SYMPLECTIC GEOMETRY: LECTURE 3

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1. Local forms

Vector fields and the Lie derivative. A *vector field* on a manifold M is a smooth assignment of a vector tangent to M at each point. We think of M as embedded in $\mathbb{R}^N$; a vector field then is a smooth map

$$v\colon M \to \mathbb{R}^N$$

such that $v(p) \in T_pM$ for every p.

More generally, a *time-dependent vector field* is a smooth family of vector fields v_t, $t \in \mathbb{R}$. We think of a vector field v as v_t such that $v_t = v$ for all $t \in \mathbb{R}$. Imagine that we allow every point $p \in M$ to flow along the vector field v for some time t, we thus obtain a map of M whose fixed points (if t is small enough) are exactly the zeros of v. The *flow* of v_t is the map

$$\Phi\colon M \times \mathbb{R} \to M,\ \Phi_t(p) := \Phi(p,t),$$

with $\Phi_0 = \mathrm{Id}_M$ and each $\Phi_t\colon M \to M$ is a diffeomorphism (such a map is called an *isotopy*), such that

$$\frac{d}{dt}\Phi_t = v_t \circ \Phi_t.$$

(Locally exists by Picard's theorem of the existence and uniqueness of solutions of first-order equations with given initial conditions, globally exists when M is compact or v is of compact support.)

Examples (draw): $M = \mathbb{R}$, $v = \gamma \in \mathbb{R}, \Phi_t(m) = e^{\gamma t}m$, $M = S^1$, $v = i\gamma$, $\Phi_t(m) = e^{i\gamma t}m$, $M = S^2$, $v = i\gamma, \Phi_t(m) = e^{i\gamma t}m$. Note that in the last example the flow is along the geodesic (locally minimizing length of constant velocity) curve starting from x whose tangent at x is v; this generalizes to (locally) define a flow given a Riemann metric on a manifold, using the exponential map, in case $v_t = v$ is independent of t.

The *Lie derivative* of a time dependent vector field v_t is

$$\mathcal{L}_v\colon \Omega^k(M) \to \Omega^k(M),\ \mathcal{L}_v\alpha = \frac{d}{dt}(\Phi_t)^*\alpha|_{t=0}.$$

(For the definition of the Lie derivative it is enough that the isotopy exists locally, by Picard's theorem.)

We will use the properties of the Lie derivative.

- *Cartan's magic formula*
$$\mathcal{L}_v\alpha = \iota_v d\alpha + d\iota_v\alpha,$$
- $$\frac{d}{dt}\Phi_t^*\alpha = \Phi_t^*\mathcal{L}_{v_t}\alpha.$$

For a guided exercise towards the proof that the properties hold, see p. 42 in [2]. Using the chain rule and the linearity of the pullback, we deduce a stronger version of the second property.

1.1. **Lemma.** *For a smooth family ω_t, $t \in \mathbb{R}$ of k-forms we have*
$$\frac{d}{dt}\Phi_t^*\omega_t = \Phi_t^*\Big(\mathcal{L}_{v_t}\omega_t + \frac{d}{dt}\omega_t\Big). \tag{1.2}$$

Proof: [2, Proposition 6.4, pp. 42–43].

Moser's trick.

1.3. **Lemma** (Moser's trick)**.** *Let ω_t, $0 \le t \le 1$ be a smooth family of symplectic forms on a manifold M. Assume that*

(1) *$\frac{d}{dt}\omega_t = d\alpha_t$ for a smooth family of 1-forms on M, and*
(2) *there is a vector field v_t such that $\iota_{v_t}\omega_t = -\alpha_t$, and an isotopy ϕ_t, $t \in \mathbb{R}$ such that $\frac{d}{dt}\phi_t = v_t \circ \phi_t$ and $\phi_0 = \mathrm{Id}$.*

Then $\phi_t^*\omega_t = \omega_0$.

If a Lie group G acts on M, and the action is symplectic with respect to ω_t and α_t for every t, then v_t (defined by ω_t and α_t) is G-equivariant and so is its flow, ϕ_t i.e., for every t we have $\phi_t(g.x) = g.\phi_t(x)$ for all $g \in G$.

Proof.
$$\begin{aligned}\frac{d}{dt}(\phi_t^*\omega_t) &= \phi_t^*\Big(\mathcal{L}_{v_t}\omega_t + \frac{d}{dt}\omega_t\Big)\\ &= \phi_t^*\Big(d\iota_{v_t}\omega_t + \iota_{v_t}d\omega_t + d\alpha_t\Big) = 0\end{aligned}$$

The first equality is by (1.2), the second equality is by Cartan's magic formula, and the third is by the setting of v_t and α_t and the assumption that ω_t are symplectic and in particular closed.

