SYMPLECTIC GEOMETRY: LECTURE 2

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1. Symplectic Manifolds

1.1. Basic definitions.

1.1. Recall: manifolds, immersion, submersion, embedding, tangent bundle, co-tangent bundle ([3]). In particular, we have partition of unity.

Let ω be a differential 2-form on M, i.e, an assignment of anti-symmetric bilinear 2-form ω_p on T_pM for each $p \in M$, that varies smoothly in p. We say that ω is *closed* if $d\omega = 0$ where d is the exterior derivative,

We say that ω is *symplectic* if it is closed and ω_p is symplectic (non-degenerate) for all $p \in M$.

Note: If ω is symplectic then $\dim T_pM = \dim M$ must be even.

A *symplectic manifold* is a pair (M, ω) where M is a manifold and ω a symplectic form.

Example. Let $M = \mathbb{R}^{2n}$, or an open subset of $\mathbb{R}^{2n}$, with linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic. Is ω_0 closed? YES (as follows from the linearity and the multiplicatiive law of the exterior derivative). It is even exact: $\omega_0 = d\sum_{i=1}^{n} x_i dy_i$. The ordered set

$$\left(\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p, \left(\frac{\partial}{\partial y_1}\right)_p, \ldots, \left(\frac{\partial}{\partial y_n}\right)_p\right)$$

is a symplectic basis of T_pM

1.2. We saw that the nth exterior power ω^n is a top degree non-vanishing form, i.e., a volume form. Hence (M, ω) is canonically oriented by the symplectic structure. (Does the mobius strip admit a symplectic form?)

Claim 1.1. *Assume that M is compact and with no boundary.*

- *The de Rham cohomology class $[\omega^n] \in H^{2n}(M;\mathbb{R})$ is non-zero. Proof: otherwise $\omega^n = d\Omega$ for a $2n-1$-form. Then*
$$0 = \int_{\partial M} \Omega = \int_M d\Omega = \int_M \omega^n \neq 0,$$
where the first equation is since M is compact; the second by Stokes' theorem; the third by assumption and the last inequality since ω^n is non-vanishing. (Remark: Stokes Theorem holds here since M is compact.)
- *Therefore $[\omega]$ is non zero (i.e., ω is not exact). (Proof: $[\omega]^n = [\omega^n]$.)*
- *In particular, if $n > 1$ there are no symplectic forms on S^{2n}. There is a symplectic form (the area form) on S^2 and you will study it in Problem Set 2.*

A symplectomorphism between (M_1, ω_1) and (M_2, ω_2) is a diffeomorphism $\phi\colon M_1 \to M_2$ such that $\phi^*\omega_2 = \omega_1$, i.e., . The group of symplectomorphisms of M onto itself is denoted by $\mathrm{Symp}(M, \omega)$.

Example. Let Σ be an orientable 2-manifold and ω a volume form. Then ω is non-degenerate (since $\omega^n = \omega \neq 0$ everwhere) and closed (since it is a top degree form). a symplectomorphism in this case is a volume-preserving diffeomorphism. By a result of Moser, any two volume forms on a compact manifold M, defining the same orientation and having the same total volume are related by a diffeomorphism of M. In particular, every closed symplectic 2-manifold is determined up to symplectomorphism by its genus and total volume. (You will prove this result of Moser in PS3.)

1.2. **Almost complex structures.** An almost complex structure on a (real) manifold M is an automorphism $J\colon TM \to TM$ such that $J^2 = -\mathrm{Id}$ (i.e., it is an almost complex structure on every T_pM that varies smoothly). It is *integrable* if it comes from a complex atlas on the manifold. An almost complex structure J is *compatible* with a symplectic form ω if $\omega(\cdot, J\cdot)$ is a Riemannian metric on M, i.e., J is ω-compatible on every tangent space T_pM.

