SYMPLECTIC GEOMETRY: LECTURE 1

LIAT KESSLER

1. Symplectic Linear Algebra

Symplectic vector spaces. Let V be a finite dimensional real vector space and $\omega \in \wedge^2 V^*$, i.e., ω is a bilinear antisymmetric 2-form:

$$\omega\colon V \times V \to \mathbb{R},\ \omega(u,v) = -\omega(v,u)$$

(hence $\omega(v,v) = 0$ for all $v \in V$). We say that ω is *symplectic* if it is non-degenerate: for every $v \neq 0$ there is u such that $\omega(v,u) \neq 0$. We call (V,ω) a *symplectic vector space.*

Claim 1.1. *Let $\omega \in \wedge^2 V^*$. The following are equivalent.*

- *The form ω is symplectic.*
- *The kernel*

  $$\ker \omega := \{v \in V \ :\ \omega(v,u) = 0 \text{ for all } u \in V\}$$

  is trivial.
- *The map*

  $$w^\flat\colon V \to V^*, \omega^\flat(v)(u) = \omega(v,u)$$

  is an isomorphism.

A *symplectomorphism* ϕ between symplectic vector spaces (V,ω) and (V',ω') is a linear isomorphism $\phi\colon V \to V'$ such that $\phi^*\omega' = \omega$. (By definition, $(\phi^*\omega')(u,v) = \omega'(\phi(u),\phi(v))$.) If a symplectomorphism exists, (V,ω) and (V',ω') are said to be *symplectomorphic*. Note that being symplectomorphic is an equivalence relation on vector spaces of finite dimension. The group of symplectomorphisms of (V,ω) is denoted $\mathrm{Sp}(V)$.

Example. **The standard symplectic vector space.** Consider $V = \mathbb{R}^{2n}$ with basis $(e_1,\dots,e_n,f_1,\dots,f_n)$. Then the antisymmetric bilinear form defined by

$$\omega_0(e_i,e_j) = 0,\ \omega_0(f_i,f_j) = 0,\ \omega_0(e_i,f_j) = \delta_{i,j}$$

is a symplectic form. Can you give examples of symplectomorphisms? E.g., $A(e_j) = f_j$, $A(f_j) = -e_j$, or $A(e_j) = e_j + f_j$, $A(f_j) = f_j$.

We will show that every symplectic vector space is symplectomorphic to the standard one.

Lemma 1.1. *For (V, ω), there exists a basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ of V such that $\omega(e_i, f_j) = \delta_{i,j}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$.*

Such a basis is called a *symplectic basis.*

Remarks.

(1) A choice of a symplectic basis for $V, \omega)$ yields a symplectomorphism to $(\mathbb{R}^{2n}, \omega_0)$. Hence the dimension of a symplectic vector space is the only invariant of its isomorphism type.

(2) A symplectic basis is not unique, but is called a "canonical" basis.

(3) Using a symplectic basis, we can write $\omega = e_1^* \wedge f_1^* + \dots + e_n^* \wedge f_n^*$ where $e_1^*, \dots, e_n^*, f_1^*, \dots, f_n^*$ is a basis of V^* dual to the symplectic basis. It is easy to see (e.g., by induction) then that the n-th exterior power

$$\omega^n = \omega \wedge \dots \wedge \omega = \operatorname{alt}(\omega \otimes \dots \otimes \omega) = n! e_1^* \wedge f_1^* \wedge \dots \wedge e_n^* \wedge f_n^*,$$

in particular, $\omega^n(e_1, f_1, \dots, e_n, f_n) = n! \neq 0$. So ω^n is a non-vanishing top degree form.

Before we prove the lemma, we define some of the terms that we will use in the proof and later on.

Subspaces of a symplectic vector space. The *symplectic orthogonal complement* of a linear subspace $W \subseteq V$ is defined as the subspace

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

We note that W^ω is the pre-image of the annihilator $\operatorname{ann}(W) \subseteq V^*$ under the isomorphism $\omega^\flat$. Therefore

$$\dim W^\omega + \dim W = \dim V, \ (W^\omega)^\omega = W.$$

A linear subspace W is called *isotropic* if $W \subseteq W^\omega$; *coisotropic* if $W^\omega \subseteq W$; *symplectic* if $W \cap W^\omega = 0$; *Lagrangian* if $W = W^\omega$. For example, in $(\mathbb{R}^{2n}, \omega_0)$, the span of e_1, e_2 is isotropic; the span of e_1, f_1 is symplectic; the span of $(e_1, \dots, e_n)$ (or of $(f_1, \dots, f_n)$) is Lagrangian.