□

We first apply Moser's trick to get symplectomorphisms on a compact manifold M.

1.4. **Corollary.** *Assume that M is compact and that $[\omega_0] = [\omega_1]$ and the 2-form*

$$\omega_t = (1-t)\omega_0 + t\omega_1$$

is symplectic for each $t \in [0,1]$. Then there exists an isotopy $\phi\colon M \times \mathbb{R} \to M$ such that $\phi_t^* \omega_t = \omega_0$ for all $t \in [0,1]$. In particular $\phi_1^* \omega_1 = \omega_0$.

If a compact Lie group G acts on M, and the action is symplectic with respect to ω_t for every t, then ϕ_t is G-equivariant for every t.

Proof. Since $[\omega_0] = [\omega_1]$ there exists a 1-form α such that

$$\omega_1 - \omega_0 = d\alpha.$$

If a compact Lie group G acts on M, and the action is symplectic with respect to ω_0, ω_1 then for every $a \in G$ we have

$$d\tau_a^* \alpha = \tau_a^* d\alpha = \tau_a^*(\omega_0 - \omega_1) = \omega_1 - \omega_0.$$

Averaging α w.r.t the G-action, we get the 1-form

$$\int_G \alpha({\tau_a}^* u) d\tau_a,$$

for $u \in TM$, where $d\tau_a$ is given by any measure on the group G, invariant by left translations, e.g., the Haar measure. By the above equation, $d\int_G \alpha({\tau_a}^* u) d\tau_a = \omega_1 - \omega_0$, so we can assume that α is G-invariant.

We get that

$$\frac{d}{dt}\omega_t = \frac{d}{dt}(1-t)\omega_0 + t\omega_1 = \omega_1 - \omega_0 = d\alpha.$$

So it remains to find v_t such that

$$\iota_{v_t}\omega_t = \alpha.$$

By the non-degeneracy of ω_t we can solve this pointwise, to obtain a unique (smooth) v_t. The existence of the flow integrating v_t is guaranteed since M is compact.

If ω_t and α are G-invariant then so is v_t and so is the flow $\phi\colon M \times \mathbb{R} \to M$ integrating v_t.

□

1.5. Note that if ω_0 and ω_1 are symplectic forms on M then $\omega_t = (1-t)\omega_0 + t\omega_1$ are necessarily closed but not necessarily non-degenerate. However (as we showed in PS2), if ω_0 and ω_1 are compatible with a fixed almost complex structure J on M, then ω_t is non-degenerate.

1.6. If N is a compact complex manifold and h_0, h_1 Hermitian metrics on M, then for every $t \in [0,1]$,

$$h_t = (1-t)h_0 + th_1$$

is again a Hermitian structure (CHECK). Therefore

$$\omega_t = \operatorname{Im} h_t = (1-t)\omega_0 + t\omega_1$$

is symplectic for every $t \in [0,1]$.

As a result of the last observation and Corollary 1.4, we get the following theorem.

1.7. **Theorem** (Banyaga)**.** *Let N be a compact complex manifold and h_0, h_1 Hermitian metrics on N (w.r.t to the given complex structure). Assume that $[\omega_0] = [\omega_1]$. then there exists a diffeomorphism ϕ of N (as a real manifold) such that $\phi^*\omega_1 = \omega_0$.*

Darboux and Weinstein Theorems. By the Darboux theorem, the dimension is the only local invariant of symplectic manifolds, up to symplectomorphisms.

1.8. **Theorem** (Darboux Theorem)**.** *Let (M, ω) be a $2n$-dimensional symplectic manifold, and let p be any point in M. Then there is a coordinate chart $(U, x_1, \dots, x_n, y_1, \dots, y_n)$, with $p \in U$, such that on U, $\omega = \sum_{i=1}^n dx_i \wedge dy_i$.*

Such a chart is called a *Darboux chart*.

Darboux first proved the theorem using induction on dimension. The proof given here is due to Weinstein, and is valid in many infinite-dimensional cases. Another recent proof is one of the suggested topic for students' presentations [1].

The proof applies Moser's trick to get a symplectomorphism, not on the manifold as in Corollary 1.4 but on a neighbourhood of a point. We will also use Poincaré lemma.