We denote by $\mathcal{J}(M,\omega)$ the space of ω-compatible almost complex structures on M. Our proof from symplectic linear algebra can be carried fiberwise, to get a canonical surjective map $\mathrm{Riem}(M) \to \mathcal{J}(M,\omega)$ which is a left inverse to the map $\mathcal{J}(M,\omega) \to \mathrm{Riem}(M)$ associating to J_p the corresponding inner product $\omega(\cdot, J_p\cdot)$ on T_pM. Since the polar decomposition is canonical, the obtained J is smooth. Therefore, $\mathcal{J}(M,\omega)$ is not empty. Moreover, any $J_0, J_1 \in \mathcal{J}(M,\omega)$ can be smoothly deformed within $\mathcal{J}(M,\omega)$: use a convex combination

$$g_t := (1-t)g_0 + tg_1$$

of the corresponding Riemannian metrics, and apply the polar decomposition to (ω, g_t) to obtain a smooth family of J_ts joining J_0 to J_1.

Corollary 1.1. *The space $\mathcal{J}(M,\omega)$ is path-connected and not empty.*

Exercise.
(1) Fix a symplectic form ω on M. Let $J_0, J_1 \in \mathcal{J}(M,\omega)$ and $0 \le t \le 1$. Show that $J_t := (1-t)J_0 + tJ_1$ is not necessarily in $\mathcal{J}(M,\omega)$.
(2) Fix an almost complex structure J on M. Let ω_0 and ω_1 be symplectic forms that are compatible with J, and $0 \le t \le 1$. Show that $\omega_t = (1-t)\omega_0 + t\omega_1$ is a symplectic form that is compatible with J.

This exercise is in PS2.

If J is integrable and ω-compatible, the triple (M,ω,J) is called a Kähler manifold.

Example. For example, in an open set $U \subset \mathbb{C}^n (\cong \mathbb{R}^{2n})$ with coordinates $z_j = x_j + iy_j$. The almost complex structure induced from multiplication by i is, in the symplectic basis:

$$J_0\Big(\frac{\partial}{\partial x_j}\Big)_p = \Big(\frac{\partial}{\partial y_j}\Big)_p,$$

$$J_0\Big(\frac{\partial}{\partial y_j}\Big)_p = -\Big(\frac{\partial}{\partial x_j}\Big)_p.$$

For a complex manifold, this defines a complex structure locally, in a chart, which is well-defined globally, since the transiition maps are holomorphic (using the Cauchy-Riemann equations). [1, p.102].

1.3. **Kähler manifolds.**

Let M be a complex manifold (a manifold with an atlas consisting of open subsets of $\mathbb{C}^n$ such that the transition functions are holomorphic) with a Hermitian h metric on TM. As before, we can get a nondegenerate 2-form by assigning $\omega_p = \mathrm{Im}(h_p)$ at every $p \in M$. If this form is closed ($d\omega = 0$) we get a Kähler structure. Do we always have a Hermitian metric on a complex manifold? Yes, using partition of unity.

Claim 1.2. *If (M,ω,J) is a Kähler manifold, then there is a Hermitian metric h on the complex manifold such that $\omega = \mathrm{Im}\, h$.*

Proof: Exercise (PS2). Read 16.1.

Example. For example, $\mathbb{C}^n(\cong \mathbb{R}^{2n})$ with coordinates $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$. We have

$$d\bar{z}_j \otimes dz_j = (dx_j - idy_j) \otimes (dx_j + idy_j)$$
$$= (dx_j \otimes dx_j + dy_j \otimes dy_j) + i(dx_j \otimes dy_j - dy_j \otimes dx_j)$$
$$= (dx_j \otimes dx_j + dy_j \otimes dy_j) + idx_j \wedge dy_j.$$

Therefore the Hermitian metric

$$h(z) = \sum_{j=1}^{n} d\bar{z}_j \otimes dz_j = g_0 + i\omega_0.$$

($g_0 = \sum_{j=1}^{n}(dx_j \otimes dx_j + dy_j \otimes dy_j)$, $\omega_0 = \sum_{i=1}^{n} dx_j \wedge dy_j = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$.) So $(\mathbb{C}^n, \omega_0)$ is Kähler.

