SYMPLECTIC GEOMETRY: LECTURE 4

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1. Group Actions on Symplectic Manifolds: basic definitions

A *Lie group* G is a manifold equipped with a group structure, where the group operations: $G \times G \to G$ sending $(a, b) \mapsto ab$, and $G \to G$, sending $a \mapsto a^{-1}$ are smooth maps.

Examples:

- $\mathbb{R}$ (with addition).
- S^1 regarded as unit complex numbers with multiplication, represents rotations of the plane: $S^1 = \mathrm{U}(1) = \mathrm{SO}(2)$.
- $T^n = \mathbb{R}^n/\mathbb{Z}^n$, the compact torus.
- $(\mathbb{C}^*)^n$, the complex torus.
- $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{GL}(n, \mathbb{C})$.
- $\mathrm{U}(n)$, unitary linear transformations of $\mathbb{C}^n$.
- $\mathrm{SU}(n)$, unitary linear transformations of $\mathbb{C}^n$ with $\det = 1$.
- $O(n)$, orthogonal linear transformations of $\mathbb{R}^n$; $\mathrm{SO}(n)$.
- $\mathrm{GL}(V)$, invertible linear transformations of a vector space V.

An action of a Lie group G on a manifold M is a group homomorphism

$$\tau \colon G \to \mathrm{Diff}(M)$$
$$g \mapsto \tau_g.$$

We will write g instead of τ_g, and $g \cdot m$ instead of $\tau_g(m)$. An action of G on M is smooth if the associated map

$$G \times M \to M, \quad (g, m) \mapsto g \cdot m$$

is smooth.

Example. If M is a linear vector space and τ_a are linear transformations, this is a *linear representation* of G.

We will usually, but not always, consider compact Lie groups. Sometimes the compactness requirement can be replaced by the following condition.

An action of G on M is *proper* if the map

$$G \times M \to M \times M, \quad (a, m) \mapsto (a \cdot m, m),$$

is proper. i.e., the preimage of any compact set is compact. Recall that a proper map between locally compact Hausdorff spaces (as are manifolds) is closed (exercise). If G is compact, the action is proper. If G is not compact, but M is compact, the action is not proper.

If the group G acts on two manifolds M and N, and if $\phi\colon M \to N$ is a smooth map, it is said that ϕ is an *equivariant map* if

$$\forall m \in M, \quad \forall g \in G, \quad \phi(g \cdot m) = g \cdot \phi(m).$$

Clearly, if an equivariant map ϕ is a diffeomorphism, the inverse map ϕ^{-1} is also equivariant.

2. Orbits and stabilizers

Let a Lie group G act on a manifold M. The orbit of a point $m \in M$ is

$$G \cdot m = \{g \cdot m | g \in G\} \subseteq M,$$

and its stabilizer G_m,

$$G_m = \{g \in G | g \cdot m = m\} \subseteq G.$$

Being a closed subgroup of the Lie group G, the stabilizer G_m is a Lie group. See [1, Theorem 2.27]. If the action is proper the stabilizer G_m is compact. Notice that stabilizers of points in the same orbit are conjugate to each other, as $G_{g \cdot m} = gG_m g^{-1}$ (this equality also shows that all possible conjugates appear). Also, if $\phi\colon M \to N$ is an equivariant map, $G_m \subseteq G_{\phi(m)}$.

The action is said to be *free* if $G_m = \{e\}$ for all $m \in M$.

The action is said to be *effective* if the homomorphism $G \to \mathrm{Diff}(M)$ that defines the action is one-to-one, or, equivalently, if $\cap_{m \in M} G_m = \{e\}$. A G-action descends to an effective G/K action, where $K = \ker \tau$ is the kernel of the action.

The evaluation map $ev_m\colon g \to g \cdot m$ induces a G-equivariant bijection from the quotient G/G_m to the orbit $G \cdot m$, where G acts on G/G_m by left multiplication: $a \in G$ sends the coset $gG_m \in G/G_m$ to the coset agG_m.

The manifold M decomposes into a disjoint union of orbits. The quotient M/G is the set of orbits with the quotient topology: the topology such that $\pi\colon M \to M/G$ is continuous, i.e., the preimage of an open set is open.