1.9. **Lemma** (Poincaré Lemma)**.** *Let $U \subset \mathbb{R}^{2n}$ be an open ball around 0. Then for every closed form $\alpha \in \Omega^k(U)$ with $k > 0$ there is $\beta \in \Omega^{k-1}(U)$ such that $\alpha = d\beta$.*

Proof of the Darboux theorem. First, consider the exponential

$$\exp = \exp_p \colon T_pM \to M$$

of some Riemann metric on M. (The exponential map sends $v \in T_pM$ to $\gamma_v(1)$ where γ_v is the unique geodesic (locally length-minimizing curve) satisfying $\gamma_v(0) = p$ and the initial tangent vector $\gamma_v'(0) = v$. Note that exp is locally defined since Picard's theorem of the existence and uniqueness of solutions of first-order equations with given initial conditions holds locally.) The map exp is a diffeomorphism between a neighbourhood V_0 of 0 in the vector space T_pM and a neighbourhood U_p of p in the manifold M. Wlg, V_0 is an open ball.

We get two symplectic forms on V_0: $\omega_0 = \omega_p$ and $\omega_1 = \exp^* \omega$. Then ω_0 and ω_1 agree at the origin. By Poincaré lemma, there is $\beta \in \Omega^1(V_0)$ such that $\omega_1 - \omega_0 = d\beta$. Since ω_0 and ω_1 agree at 0, the form $\beta|_0 = 0$. Let

$$\omega_t = \omega_0 + t d\beta.$$

For $0 \leq t \leq 1$, the 2-form ω_t is closed and agrees with ω_0 at 0 hence non-degenerate at 0. Non-degeneracy of a 2-form σ on a $2n$-manifold means that σ^n is nowhere vanishing, hence is an open condition. Therefore, shrinking V_0 if necessary, we may assume that ω_t is non-degenerate on U for all $t \in [0,1]$. The non-degeneracy of ω_t implies, as before, that there is a unique vector field v_t such that

$$\iota_{v_t} \omega_t = \beta.$$

By shrinking V_0 we can assume that v_t integrates to a flow ϕ_t. Furthermore, since $\beta|_0 = 0$ also $v_t|_0 = 0$ hence $||v_t||$ is small near 0, there is a neighbourhood $0 \in V' \subset V_0$ such that the flow ϕ_t integrating v_t in V' does not map V' out of V_0 for $t \in [0,1]$. Since $v_t|_0 = 0$ we have $\phi_t(0) = 0$.

Now apply Moser's trick to show that $\phi_t^* \omega_t = \omega_0$, hence $\phi_1^*(\exp^* \omega) = {\phi_1}^* \omega_1 = \omega_0 = \omega_p$. The chart $\exp \circ \phi$ transforms coordinates of T_pM with respect to an ω_x-symplectic basis into local coordinates of M in U in which $\omega = \sum_{i=1}^n dx_i \wedge dy_i$.

□

1.10. **Theorem** (Moser relative theorem). *Let M be a manifold, $\iota\colon X \to M$ a compact submanifold of M, and ω_0, ω_1 symplectic forms on M. Assume that*

$$\omega_0|_X = \omega_1|_X.$$

Then there exist neighbourhoods U_0, U_1 of X in M and a diffeomorphism $\Phi\colon U_0 \to U_1$ such that $\iota = \Psi \circ \iota$ on X, and $\Phi^* \omega_1 = \omega_0$ on U_0.

If G is a compact Lie group acting on M with ω_0 and ω_1 both invariant under the G-action, and if Y is invariant under the action, then the map Φ can be chosen to commute with the action of G.

1.11. **Notation.** *If σ is a differential form on M then $\sigma|_X$ denotes the restriction of σ to $\wedge(TM)|_X$. Thus $\sigma|_X$ can be evaluated on vectors that are not necessarily tangent to X.*

The proof is similar to the proof of Darboux theorem: the identification of a neighbourhood of a point p in M with a neighbourhood of 0 in T_pM will be replaced with the tubular neighbourhood theorem, using the exponential map, and Poincaré lemma will be generalized to the homotopy formula. See [2, §6.2, 6.3].

1.12. Recall: The *normal bundle* of a k-dimensional submanifold $\iota : X \hookrightarrow M$ of an n-dimensional manifold M (with $k < n$) is

$$NX = \{(x, v) \mid x \in X,\, v \in T_xM/T_xX\},$$

where we identify x with $\iota(x)$ and consider the tangent space to X as a subspace of the tangent space to M through the linear inclusion $d\iota_x \colon T_xX \to T_xM$. The quotient $N_xX := T_xM/T_xX$ is an $n-k$-dim vector space. NX is a vector bundle over X with the projection $\pi \colon NX$, $\pi(x, v) = x$, hence a manifold of dimension n. The zero section

$$\iota_0 \colon X \to NX;\, x \mapsto (x, 0)$$

embeds X as a closed submanifold of NX.