Notice that every 1-dimensional subspace is isotropic (ask: $\omega(v, v) = 0$), and that W is isotropic iff W^ω is coisotropic. We will use these facts to prove the following lemma.

Lemma 1.2. *In any symplectic vector space (V, ω) there is a Lagrangian subspace.*

Proof. As noted, there exists an isotropic subspace. Let L be an isotropic subspace that is not contained in any isotropic subspace of strictly larger dimension. Then L must be Lagrangian: otherwise, there is $v \in L^\omega \setminus L$ and $L \oplus \operatorname{span}(v) > L$ is isotropic. □

Consequently, the dimension of a symplectic vector space is even: $\dim V = \dim L + \dim L^\omega = 2 \dim L$ for a Lagrangian subspace L. Note that the dimension of an isotropic subspace is $\leq \frac{1}{2} \dim V$.

You will prove a stronger version of the lemma in Problem Set 1:

Lemma 1.3. *Given a Lagrangian subspace M in (V, ω), there is a Lagrangian subspace L such that $L \cap M = \{0\}$.*

Sketch of proof: Now let L be a maximal isotropic subspace with $L \cap M = \{0\}$. If it is not Lagrangian consider the quotient $\pi\colon L^\omega \to L^\omega/L$. The image $\pi(M \cap L^\omega)$ is isotropic hence of positive codimension. Therefore, one can choose an 1-dimensional subspace $F \subset L^\omega/L$ such that $F \cap \pi(M \cap L^\omega) = \{0\}$. Then $L' = \pi^{-1}(F)$ is an isotropic subspace that contains L and whose intersection with M is empty.

Proof of Lemma 1.1. Take two Lagrangian subspaces L, M of (V, ω) such that $L \cap M = \{0\}$. Since $L^\omega = L \cap M = \{0\}$, the composition

$$M \hookrightarrow V \xrightarrow{\omega^\flat} V^* \to L^*,$$

where the last map is the dual to the inclusion $L \hookrightarrow V$, is an isomorphism. Let $e_1, \ldots, e_n$ be a basis for L and $f_1, \ldots, f_n$ the dual basis for $L^* \cong M$. □

We will give another proof of the lemma using compatible complex structures.

Compatible complex structures.

Example. Let V be a complex vector space of complex dimension n, with a Hermitian metric (complex positive definite inner product, complex linear with respect to the second entry and complex anti-linear with respect to the first entry) $h\colon V \times V \to \mathbb{C}$. Then $\omega = \operatorname{Im}(h)$ is a symplectic form on V (considered as a real vector space) (check). Every unitary map $V \to V$ is a symplectomorphism. Note that $g = \operatorname{Re}(h)$ is a real positive definite inner product (check).

Can you always find a Hermitian metric on a finite dim complex vector space? (Sure. Since the answer is clearly yes in $\mathbb{C}^n$, e.g., $h(u, v) = \bar{u}^T \operatorname{Id} v$.)

Lemma 1.4. Wirtinger's inequality: *Let V be a complex linear space of dimension n with a positive definite Hermitian form h on V. Let $g = \mathrm{Re}(h)$ and $\omega = \mathrm{Im}(h)$.*

For $X_1, \ldots, X_{2k}$ in V that are orthonormal with respect to g, we have

$$|\omega^k(X_1, \ldots, X_{2k})| \leq k!,$$

with equality holding precisely when $W = \mathrm{span}(X_1, \ldots, X_{2k})$ is a complex k-dimensional subspace of L.

Proof. Note that the value of $|\omega^k(X_1, \ldots, X_{2k})|$ does not depend on the choice of an orthonormal basis to W, and that there is an orthonormal basis $X_1, \ldots, X_{2k}$ such that

$$\omega|_W = \sum_{j=1}^{k} \omega(X_{2j-1}, x_{2j})(X^*_{2j-1} \wedge X^*_{2j}),$$

where X^*_i is the dual to X_i. (Check!) So

$$|\omega^k(X_1, \ldots, X_{2k})| = k!\pi^k_{j=1}|\omega(X_{2j-1}, X_{2j})|.$$

Therefore it is enough to check the case $k = 1$.

