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Math 6630: Symplectic Geometry, Spring 2018
Problem Set 2

The HW and Section numbers refer to Homeworks and sections in *Lectures on Symplectic Geometry*, A. Cannas da Silva, Lecture Notes in Mathematics, Springer-Verlag, 2008.

1. The standard area form on $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ is $\omega_x(u,v) = \langle x, u \times v \rangle$ for $u, v \in T_x S^2$. Prove that, on $S^2 \setminus \{\text{the north and south poles}\}$, the standard area form equals $d\theta \wedge dx_3$ where θ, x_3 are the cylindrical coordinates ($x_1 = r\cos\theta$, $x_2 = r\sin\theta$ for $r = \sqrt{x_1^2 + x_2^2}$).

2. (a) Fix a symplectic form ω on M. Let $J_0, J_1 \in \mathcal{J}(M,\omega)$ and $0 \le t \le 1$. Show that $J_t := (1-t)J_0 + tJ_1$ is not necessarily in $\mathcal{J}(M,\omega)$.

   (b) Fix an almost complex structure J on M. Let ω_0 and ω_1 be symplectic forms that are compatible with J, and $0 \le t \le 1$. Show that $\omega_t = (1-t)\omega_0 + t\omega_1$ is a symplectic form that is compatible with J. Is ω_t symplectic if ω_0 and ω_1 are symplectic forms not necessarily compatible with a fixed J?

3. Prove that if (M,ω,J) is a Kähler manifold, then there is a Hermitian metric h on the complex manifold such that $\omega = \operatorname{Im} h$. Guidance: Section 16.1.

4. HW10: 1

5. HW11: 1

6. HW3: 1,2,3

7. Let X be a smooth manifold, T^*X be the cotangent bundle of X equipped with the canonical 1-form α and the standard symplectic form $\omega_0 = -d\alpha$. Let $\pi : T^*X \to x$ be the canonical projection.

   (a) Let $\gamma \in \Omega^1(X)$ be a 1-form on X. Let $G_\gamma \colon T^*X \to T^*X$ be the diffeomorphism obtained by adding γ. Show that

   $$G_\gamma^*(\alpha) - \alpha = \pi^*(\gamma).$$

   Deduce that G_γ is a symplectomorphism of (T^*X, ω_0) if and only of $d\gamma = 0$, i.e., $\gamma \in Z^1(X)$. (Hint: For every $\beta \in \Omega^1(X)$, compare $\beta^* G_\gamma^* \alpha$ and $\beta^*\alpha = \beta$.)

(b) Let $\sigma \in \Omega^2(X)$ be a closed 2-form on X. Show that the 2-form

$$\omega = \omega_0 + \pi^*\sigma$$

defines a symplectic form on T^*M, and that the Liouville form of $\omega_0 + \pi^*\sigma$ equals the Liouville form of ω_0. (Hint. First show that the kernel of $\pi^*\sigma$ at $p \in T^*X$ contains a Lagranigian subspace. Then show that in a symplectic vector space (V, ω) of dimension n, for every $\tau \in \cap^2 V^*$ such that $\ker \tau$ contains a Lagrangian subspace, the form $(\omega + \tau)^n = \omega^n$, in particular $\omega + \tau$ is non-degenerate.)