1.13. A neighbourhood U_0 of the zero section X is called *convex* if for every $x \in X$, the intersection $U_0 \cap N_xX$ is convex. Note that on a convex neighbourhood U of X in NX we can set

$$\phi_0(x, v) = (x, 0),\ \phi_1(x, v) = (x, v),\ \phi_t(x, v) = (x, tv) \text{ for } 0 \le t \le 1$$

and get a smooth retraction $\phi \colon U_0 \times [0, 1] \to U_0$ of U_0 onto X (well defined since U_0 is convex and includes the zero section). Let v_t be the time-dependent vector field on U_0 that generate ϕ_t, i.e., $\frac{d}{dt}\phi_t = v_t \circ \phi_t$, $v_t(p)$ is the tangent vector to the curve $\phi(p, \cdot)$ at t. For a form σ on U_0 we have the *homotopy formula*

$$\sigma - \phi_0^*\sigma = \int_0^1 \frac{d}{dt}(\phi_t^*\sigma)dt = \int_0^1 \phi_t^*\mathcal{L}_{v_t}\sigma = \int_0^1 \phi_t^*(\iota_{v_t}d\sigma + d\iota_{v_t}\sigma)dt = Id\sigma + dI\sigma,$$

where

$$I\beta = \int_0^1 (\phi_t^*(\iota(v_t)\beta)dt \tag{1.14}$$

and

$$\phi_t^*\iota(v_t)\sigma(u_1, \dots, u_k) = (\iota_{v_t(p)}\sigma)(d\phi_t u_1, \dots, d\phi_t u_k).$$

The first equality is by the fundamental theorem of calculus and the fact that ϕ_1 is the identity map, the second equality is by the pull-back property property of the Lie derivative stated last week, the third equality is by Cartan's magic formula, and the fourth is by the fundamental theorem of calculus again. This is the *homotopy formula* in a convex neighbourhood of X in NX. In particular, if σ is closed and $\phi_0^*\sigma = 0$ then $\sigma = d(I\sigma)$. Moreover, since $\phi_t(x, 0) = (x, 0)$ for every t, we get that $v_t|_X = 0$, hence $(I\sigma)_X = 0$.

1.15. **Homotopy operator.** The operator (1.14) is a special case of a homotopy operator. In general, in the category of smooth manifolds, let f_0, $f_1\colon M_1 \to M_2$, such that there is a smooth homotopy $f\colon [0,1]\times M_1 \to M_2$ between f_0 and f_1. Then there is a chain homotopy $I^k\colon \Omega^k(M_2) \to \Omega^{k-1}(M_1)$ such that the homotopy formula

$$d\circ I + I\circ d = f_1^* - f_0^* \tag{1.16}$$

holds.

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & \Omega^{k-1}(M_2) & \xrightarrow{d} & \Omega^{k}(M_2) & \xrightarrow{d} & \Omega^{k+1}(M_2) & \xrightarrow{d} & \cdots \\
 & \swarrow^{I^{k-1}} & f_0^*\Big\downarrow\Big\downarrow f_1^* & \swarrow^{I^{k}} & f_0^*\Big\downarrow\Big\downarrow f_1^* & \swarrow^{I^{k+1}} & f_0^*\Big\downarrow\Big\downarrow f_1^* & & \\
\cdots & \xrightarrow[d]{} & \Omega^{k-1}(M_1) & \xrightarrow[d]{} & \Omega^{k}(M_1) & \xrightarrow[d]{} & \Omega^{k+1}(M_1) & \xrightarrow[d]{} & \cdots
\end{array}$$

The operator I is defined by

$$I(\sigma) = \int_{[0,1]} f^*\sigma.$$

The homotopy formula (1.16) follows from the fundamental theorem of calculus, as seen in the special case (1.14). If σ is closed, we deduce that $f_1^*\sigma - f_0^*\sigma = d(I\sigma)$. We conclude that if there is a homotopy between f_0 and f_1, then f_0^* and f_1^* induce the same map on cohomology.

