MORSE-BOTT THEORY

The goal of this talk is to introduce elements of Morse-Bott theory and some of its applications to the geometry of moment maps.

Let $f : M \to \mathbb{R}$ be a smooth function on a manifold M.

Definition 1. A point $p \in M$ is called a *critical point* of f if the induced map $df_p : T_pM \to T_{f(p)}\mathbb{R}$ is zero. Suppose M is an $n-$dimensional manifold and $(U; x_1, \cdots, x_n)$ is any coordinate chart containing p. If p is a critical point, then $\frac{\partial f}{\partial x_i}(p) = 0$ for all $i \in \{1, \cdots, n\}$. For a critical point p of f, the value $f(p)$ is called a *critical value*, otherwise $f(p)$ is called a *regular value.*

Let p be a critical point of f. Define a bilinear form H_pf on T_pM as follows: For $v, w \in T_pM$, $H_pf(v, w) := V_p(W(f))$ where V and W are vector field extensions of v and w respectively. One can show that H_pf is symmetric and well defined on T_pM and is called the *Hessian* of f at p. With a local coordinate system $(U; x_1, \cdots, x_n)$ containing p, the tangent space T_pM has the basis $\frac{\partial}{\partial x_1}|_p, \cdots, \frac{\partial}{\partial x_n}|_p$ with respect to which the Hessian is represented by the matrix

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right). \tag{0.1}$$

Definition 2. The *index* of f at p is defined to be the index of H_pf, namely, the maximal dimension of a subspace of T_pM on which H_pf is negative definite. The *nullity* of f at p is the nullity of H_pf, namely, the dimension of the subspace of T_pM consisting of vectors $v \in T_pM$ such that $H_pf(v, w) = 0$ for all $w \in T_pM$.

Definition 3. A critical point p of f is said to be non-degenerate if the nullity of f at p is trivial, that is, there is a local chart containing p with respect to which the Hessian matrix given by (0.1) is non-singular. If all the critical points of f are non-degenerate, then f is called a *Morse function.*

Lemma 4. *(Morse Lemma) Let p be a non-degenerate critical point of f. Then there is a local coordinate system $(U; x_1, \cdots, x_n)$ containing p with $x_i(p) = 0$ for all i and $f = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$ holds throughout U, where λ is the index of f at p.*

Corollary 5. *Non-degenerate critical points are isolated. In particular, if M is compact, then smooth real valued functions on M have finitely many non-degenerate critical points.*

1. Changing Homotopy type

Famous motivating example: Consider a torus M tangent to a plane V at a point. Let $f : M \to \mathbb{R}$ be the function giving the height of M above the plane V. Denote by M^a the set of all points $x \in M$ such that $f(x) \leq a$. Note that if a is a regular value of f, then M^a is a submanifold of M. The height function f has four critical points $p_1 < p_2 < p_3 < p_4$ which are all non-degenerate. As a varies among regular values of f, have the following submanifolds M^a:

(1) If $a < 0 = f(p_1)$, then M^a is empty.
(2) If $f(p_1) < a < f(p_2)$, then M^a is homeomorphic to a $2-$cell.
(3) If $f(p_2) < a < f(p_3)$, then M^a is homeomorphic to a cylinder.
(4) If $f(p_3) < a < f(p_4)$, then M^a is homeomorphic to a punctured torus.
(5) If $a > f(p_4)$, then M^a is homeomorphic to M.

The non-degenerate critical points of f seem to encode the information of the changing homotopy type of M^a. In particular, the change of homotopy type (1) $\to$ (2) is precisely the attaching of a $0-$cell which is homotopic to a $2-$cell. The change (2) $\to$ (3) is the attaching of a $1-$cell and the resulting space is homotopic to a cyclinder. The change (3) $\to$ (4) is again the attaching of a $1-$cell and the resulting space is homotopic to a punctured torus. Lastly, the change (4) $\to$ (5) is the attaching of a $2-$cell. Note that the dimension of the attached cell when "passing" a critical point corresponds to the index of of that critical point. These observations are generalized in the following results.

Let f be a real-valued function on an $n-$dimensional manifold M and let $M^a = f^{-1}(-\infty, a]$.

Theorem 6. *If $a < b$ and $f^{-1}[a,b]$ is compact and contains no critical points of f, then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b, so that the inclusion map $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

Proof. Take a Riemannian metric on M and let $\langle \cdot, \cdot \rangle$ be the inner product induced by the metric. Consider the vector field ∇f, called the gradient vector field of f, that is characterized by $\langle X, \nabla f \rangle = X(f)$ for every vector field X. Define a smooth function $\rho : M \to \mathbb{R}$ which takes the value $1/\|\nabla f\|^2$ on $f^{-1}[a,b]$ and vanishes outside a compact neighbourhood of $f^{-1}[a,b]$. The vector field X defined by $X_p := \rho(p)(\nabla f)_p$ vanishes outside a compact set and therefore generates a global flow $\varphi_t : M \to M$.