Claim 1.3. *Let (M, ω, J) be a Kähler manifold. Let (N, J_N) be a complex manifold and $\iota \colon N \to M$ a complex immersion, i.e., $J \circ d\iota = d\iota \circ J_N$. Then $(N, \iota^*\omega, J_N)$ is a Kähler manifold.*

Proof. It is enough to note that a complex subspace of a Hermitian vector space is Hermitian. Applying this to each $d\iota_n(T_nN) \subset T_{\iota(n)}M$ we see that the closed 2-form $\iota^*\omega$ ($d \circ \iota^* = \iota^* \circ d$) is non-degenerate, and $J_N \in \mathcal{J}(N, \iota^*\omega)$. □

In particular, every complex submanifold of $\mathbb{C}^n$, with the pullback of ω_0, is Kähler.

1.4. Wirtinger's inequality (proved in a previous lecture) says that for $X_1, \ldots, X_{2k}$ in $\mathbb{R}^{2n}$ that are orthonormal with respect to $g_0 = \omega_0(\cdot, J_0\cdot)$, we have

$$|\omega_0^k(X_1 \wedge \ldots \wedge X_{2n})| \leq k!,$$

with equality holding precisely when $\mathrm{span}(X_1, \ldots, X_k)$ is a complex k-dimensional subspace of $\mathbb{C}^n$.

Therefore, if N is a smooth $2k$-dimensional manifold (smoothly) immersed in $\mathbb{C}^n$, Wirtinger's inequality implies that

$$\int_N \frac{1}{k!}\omega_0^k \leq \int_N dV = \mathrm{Vol}(N)$$

with equality precisely when N is an immersed complex k-dimensional submanifold of $\mathbb{C}^n$. (The volume is with respect to the Riemannian metric g_0.)

(The proof is by partition of unity, on each chart using the Euclidean coordinate basis that we specified in the first example in the lecture.)

Note that the form $\frac{\omega_0^k}{k!}$ on $\mathbb{C}^n$ is exact (since ω_0 is), i.e., $\frac{\omega_0^k}{k!} = d\alpha$.

Corollary 1.2. *If (N_1, ∂) and (N_2, ∂) are compact $2k$-manifolds with boundary immersed in $\mathbb{C}^n$ and having the same boundary ∂, and N_1 is a complex k-manifold then*

$$\mathrm{Vol}(N_1) = \int_{N_1} \frac{\omega^k}{k!} = \int_{\partial} \alpha = \int_{N_2} \frac{\omega^k}{k!} \leq \mathrm{Vol}(N_2),$$

where the equalities in the middle are due to Stokes' theorem.

Corollary 1.3. [4, Theorem B]. *If a k-dimensional subvariety of a ball of radius R in $\mathbb{C}^n$ passes through the center of the ball then its $2k$-volume is at least the volume of the unit ball in an Euclidean $2k$-space times R^{2k}.*

This corollary is used in the proof of Gromov's non-squeezing theorem that we will discuss later in the semester.

Example. Any almost complex structure on a (real) 2-dimensional manifold is integrable. (Theorem, proof: PS2.) As a result, an orientable 2-manifold with a volume form and a compatible complex structure is Kähler.

Example. The complex projective space

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \smallsetminus \{0\})/(\mathbb{C} \smallsetminus \{0\}) = S^{2n+1}/S^1$$

is the space of complex lines in $\mathbb{C}^{n+1}$: $\mathbb{CP}^n$ is obtained from $\mathbb{C}^{n+1} \smallsetminus \{0\}$ by making the identifications $(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$ for all $\lambda \in \mathbb{C} \smallsetminus \{0\}$. It is a complex manifold. One denotes by $[z_0, \dots, z_n]$ the equivalence class of $(z_0, \dots, z_n)$, and calls $z_0, \dots, z_n$ the homogeneous coordinates of the point $p = [z_0, \dots, z_n]$.