Example. The Lie group $\mathbb{R}$ acts on the manifold $\mathbb{R}$ by $t.m \mapsto me^t$. There are three orbits: $\mathbb{R}^+$, $\mathbb{R}^-$ and $\{0\}$. The point $\{0\}$ in the three-point orbit space is not open, so the orbit space with the quotient topology is not Hausdorff.

Example. Let S^1 act on the unit sphere $S^2 \subset \mathbb{R}^3$ by rotations around the $z - axis$:

$$\theta \mapsto \text{ rotation in angle } \theta \text{ around the } z\text{-axis.}$$

Then the quotient S^2/S^1 is the closed interval $[-1, 1]$.

Example. The circle S^1 acts on $\mathbb{C}^n$ by complex multiplication

$$t \cdot (z_1, \ldots, z_n) = (tz_1, \ldots, tz_n).$$

The point 0 is fixed; all the other orbits are circles.

Exercise. The circle S^1 also acts on $S^3 \subset \mathbb{C}^2$ by

$$t \cdot (z_1, z_2) = (t^{m_1} z_1, t^{m_2} z_2),$$

where $m_1, m_2 \in \mathbb{Z}$. Show that this action is effective if and only if m_1 and m_2 are relatively prime.

Exercise. The complex torus $(\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by coordinate-wise multiplication

$$(\lambda_1, \ldots, \lambda_n)(z_1, \ldots, z_n) = (\lambda_1 z_1, \ldots, \lambda_n z_n).$$

Is the action proper? What are the orbits? What are the stabilizers? Describe the quotient $\mathbb{C}^n/(\mathbb{C}^*)^n$. Describe the orbits, the stabilizers, and the quotient for the action of the compact torus $(S^1)^n \subseteq (\mathbb{C}^*)^n$ that is given by the same formula.

Exercise. Fix a real number α and let $\mathbb{R}$ act on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ by

$$t \cdot (x, y) = (x + t, y + \alpha t).$$

Find the orbits and show that these are submanifolds of T^2 if and only if α is rational. (Hint: if α is irrational, all the orbits are dense in T^2).

If the action is proper, every orbit is a closed subset of M, and the orbit space M/G is Hausdorff. (Not true in general, e.g., the irrational flow on the torus T^2.) For a proof, see [3, Proposition B.8].

3. The exponential map

Let G be a Lie group with unit e. Denote by $\mathfrak{g}$ its tangent vector space $T_e G$ at e. Alternatively, consider the left multiplication

$$L_a \colon G \to G, \;\; g \mapsto ag.$$

A vector field $X \colon G \to TG$ is *left-invariant* if $L_{a*} X_h = X_{ah}$ for every $h \in G$, where L_{a*} denotes the derivative of L_g. Any left-invariant vector field is determined by its value at the identity element. Thus evaluation at e identifies the vector space of left-invariant vector fields on G with T_eG.

The *exponential* map

$$\exp\colon \mathfrak{g} \to G$$

is characterized by the properties:

(1) for each $\xi \in \mathfrak{g}$, the map $t \mapsto \exp(t\xi)$ is a group homomorphism from $(\mathbb{R}, +)$ to G, and
(2) $\frac{d}{dt}|_{t=0} \exp(t\xi) = \xi \in T_eG$.

Explicitly, it is the flow associated to the vector field ξ, starting from e, at $t = 1$. As before, the exponential map is a diffeomorphism from a neighbourhood of $0 \in \mathfrak{g}$ onto a neighbourhood of $e \in G$.

Example. If G is a group of matrices, then $\exp(A) = I + A + \frac{1}{2}A^2 + \frac{1}{3}A^3 + \dots$

Example. If G is the 1-torus $\mathbb{R}/\mathbb{Z}$, then $\mathfrak{g} = \mathbb{R}$, and $\exp\colon \mathbb{R} \to S^1$ is the exponential mapping $t \mapsto e^{2\pi i t}$. In this case exp is a group homomrphism with respect to addition on $\mathfrak{g}$. Usually exp is not a group homomorphism.

