^n/(\mathbb{C}^*)^n$. Describe the orbits, the stabilizers, and the quotient for the action of the compact torus $(S^1)^n \subseteq (\mathbb{C}^*)^n$ that is given by the same formula.

Exercise. Fix a real number α and let $\mathbb{R}$ act on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by

$$t \cdot (x, y) = (x + t, y + \alpha t).$$

Find the orbits and show that these are submanifolds of T^2 if and only if α is rational. (Hint: if α is irrational, all the orbits are dense).

If the action is proper, every orbit is a closed subset of M, and the orbit space M/G is Hausdorff. (Not true in general, e.g., the irrational flow on the torus T^2.) For a proof, see Proposition B.8 (in Appendix b) in the book *Moment maps, Cobordisms, and Hamiltonian Group Actions* by V. Ginzburg, V. Guillemin, and Y. Karshon, Amer. Math. Soc. Math. Surveys and Monographs **98**, 2002.

3. The Lie algebra

Let G be a Lie group with unit e. Denote by $\mathfrak{g}$ its tangent vector space T_eG at e. We define a Lie bracket $[X, Y]$ on $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is anti-symmetric, and satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

as follows. The group acts on itself by conjugation for all $g \in G$, $G \to G$:

$$h \mapsto ghg^{-1}.$$

By differentiating this map at e we get the *Adjoint action*, for all $g \in G$, $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$:

$$X \mapsto \mathrm{Ad}_g(X),$$

which we may consider as a map $G \to \mathrm{End}(\mathfrak{g})$:

$$g \to \mathrm{Ad}_g,$$

and differentiate at e to get $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$:

$$X \to \mathrm{ad}_X\,.$$

The Lie bracket is defined by $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$(X, Y) \mapsto \mathrm{ad}_X(Y),$$

i.e., $[X, Y] = \mathrm{ad}_X(Y)$.

Example. For $G = T^n$, the Lie algebra is $\mathbb{R}^n$, and the Lie bracket is trivial $[X, Y] = 0$, reflecting the fact that the group is Abelian and thus the Adjoint action trivial.

Alternatively, the Lie algebra is identified with the vector space of left-invariant vector fields on G (by the linear map associating to a vector field its value at e). The Lie bracket defined here coincides then with the Lie bracket of vector fields, given by

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \text{ for any } f \in C^\infty(G).$$

Example. If G is a group of matrices, the Lie bracket is simply the commutator of matrices $[A, B] = AB - BA$.

4. The exponential map

The *exponential* map

$$\exp\colon \mathfrak{g} \to G$$

is characterized by the fact that for each $\xi \in \mathfrak{g}$, the curve $t \mapsto \exp(t\xi)$ is a group homomorphism from $(\mathbb{R}, +)$ to G, and $\frac{d}{dt}|_{t=0} \exp(t\xi) = \xi \in T_eG$. The exponential map is a diffeomorphism from a neighbourhood of $0 \in \mathfrak{g}$ onto a neighbourhood of $e \in G$.

Example. If G is the 1-torus $\mathbb{R}/\mathbb{Z}$, then $\mathfrak{g} = \mathbb{R}$, and $\exp\colon \mathbb{R} \to S^1$ is the exponential mapping $t \mapsto e^{2\pi i t}$. We can consider this torus as being the unit circle

$$S^1 = \{u \in \mathbb{C} | |u| = 1\},$$

and the exponential mapping $\theta \mapsto e^{i\theta}$. They are not exactly the same since the former has period 1 and the latter has period 2π. To solve this inconsistency, in the sequel we will see the 1-torus as $\mathbb{R}/2\pi\mathbb{Z}$.

In general, for $G = T \cong (S^1)^k$, the Lie algebra $\mathfrak{t} = T_eT$ is identified with $\mathbb{R}^k$, and the exponential mapping

$$\exp\colon \mathfrak{t} \to T$$

becomes $\mathbb{R}^k \to \mathbb{R}^k/2\pi\mathbb{Z}^k$. It can be identified with the covering map $\mathbb{R}^n \to T^n$.