If $k = 1$ then $\omega(X_1, X_1) = 0 = \omega(x_2, X_2)$ (since antisymmetric) hence $h(X_1, X_1) = g(X_1, X_1) = 1 = h(X_2, X_2)$, and $h(X_1, X_2)$ is in $\sqrt{-1}\mathbb{R}$ (since $\mathrm{Re}\, h(X_1, X_2) = g(X_1, X_2) = 0$ hence $\omega(X_1, X_2) = -\sqrt{-1}h(X_1, X_2) = h(iX_1, X_2)$. Therefore, by Schwartz's inequality (applied to h):

$$|\omega(X_1, X_2)| = |h(\sqrt{-1}X_1, X_2)| \leq h(\sqrt{-1}X_1, \sqrt{-1}X_1)h(X_2, X_2) = 1$$

with equality if and only if $X_2 = cX_1$ for a complex number $c \neq 0$. Since $h(X_1, X_1) = 1 = h(X_2, X_2)$ and $\mathrm{Re}(h(X_1, X_2)) = 0$, we have $c^2 = \pm 1$ and $c \neq \pm 1$, i.e., $c = \pm i$ and $\mathrm{span}(X_1, X_2 = \pm iX_1)$ is a complex 1-dimensional subspace.

□

We now go on the reverse direction, starting from a real symplectic vector space. A *complex structure* on a real vector space V is an automorphism $J \colon V \to V$ such that $J^2 = -\,\mathrm{Id}$. A complex structure J on a symplectic vector space (V, ω) is *ω-compatible* if

$$g(u, v) = \omega(u, Jv)$$

defines a positive definite inner product. This implies that

$$\text{for } v \neq 0,\ \omega(v, Jv) = g(v, v) > 0,$$

and

$$\omega(Ju, Jw) = g(Ju, v) = g(v, Ju) = \omega(v, j^2u) = -\omega(v, u) = \omega(u, v),$$

i.e., J is a symplectomorphism. Moreover, a complex structure J is compatible with ω iff these two conditions hold: since ω is bilinear so is g, the first condition is equivalent to g being positive definite; the second condition is equivalent to g being symmetric.

Example. Consider the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$. The complex structure given by $Je_i = f_i$, $Jf_i = -e_i$ is compatible with ω_0 (we noticed before that it is a symplectomorphism; also, by definition of ω_0 and J, $\omega_0(e_i, Je_i) = \omega_0(e_i, f_i) = 1 > 0$ and $\omega_0(f_i, Jf_i) = \omega_0(f_i, -e_i) = 1 > 0$). This identifies $(\mathbb{R}^{2n}, \omega_0, J)$ with $\mathbb{C}^n$ with $\mathrm{Im}(h)$ for $h(u, v) = \bar{u}^T \,\mathrm{Id}\, v$.

In general, a compatible complex structure on (V, ω) makes V a complex vector space with Hermitian metric

$$h(u, v) = g(u, v) + \sqrt{-1}\omega(u, v).$$

(h is complex linear with respect to the second entry, $h(u, Jv) = \sqrt{-1}h(u, v)$, and complex anti-linear with respect to the first entry, $h(Ju, v) = -\sqrt{-1}h(u, v)$, and $h(v, v) > 0$ for $v \neq 0$.)

Lemma 1.5. *Let (V, ω) be a symplectic vector space. Given a positive inner product G on V, there is a linear isomorphism $A\colon V \to V$ such that*

$$J = J^G = \left(\sqrt{AA^*}\right)^{-1} A$$

is well defined and is a compatible complex structure on (V, ω).

The factorization $A = (\sqrt{AA^*})J^G$ is called the *polar decomposition* of A.

Proof. (As noted before), since ω and G are non-degenerate, the maps $\omega^\flat, G^\flat\colon V \to V^*$ are isomorphisms between V and V^*. Hence $\omega(u, v) = G(Au, v)$ for some linear isomorphism $A\colon V \to V$. The map A is skew-adjoint (with respect to G) since

$$G(A^*u, v) = G(u, Av) = G(Av, u) = \omega(v, u) = -\omega(u, v) = G(-Au, v),$$

hence $A^* = -A$. Note that AA^* is symmetric ($(AA^*)^* = AA^*$), and positive: $G(AA^*u, u) = G(A^*u, A^*u) > 0$ for $u \neq 0$. We conclude that AA^* diagonalizes with positive eigenvalues λ_i:

$$AA^* = B\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)B^{-1}.$$

So we can define

$$\sqrt{AA^*} = B\,\mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})B^{-1};$$

and it is symmetric and positive definite. Let

$$J = J^G := \left(\sqrt{AA^*}\right)^{-1} A.$$

Since A commutes with $\sqrt{AA^*} = \sqrt{-A^2}$, we get that J commutes with A. Since $A^* = -A$, also $J^* = -J$. Hence, since $JJ^* = \mathrm{Id}$ (check), we have $J^2 = -\mathrm{Id}$, i.e., J is a complex structure. We get

$$\omega(Ju, Jv) = G(AJu, Jv) = G(JAu, Jv) = G(JJ^*Au, v) = G(Au, v) = \omega(u, v),$$

and

$$\omega(u, Ju) = G(Au, Ju) = G(J^*Au, u) = G(\sqrt{AA^*}u, u) > 0$$

for $u \neq 0$ (since $\sqrt{AA^*}$ is positive definite). Hence J is ω-compatible. □