We can consider this torus as being the unit circle

$$S^1 = \{u \in \mathbb{C} | |u| = 1\},$$

and the exponential mapping $\theta \mapsto e^{i\theta}$. They are not exactly the same since the former has period 1 and the latter has period 2π. To solve this inconsistency, in the sequel we will see the 1-torus as $\mathbb{R}/2\pi\mathbb{Z}$.

In general, for $G = T \cong (S^1)^k$, the Lie algebra $\mathfrak{t} = T_eT$ is identified with $\mathbb{R}^k$, and the exponential mapping

$$\exp\colon \mathfrak{t} \to T$$

becomes $\mathbb{R}^k \to \mathbb{R}^k/2\pi\mathbb{Z}^k$. It can be identified with the covering map $\mathbb{R}^n \to T^n$.

4. The Lie algebra and the adjoint and co-adjoint representations

We define a Lie bracket $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is anti-symmetric $[X, Y] = -[Y, X]$, and satisfies the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

as follows. The group acts on itself by conjugation: for all $g \in G$, $G \to G$:

$$h \mapsto ghg^{-1}.$$

The derivative at e of this map is an invertible linear map $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$. We get the *Adjoint action*: for $g \in G$, $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$:

$$X \mapsto \mathrm{Ad}_g(X),$$

which we may consider as a map $G \to \mathrm{GL}(\mathfrak{g})$:

$$g \to \mathrm{Ad}_g .$$

Exercise. Show that $\mathrm{Ad}_g \circ \mathrm{Ad}_h = \mathrm{Ad}_{g \circ h}$.

Define

$$\mathrm{ad}_X(Y) := \frac{\partial}{\partial t}|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y).$$

We get $\mathfrak{g} \to \mathrm{GL}(\mathfrak{g})$:

$$X \to \mathrm{ad}_X .$$

The Lie bracket is defined by $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$:

$$(X, Y) \mapsto \mathrm{ad}_X(Y) =: [X, Y]. \tag{4.1}$$

This is consistent with the definition

$$[X, Y] = \mathcal{L}_X(Y) = \frac{\partial}{\partial t}(\phi_t)_*(Y)$$

when we consider X and Y as left-invariant vector fields (and $\mathcal{L}$ is the Lie derivative, as before).

Exercise. Show that the if X and Y are left-invariant vector field, then $[X, Y]$ is a left-invariant vector field.

Exercise. Show that the brackets $[\cdot, \cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined in (4.1) are indeed Lie brackets: anti-symmetric and satisfies the Jacobi identity.

Example. For $G < \mathrm{GL}(n; \mathbb{R})$ (for some n) we have

$$\mathrm{Ad}_g(Y) = gYg^{-1}, \ g \in G, \, Y \in \mathfrak{g}$$

and

$$[X, Y] = XY - YX, \ X, Y \in \mathfrak{g}.$$

Example. For $G = T^n$, the Lie algebra is $\mathbb{R}^n$, and the Lie bracket is trivial $[X, Y] = 0$, reflecting the fact that the group is Abelian and thus the Adjoint action trivial.

4.1. Given $\eta \in \mathfrak{g}^*$, we set

$$\mathrm{Ad}_g^* \eta(X) := \eta(\mathrm{Ad}_{g^{-1}} X), \quad \text{for any} X \in \mathfrak{g}.$$

The collection of maps Ad_g^* forms the *coadjoint representation*, or *coadjoint action* of G on $\mathfrak{g}^*$:

$$\mathrm{Ad}_g^* \colon G \to \mathrm{GL}(\mathfrak{g}^*), \ \ g \mapsto \mathrm{Ad}_g^* .$$

We take g^{-1} in the deffinition in order to get a group homomorphism and not a group anti-homomorphism.

Exercise. Show that $\mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h = \mathrm{Ad}^*_{gh}$.

5. Symplectic and Hamiltonian actions

Definition 5.1. Consider a G-action on a manifold M. For $\eta \in \mathfrak{g}$, the *generating vector field* η_M is the unique vector field with flow

$$m \mapsto \exp(t\eta).m,$$

i.e.,

$$\eta_M|_m = \frac{d}{dt}|_{t=0} \exp(t\eta).m$$

Definition 5.2. A G action $g \mapsto \tau_g$ on a symplectic manifold (M, ω) is called *symplectic* if $\tau_g \in \mathrm{Symp}(M, \omega)$ for all g. A $\mathfrak{g}$-action $\eta \to \eta_M$ is called *symplectic* if η_M is a symplectic vector field, i.e., $\mathcal{L}_{\eta_M}\omega = 0$.