Fix $q \in M$ and consider the map $t \mapsto f(\varphi_t(q))$. If $\varphi_t(q) \in f^{-1}[a,b]$, then

$$\frac{df(\varphi_t(q))}{dt} = \left\langle \frac{d\varphi_t(q)}{dt}, (\nabla f)_q \right\rangle = \left\langle X_q, (\nabla f)_q \right\rangle = 1.$$

In particular, if $a \leq f(\varphi_t(q)) \leq b$, then $f(\varphi_t(q)) = t + C$ for some constant C. For $q \in M$ with $f(q) = a$, then $a = f(q) = f(\varphi_0(q)) = C$ and $f(\varphi_{b-a}(q)) = b$. Hence φ_{b-a} maps M^a diffeomorphically to M^b and similarly, φ_{a-b} maps M^b diffeomorphically to M^a. The family $r_t : M^b \to M^b$ given by

$$r_t(q) = \begin{cases} q & \text{if } f(q) \leq a \\ \varphi_{t(a-f(q))} & \text{if } a \leq f(q) \leq b \end{cases}$$

gives a deformation retract of M^b to M^a. □

Theorem 7. *Let $f : M \to \mathbb{R}$ be a smooth function, and let p be a non-degenerate critical point with index λ. Setting $f(p) = c$ and suppose that $f[c-\varepsilon, c+\varepsilon]$ is compact and does not contain any critical point of f other than p, for some $\varepsilon > 0$. Then for all sufficiently small ε, $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a $\lambda-$cell attached.*

See [Mil] for a careful proof.

2. Morse-bott functions

Let f be a smooth real-valued function on an $n-$dimensional manifold M. Denote by C_f the set of critical points of f and let C be a connected component of C_f. With respect to some Riemannian metric, the tangent space at every point $p \in C$ decomposes into $T_pM = T_pC \oplus N_pC$, where N_pC is the normal bundle at p with respect to the chosen Riemannian metric. Note that the T_pC vanishes under the Hessian. Indeed, if $v, w \in T_pC$, $H_pf\,(v, w) = V_p\,(W\,(f))$ and with respect to the chosen metric, $W\,(f) = \langle W, \nabla f\rangle$, where ∇f is the gradient vector field corresponding to f. Since points in C are critical points, the gradient vector field vanishes on C. Therefore H_pf induces a symmetric bilinear form on N_pC which we denote by h_pf.

Definition 8. A smooth submanifold $C \hookrightarrow M$ is said to be a *non-degenerate critical submanifold* if $C \subset C_f$, C is connected, and for all $p \in C$, the induced symmetric form h_pf is non-degenerate. Note that h_pf is non-degenerate if and only if $T_pC = \operatorname{Ker}(H_pf)$, that is H_pf be non-degenerate in the direction normal to C at p. We say that f is a *Morse-Bott function* if the connected components of C_f are non-degenerate critical submanifolds.

Lemma 9. *(Morse-Bott) Let $f : M \to \mathbb{R}$ be a Morse-Bott function and C a connected component of C_f of dimension k as a manifold. Then for $p \in C$, there exists a local coordinate system $\left(U; \varphi = (x_1, \cdots, x_k, y_1, \cdots, y_{n-k}) : U \to V \subset \mathbb{R}^k \times \mathbb{R}^{n-k}\right)$ containing p such that $\varphi\,(p) = 0$, $\varphi\,(U \cap C) = \{(x, y) \in V : y = 0\}$ and the identity $f = f\,(C) - y_1^2 - y_2^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_{n-k}^2$ holds throughout U, where λ is the index of h_pf.*

An immediate consequence the Morse-Bott lemma is the fact that the index of h_pf is locally constant and is therefore an invariant of C called the *index of* C. See [BH] for many proofs of the Morse-Bott lemma.

Examples:

1: Let $M = \mathbb{R}^n$ and I, J, K disjoint subsets of $\{1, \cdots, n\}$ such that $I \cup J \cup K = \{1, \cdots, n\}$. Define $f : M \to \mathbb{R}$ by $f\,(x) = \frac{1}{2}\sum_{i\in I} x_i^2 - \frac{1}{2}\sum_{i\in J} x_j^2$. Then f is Morse-Bott but not Morse.

2: The height function of a torus tangent to a plane at a point is Morse and hence Morse-Bott.

3: The height function of a torus tangent to a plane along a circle is Morse-Bott but not Morse since its critical submanifolds are circles.

3. Stable and unstable cell bundles

Now, we assume that M is a compact connected manifold equipped with a Riemmanian metric and $f : M \to \mathbb{R}$ is Morse-Bott. Let ∇f be the gradient vector field of f on M and consider its associated flow $\psi_t : M \to M$, $t \in \mathbb{R}$. Define for each critical submanifold C_i the sets

$$W_i^+ = \left\{p \in M : \lim_{t\to+\infty} \psi_t\,(p) \in C_i\right\} \quad \text{and} \quad W_i^- = \left\{p \in M : \lim_{t\to-\infty} \psi_t\,(p) \in C_i\right\}$$

called the *stable* and *unstable* sets of C_i respectively.