In the spacial case that $M_1 = \{0\}$ and $M_2 = U \subset \mathbb{R}^n$ is an open ball around 0 in $\mathbb{R}^n$, let $\iota\colon \{0\} \to U$ be the inclusion, and $\pi\colon U \to \{0\}$ be the projection. We have a homotopy $f_t\colon U \to U$, $u \mapsto tu$, $0 \le t \le 1$ such that $f_0 = \iota\circ\pi$ and $f_1 = \mathrm{Id}$, i.e., a retraction of U onto $\{0\}$. Using the homotopy formula, we get that $\pi^*\circ\iota^* = (\iota\circ\pi)^*$ and Id^* induce the same map on cohomology. Therefore $\iota^*\colon H^k(U) \to H^k(\{0\})$ is an isomorphism with inverse π^*. Hence, for $k > 0$, every closed k-form on U is exact. This proves Poincaré lemma 1.9 that we used in the proof of Darboux theorem.

1.17. To get a convex neighbourhood of X in NX and a diffeomorphism from U_0 to a neighbourhood U, we choose a Riemann metric g on M and use the exponential map. Note that the vector space N_xX is identified with the orthogonal complement $\{v \in T_xM \mid g_x(v,w) = 0 \text{ for any } w \in T_xX\}$. – DRAW. Let $NX^\epsilon = \{(x,v) \in NX \mid \sqrt{g_x(v,v)} < \epsilon\}$. This is a convex neighbourhood of X in NX (CHECK, use Cauchy-Schwarz inequality). Consider the exponential map $\exp\colon NX^\epsilon \to M$ that sends (x,v) to $\gamma(1)$ where $\gamma\colon [0,1] \to M$ is the geodesic (locally minimizing length of constant velocity) curve starting from x whose tangent $\frac{d\gamma}{dt}(0) = v$ at 0 is v. By Picard's uniqueness and existence theorem, if X is a compact submanifold of M, then for ϵ small enough, the map exp is well defined. Then exp maps NX^ϵ diffeomorphically to a neighbourhood U^ϵ of X in M, and is the identity on the zero section X. (If X is not compact, replace ϵ by a continuous map $X \to \mathbb{R}^+$ that tends to zero fast enough as x tends to infinity.) This is the *tubular neighbourhood theorem*, [2, Theorem 6.5]. It allows us to apply the homotopy formula in a tubular neighbourhood of X.

Proof of Moser's relative theorem. On a tubular neighbourhood U_0 of X, the 2-form $\omega_1 - \omega_0$ is closed and its restriction to X is zero. By the homotopy formula, there is a 1-form α on U_0 such that $\omega_1 - \omega_0 = d\alpha$ and $\alpha|_X = 0$. Now, as in the proof of Darboux theorem set $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + td\alpha$. By shrinking U_0 we can assume that ω_t is symplectic $\forall 0 \leq t \leq 1$, so we can solve the Moser equation $\iota_{v_t}\omega_t = -\alpha$ with $v_t = 0$ on X, and integrate v_t to a flow ρ_t. By Moser's trick, ρ_t is an isotopy between ω_0 and ω_1, i.e., $\rho_t^*\omega_t = \omega_0$ for every $0 \leq t \leq 1$. Now let $U_1 = \rho_1(U_0)$ and $\Phi = \rho_1$.

□

1.18*.* If a compact Lie group acts on U and X is invariant under the G-action, we can choose an invariant Riemann metric on U_0. Such a metric is obtained from some Riemmann metric g' by averaging with respect to the compact group G action to get

$$g(u,v) = \int_G g'({\tau_a}^*u, {\tau_a}^*v)d\tau_a,$$

for $u, v \in T_xM$, where $d\tau_a$ is given by any measure on the group G, invariant by left translations, e.g., the Haar measure. Then the exponential map and hence the retraction ϕ_t will commute with the action of G, and hence so will I, i.e., $I{\tau_a}^*\beta = {\tau_a}^*I\beta$ for every $a \in G$. This gives the equivariant tubular neighbourhood and homotopy formula, and therefore an equivariant version of Moser's relative theorem.

1.19. *Remark.* Note that Darboux theorem has two parts:

(1) Locally there is a change of coordinates so that the transformed symplectic form is constant.
(2) There is a further change of coordinates yielding the "canonical" symplectic form $\sum dx_i \wedge dy_i$. (This is the existence of a symplectic basis result in symplectic linear algebra we showed in the first lectures.)

The equivariant version of the first part, the "locally constant" result still holds, moreover, the equivariant version of Moser relative theorem holds, as noted before. However the second part is not correct in the equivariant setting. This is related to the representation of the group.