Remarks.
1. In general, the positive inner product defined by $\omega(u, Jv)$ is different from $G(u, v)$. However, if J is given and $G(u, v) = \omega(u, Jv)$ then $J^G = J$. (In that case, $A = J^*$ and $AA^* = \mathrm{Id}$.)
2. The construction is canonical. The map $\sqrt{AA^*}$ does not depend on the choice of B nor on the ordering of the eigenvalues in the diagonal map.

Denote by $\mathcal{J}(V, \omega)$ the space of compatible complex structures. We equip it with the subset topology induced from the topology on the space of linear endomorphisms of V. Let $\mathrm{Riem}(V)$ denote the space of positive definite inner products; it is a convex open subset of the space of symmetric bilinear forms.

The previous lemma and Remarks imply the following result.

Corollary 1.1. *The map $G \mapsto J^G$ is a continuous and surjective map $F\colon \mathrm{Riem}(V) \to \mathcal{J}(V, \omega)$. Furthermore (by construction) for $H\colon \mathcal{J}(V, \omega) \to \mathrm{Riem}(V)$ associating to J the inner product $\omega(u, Jv)$ is a section, i.e., $F \circ H(J) = J$.*

Theorem 1.1. *The space $\mathcal{J}(V, \omega)$ is contractible and not empty.*

Proof. The existence of a compatible complex structure follows directly from the previous Corollary and the fact that $\mathrm{Riem}(V)$ is not empty. Since $\mathrm{Riem}(V)$ is a convex subset of a vector space, it is contractible. If $\Phi\colon I \times \mathrm{Riem}(v) \to \mathrm{Riem}(V)$ is a contraction: $\Phi_0 = \mathrm{Id}_{\mathrm{Riem}(V)}$ and Φ_1 is a map onto a point, then $F \circ \Phi \circ (\mathrm{Id} \times H)\colon I \times \mathcal{J}(M, \omega) \to \mathcal{J}(M, \omega)$ is a contraction of $\mathcal{J}(M, \omega)$ to a point. □

We deduce a second proof for the existence of a symplectic basis for (V, ω).

Proof of Lemma 1.1. Let J be an ω-compatible complex structure on V (exists by the previous theorem). Let h be the Hermitian form:

$$h(u, v) = \omega(u, Jv) + \sqrt{-1}\omega(u, v).$$

Pick an orthonormal (w.r.t h) basis of (V, J) as a complex space: $(e_1, \dots, e_n)$. Let $f_i = Je_i$ Then $(e_1, \dots, e_n, f_1, \dots, f_n)$ is a symplectic basis:

$$\omega(e_i, f_j) = \mathrm{Im}(h(e_i, Je_j)) = \mathrm{Im}(\sqrt{-1}h(e_i, e_j)) = \mathrm{Im}(\sqrt{-1}\delta_{i,j}) = \delta_{i,j},$$
$$\omega(e_i, e_j) = \mathrm{Im}(h(e_i, e_j)) = \mathrm{Im}\,\delta_{i,j} = 0,$$

and similarly $\omega(f_i, f_j) = 0$. □

Remark 1.1. The two proofs we gave to Lemma 1.1 are related. The transition from the second proof to the first one is by the following claim: If J is ω-compatible and L is a Lagrangian subspace of (V, ω) then JL is also Lagrangian and $JL = L^{\perp}$, where $\perp$ denotes orthogonallity with respect to the positive inner product $g(u, v) = \omega(u, Jv)$.

The two proofs used two important tools of symplectic geometry: Lagrangian sub-manifolds and compatible almost complex structures, which we will discuss in the next talk, when we move from symplectic vector spaces to symplectic manifolds.

References

[1] Eckhard Meinrenken, *Symplectic Geometry - Lecture Notes*, University of Toronto.
[2] Ana Cannas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics, Springer-Verlag, 2008.