Recall that $\mathcal{L}_v \alpha = \frac{d}{dt}(\rho_t)^* \alpha|_{t=0}$, where ρ_t is the flow associated with the vector field v. In particular $\mathcal{L}_{\eta_M}\omega = \frac{d}{dt}(m \mapsto \exp(t\eta).m)^*\omega|_{t=0}$. Therefore the $\mathfrak{g}$-action $\eta \to \eta_M$ defined by a symplectic G-action is symplectic.

Definition 5.3. A symplectic G-action on a symplectic manifold (M, ω) is called *weakly Hamiltonian* if there exists a G-invariant map, called the *moment map*

$$\Phi \colon M \to \mathfrak{g}^*$$

whose coordinates

$$\Phi^\eta := \langle \Phi, \eta \rangle \colon M \to \mathbb{R}, \ \ \eta \in \mathfrak{g}$$

satisfy Hamilton's equation

$$d\Phi^\eta = -\iota(\eta_M)\omega.$$

A map Φ on M is *G-invariant* if it is constant on the orbits, i.e., $\Phi(g.m) = \Phi(m)$ for every $g \in G$.

A weakly Hamiltonian G-action is called *Hamiltonian* if Φ is equivariant with respect to the G-action on M and the co-adjoint action of G on $\mathfrak{g}^*$.

Remark 5.1. Note that if the group is Abelian, the co-adjoint action is trivial, hence a moment map that is G-invariant is automatically equivariant.

Remark 5.2. The notion of a Hamiltonian torus action comes from classical mechanics. Let X be the space of possible positions of a physical system, viewed as a manifold. Noether's principal states that to every symmetry of a physical system X there corresponds a conserved quantity. This conserved quantity is a real-valued function H on the phase space T^*X called the Hamiltonian. Recall that there is a canonical symplectic form ω_0 on the cotangent bundle T^*X. Since ω_0 is non-degenerate, we get a vector field X_H such that $dH = -\iota(X_H)\omega_0$. If the vector field is complete, this provides a flow on the manifold. When this flow is periodic, it gives rise to a (Hamiltonian) circle action on the symplectic manifold T^*X.

Example. Let (M, ω) be the unit sphere $S^2 \subset \mathbb{R}^3$ with the symplectic form $d\theta \wedge dh$, where (θ, h) are the cylindrical polar coordinates on S^2 (see PS2). Let S^1 act on the unit sphere $S^2 \subset \mathbb{R}^3$ by rotations about its vertical axis:

$$\theta \mapsto \text{ rotation in angle } \theta \text{ around the } h\text{-axis.}$$

The generating vector field is $\partial/\partial\theta$. The associated moment map is the height function h.

Remark 5.3. Note that, by Cartan's magic formula, $d(\iota_{\eta_M}\omega) = \mathcal{L}_{\eta_M}\omega - \iota_{\eta_M} d\omega$. Hence for a symplectic action of G on (M, ω) the 1-form $\iota_{\eta_M}\omega$ is closed. Furthermore, when $H^1(M; \mathbb{R}) = 0$ (in particular when the manifold is simply connected), for any symplectic group action on M there is a moment map that satisfies Hamolton's equation. This is not true if $H^1(M; \mathbb{R}) \neq 0$. For example, let M be the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$, with coordinates (x, y) accordingly, and a symplectic form $dx \wedge dy$. The 1-torus $\mathbb{R}/\mathbb{Z}$ acts on M by translations of the first coordinate; the generating vector field is $\partial/\partial x$. Then, although in a neighbourhood of $(x_0, y_0) \in M$, the map $y +$constant satisfies Hamilton's equation, there is no global function $f : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}$ with $df = dy$. On the other hand, if a compact connected symplectic manifold M admits a Hamiltonian action whose fixed points are isolated, then M is simply connected. This follows from Morse theory, as we will see next week.

References

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