For more details see [3], which is one of the suggested topics for a student presentation. There are groups with non-isomorphic invariant symplectic forms on any neighbourhood of a fixed point.

1.20. **Theorem** (Weinstein's tubular neighbourhood theorem)**.** *Let (M, ω) be a symplectic manifold and $\iota\colon L \hookrightarrow M$ a compact Lagrangian submanifold. Consider T^*L with the canonical symplectic form ω_0 and L as the zero section in T^*L, embedded by $\iota_0\colon L \hookrightarrow T^*L$. Then there exist neigbourhoods U_0 of L in T^*L, U_1 of L in M and a diffeomorphism $\theta\colon U_0 \to U_1$ such that $\theta \circ \iota_0 = \iota$ and $\theta^*\omega = \omega_0$.*

1.21. **Proposition.** *Let L be a Lagrangian sumanifold of a symplectic manifold (M, ω) Then the vector bundles NL and T^*L are canonically identified.*

Proof. For $p \in L$ define

$$\omega_p{}^\flat\colon T_pM/T_pL \to (T_pL)^*,\ \omega_p{}^\flat([v])(\cdot) = \omega_p(v, \cdot),$$

where $[v]$ is the equivalence class of v in T_pM/T_pL. Note that for $u_1, u \in T_pL$, $\omega_p(v + u_1, u) = \omega_p(v, u)$ since T_pL is a Lagrangian, in particular isotropic, subspace of (T_pM, ω_p). Hence $\omega_p{}^\flat([v])$ is well defined. Moreover $\omega_p{}^\flat$ is an isomorphism, since T_pL is Lagrangian, i.e., T_pL coincides with its symplectic orthogonal complement $(T_pL)^{\omega_p}$. □

Proof of Weinstein's tubular neighbourhood theorem. By Proposition 1.21, $T^*L \cong NL$, so we can identify the canonical form ω_0 with a symplectic form (to be denoted again ω_0) on NL.

By the tubular neighbourhood theorem in §1.17, there is a neighbourhood V of L in M, a convex neighbourhood U of L in $NL = TM|_L/TL$, and a diffeomorphism

$$\exp\colon U \to V,$$

determined by a Riemannian metric on M, such that $\exp \circ \iota_0 = \iota$. Since L is Lagrangian both in T^*L (as the zero section) and in M, we have

$$\exp^* \omega|_{TL} = \omega_0|_{TL} = 0. \tag{1.22}$$

However, to apply Moser's relative theorem we need $\exp^* \omega|_L = \omega_0|_L$ as 2-forms evaluated on vectors in $TU = T(T^*L)$ that are not necessarily tangent to L. To get this, we can choose the Riemannian metric on M wisely: take it to be $g(u, v) = \omega(u, Jv)$ where J is an ω-compatible almost complex structure on M. Now, for $p \in L$, the space N_pL is identified with the orthogonal complement of T_pL w.r.t g, which equals JT_pL (CHECK), hence $T_pV = T_pL \oplus JT_pL$. We get, by choosing an orthonormal basis w.r.t the Hermitian form $h(u, v) = \omega_p(u, Jv) +$

$\sqrt{-1}\omega_p(u,v)$ and identifying it with the cotangent coordinates, that $\exp^* \omega|_p = \omega_0|_p$ for every $p \in L$.

Alternatively, we can use (1.22) to find a linear isomorphism $L_p \colon T_pU \to T_pU$ such that $L_p|_{T_pL} = \mathrm{Id}$ and ${L_p}^*(\exp^* \omega|_p) = \omega_0|_p$, that varies smoothly in $p \in L$. Then, by Whitney's extension theorem, there is an embedding $h \colon U' \to U$, of some neighbourhood U' of L, such that $h|_L = \mathrm{Id}$ and $dh_p = L_p$ for every $p \in L$, hence

$$(h^* \exp^* \omega)_p = (dh_p)^*(\exp^* \omega|_p) = {L_p}^*(\exp^* \omega|_p) = \omega_0|_p.$$

For more details, see [2, Proof of Theorem 8.4].

Now apply Moser's relative theorem to find a neighbourhood U_0 of L in T^*L and a diffeomorphism $\psi \colon U_0 \to U_0 \subset T^*L$ such that $\psi^* \exp^* \omega = \omega_0$ and $\psi \circ \iota_0 = \iota_0$. The diffeomorphism $\theta = \exp \circ \psi \colon U_0 \to \exp(U_0) \subset M$ satisfies the required properties $\theta^* \omega = \omega_0$ and $\theta \circ \iota_0 = \iota$.

$\square$

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