Theorem 10. *If f is Morse-Bott, then each W_i^+ and W_i^- is a fiber bundle over C_i. Let λ_i be the index of C_i and k its dimension. Then the fibers of the stable* *and unstable sets are cells of dimension equal to* λ_i *and* $n-k-\lambda_i$ *respectively. Moreover,*

$$M=\bigsqcup W_i^+ \qquad \textit{and} \qquad M=\bigsqcup W_i^-.$$

Proof. Note that for any point $p\in M$, if $\lim_{t\to\pm\infty}\psi_t(p)$ exists, then it is a critical point. That such a limit always exists is a consequence of the compactness of M and the existence of "nice" neighborhoods around critical manifolds given by Morse-Bott lemma. Hence, M can be realized as a disjoint union of the stable (and unstable) sets. The maps

$$\pi_i^+ : W_i^+ \to C_i \qquad \text{and} \qquad \pi_i^- : W_i^- \to C_i$$

given by $\pi_i^+(p)=\lim_{t\to+\infty}\psi_t(p)$ and $\pi_i^-(p)=\lim_{t\to-\infty}\psi_t(p)$ respectively give the fiber bundle structure. The local description of f given by Morse-Bott lemma shows that at each $p\in C_i$, the fibers corresponding to π_i^+ are λ_i-cells and the fibers corresponding to π_i^- are $(n-k-\lambda_i)-$cells. ☐

Examples:

1: Consider f as in example (1) above. Define

$$C_K=\{p=(p_1,\dots,p_n)\in\mathbb{R}^n : p_k=0 \text{ for } k\in K\}$$

and C_I, C_J similarly. Then the critical submanifold has one connected component $C_1=C_K$ and $W_1^+=C_K\bigsqcup C_J$ which is not M. This is an example where M is not compact and the function on M is Morse-Bott but not every point converges via the gradient flow to a critical point.

2: Let $M=\mathbb{S}^2$ and $f:M\to\mathbb{R}$ is the height function. Then f is a Morse function with critical submanifolds $C_1=\{S\}$ and $C_2=\{N\}$ consisting of the south and north pole with C_1 having index 0 and C_2 having index 2. The bundle W_1^+ over C_1 is the $0-$cell $\{S\}$ and the bundle W_2^+ over C_2 is the $2-$cell $\mathbb{S}\backslash\{S\}$.

Theorem 11. *If is Morse-Bott and all critical submanifolds are of even dimension and even index, then* f *has a unique local maxima and unique local minima.*

Proof. Consider the decomposition of M into cell bundles given in Theorem 10. Then there exists some W_i^+ and W_j^- of codimension zero in M hence W_i^+ has $(n-k)-$cells as fibers over C_i and $\lambda_i=n-k$. Similarly, W_j^- has $(n-k)-$cells as fibers over C_j so $\lambda_j=0$.

Take the cell decomposition of M into unstable cell bundles

$$M=W_1^-\bigsqcup\cdots\bigsqcup W_s^-\bigsqcup W_{s+1}^-\bigsqcup\cdots\bigsqcup W_N^-$$

where W_i^- for $i\le s$ correspond to index-zero critical submanifolds and for $i>s$ correspond to critical submanifolds of index ≥ 2. Let $a_i=f(C_i)$. Then $a_1,\dots,a_s$ are all local minimas of f. Since the codimension of W_i^- is ≥ 2 for $i>s$, $M\backslash\bigsqcup_{i>s}W_i^-$ is connected and hence $s=1$ and a_1 is the unique local minima. Conversely, one can consider the cell decomposition of M into stable cell bundles. Since there exists at least one critical submanifold of index $n-k$, a similar argument shows that there is only one such critical submanifold. The image of f on this critical submanifold will be the unique local maxima. ☐

Corollary 12. *If* f *is Morse and the indices and dimensions of its critical submanifolds are even, then* $\pi_1(M)=0$.

Proof. Since f is Morse, all critical submanifolds are $0-$dimensional and M has a cell-decomposition with λ_i-cells for each critical point c_i, where λ_i is the index of c_i. Theorem 6 and 7 imply that M has the homotopy type of a finite CW complex (a careful proof of this is in chapter 1 of [Mil]). In particular, the zero skeleton M^0 has exactly one point corresponding to the unique critical point giving the relative minima. All other skeletons M^i are formed by attaching cells of even dimension ≥ 2 and hence $\pi_1(M) \cong \pi_1(M^i) = \pi_1(M^0) = 0$. □

Remark. Theorem 11 is an important ingredient in understanding the geometry of images of moment maps.

References

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[GS] V. Guillemin, S. Sternberg: *Symplectic Techniques in Physics*, Cambridge University Press, 1984. (Section 32)