Let

$$\begin{tikzcd} S^{2n+1} \arrow[r, "\iota"] \arrow[d, "\pi"'] & \mathbb{C}^{n+1} \\ \mathbb{CP}^n & \end{tikzcd}$$

where ι is an embedding and π a projection. At every point $z \in S^{2n+1}$, we have a canonical splitting of tangent spaces

$$T_z\mathbb{C}^{n+1} = T_{\pi(z)}\mathbb{CP}^n \oplus \mathrm{span}_{\mathbb{C}}\{z\}$$

as complex vector spaces. Since $T_{\pi(z)}\mathbb{CP}^n$ is a complex subspace, it is also symplectic. This induces a non-degenerate 2-form ω on $\mathbb{CP}^n$ which by construction is compatible with the complex structure. Letting $\bar{\omega}$ be the symplectic (closed) form in $\mathbb{C}^{n+1}$, we have $\iota^*\bar{\omega} = \pi^*\omega$. Therefore $\pi^* d\omega = d\pi^*\omega = d\iota^*\bar{\omega} = \iota^* d\bar{\omega} = 0$, showing that ω is closed. This shows that $\mathbb{CP}^n$ is a Kähler manifold. The 2-form ω is called the *Fubini-Study* form.

Therefore, by Claim 1.3, every non-singular (i.e., smooth) projective variety (i.e., zero locus of a collection of homogeneous polynomials) is a Kähler submanifold.

Remark 1.1. Every symplectic manifold admits a compatible almost complex structure but not necessarily an integrable one. (Examples: Kodaira, Thurston, McDuff and Gompf.)

1.3. **Cotangent bundles.** Let X be any n-dimensional manifold and $M = T^*X$ its cotangent bundle. Let

$$\pi\colon M = T^*X \to X,\ p = (x, \eta) \mapsto x,\ \eta \in T_x^*X$$

the bundle projection, and

$$d\pi\colon TM \to TX,\ d\pi_p\colon T_pM \to T_xX$$

its tangent map. The *tautoligical* 1-form α is defined point-wise by

$$\alpha_p(v) = \eta(d\pi_p(v))$$

for $v \in T_pM$.

Proposition 1.1. *The form α is the unique 1-form on T^*X with the property that for any 1-form β on the base X:*

$$\beta = \beta^*\alpha,$$

where on the right hand side, β is viewed as a section $\beta\colon X \to T^*X = M$.

Proof. The property holds for α: first note that $\beta(x) = (x, \beta_x)$. So for $u \in T_xX$ we have

$$(\beta^*\alpha)_x(u) = \alpha_{\beta(x)}(d_x\beta(u)) = \beta_x(d\pi_{(x,\beta_x)}(d_x\beta(u)) = \beta_x(u). \tag{1.5}$$

The first equality is by definition of a pullback; the second by definition of α; the third since $\pi \circ \beta\colon X \to X$ is Id, and by the chain rule.

The uniqueness is since for every $v \in T_pM \smallsetminus \ker d_p\pi$ $(p = (x, \eta))$ there is a 1-form β on X with $\beta(x) = p$ such that v is in the image of $d_x\beta$ hence, by (1.5), α is determined on v. Therefore the property determines α on $T_pM \smallsetminus \ker d_p\pi$ hence, since these vectors span T_pM, on T_pM. □

We will use this characterization of the tautological 1-form to further understand it. In local coordinates: if the manifold X is described by coordinate charts $(U, x_1, \dots, x_n)$ with $x_i\colon U \to \mathbb{R}$ then at any $x \in U$, the differentials $(dx_1)_x, \dots, (dx_n)_x)$ form a basis of T_x^*X. Namely, if $\eta \in T_x^*X$ then $\eta = \sum_{i=1}^n \eta_i(dx_i)_x$ for real functions $\eta_i, \dots, \eta_n$. The chart $(T^*U = \{(x, \eta) : x \in U, \eta \in T_x^*X\}, x_1, \dots, x_n, \eta_1, \dots, \eta_n)$ is a coordinate chart for T^*X: the coordinates are the *cotangent coordinates*.

Claim 1.4. *In the cotangent coordinates,*

$$\alpha = \sum_{i=1}^{n} \eta_i dx_i.$$

Proof. Because of Proposition 1.1, it is enough that for a 1-form $\beta = \sum_{i=1}^{n} \beta_i dx_i$ on X,

$$\beta^* \sum_{i=1}^{n} \eta_i dx_i = \sum_{i=1}^{n} \beta^* \eta_i d\beta^* x_i = \sum_{i=1}^{n} \beta_i dx_i = \beta.$$

□

Theorem 1.1. *Let $M = T^*X$ and α the canonical 1-form. Then $\omega = -d\alpha$ is a symplectic form on M.*

Proof. In cotangent coordinates, $\omega = \sum_j dx_j \wedge d\eta_j$. □

Given a diffeomorphism $f\colon X_1 \to X_2$. Then $df\colon TX_1 \to TX_2$ is a diffeomorphism and

$$F = (df^{-1})^*\colon T^*X_1 \to T^*X_2$$

is a diffeomorphism, called the *cotangent lift of f*. For every $\beta \in \Omega^1(X_1)$ there is a commutative diagram

$$\begin{array}{ccc} T^*X_1 & \xrightarrow{F} & T^*X_2 \\ {\scriptstyle \beta}\uparrow & & \uparrow{\scriptstyle (f^{-1})^*\beta} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

Proposition 1.2. *Let $F\colon T^*X_1 \to T^*X_2$ be the cotangent lift of $f\colon X_1 \to X_2$. Then F preserves the canonical 1-form: $F^*\alpha_2 = \alpha_1$, hence F is a symplectomomorphism: $F^*\omega_2 = \omega_1$.*

Proof. It is enough to show that for every $\beta \in \Omega^1(X_1)$ we have $\beta^*(F^*\alpha_2) = \beta$. Indded,

$$\beta^*(F^*\alpha_2) = (F \circ \beta)^*(\alpha_2) = ((f^{-1})^* \beta \circ f)^* \alpha_2 = f^*((f^{-1})^*\beta)^*\alpha_2 = f^*(f^{-1})^*\beta = \beta,$$

where in the equality before the last we used the property that for every $\gamma \in \Omega^1(X_2)$ we have $\gamma^*\alpha_2 = \alpha_2$. □

This gives a natural homomorphism

$$\mathrm{Diff}(X) \to \mathrm{Symp}(T^*X, \omega),\ f \to (df^{-1})^*.$$

Another subgroup of $\mathrm{Symp}(T^*X, \omega)$ is obtained from a homomorphism

$$Z^1(X) \to \mathrm{Symp}(T^*X, \omega).$$

In PS2 you will prove the following proposition.

Proposition 1.3. *Let $\sigma \in \Omega^2(X)$ be a closed 2-form on X. The 2-form*

$$\omega = -d\alpha + \pi^*\sigma$$

is a symplectic form on T^*M. The Liouville form of $-d\alpha + \pi^*\sigma$ equals the Liouville form of $-d\alpha$.

Corollary 1.4. *For any manifold X with a closed 2-form σ there is a symplectic manifold (M, ω) and $\iota\colon X \to M$ such that $i^*\omega = \sigma$.*

Take $M = T^*X$ and $\omega = -d\alpha + \pi^*\sigma$ and $\iota\colon X \to M$ the zero section embedding $x \mapsto (x, 0)$.

1.4. **Lagrangian Submanifolds and Symplectomorphisms.** [1, 3.1, 3.2, 3.4].

1.5. **Darboux Theorem.** By the Darboux theorem, the dimension is the only local invariant of symplectic manifolds, up to symplectomorphisms. [1, Theorem 8.1, p.7]. The proof is the goal of our next lectures.

References

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