SYMPLECTIC GEOMETRY: LECTURE 5

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Let (M, ω) be a connected compact symplectic manifold, T a torus, $T \times M \to M$ a Hamiltonian action of T on M, and $\Phi \colon M \to \mathfrak{t}^*$ the assoaciated moment map.

Theorem 0.1 (The convexity theorem, Atiyah, Guillemin and Sternberg, 1982)**.** *The image of Φ is a convex polytope.*

In this talk we will prove the convexity theorem. The proof given here is of Guillemin-Sternberg and is taken from [3]. We will then discuss the case $\dim T = \frac{1}{2} \dim M$.

Remark 0.1. We saw in Lecture 4 that for an Abelian group-action the equivariance condition amounts to the moment map being invariant; in case the group is a torus, the latter condition follows from Hamilton's equation: $d\Phi^{\xi} = -\iota_{\xi_M}\omega$ for all ξ in the Lie algebra, see e.g., [4, Proposition 2.9].

1. The local convexity theorem

In the convexity theorem we consider torus actions on compact symplectic manifolds. However to understand moment maps locally, we will look at Hamiltonian torus actions on symplectic vector spaces.

Example. $T^n = (S^1)^n$ acts on $\mathbb{C}^n$ by

$$(a_1, \dots, a_n) \cdot (z_1, \dots, z_n) = (a_1 z_1, \dots, a_n z_n).$$

The symplectic form is

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j = \sum_{j=1}^{n} r_j dr_j \wedge d\theta_j$$

where $z_j = x_j + i y_j = r_j e^{i\theta_j}$. With a standard basis of the Lie algebra of T^n, the action is generated by the vector fields $\frac{\partial}{\partial \theta_j}$ for $j = 1, \dots, n$ and the moment map is a map $\Phi = (\Phi_1, \dots, \Phi_n) \colon \mathbb{C}^n \to \mathbb{R}^n$. Hamilton's equation becomes $d\Phi_j = -\iota(\frac{\partial}{\partial \theta_j})\omega$. Because the right hand side is $r_j dr_j$ which is $d{r_j}^2/2$,

$$\Phi(z_1, \dots, z_n) = \Big(\frac{|z_1|^2}{2}, \dots, \frac{|z_n|^2}{2} \Big) + \text{constant}. \tag{1.1}$$

If the constant is zero, the moment map image is the positive orthant of $\mathbb{R}^2$, defined by $\{(x_1,\dots,x_n)\,|\,x_1\geq 0,\dots,x_n\geq 0\}$.

Draw when $n=1$. This is the basic building block for arbitrary Hamiltonian torus actions.

Example. An inclusion

$$\mathrm{inc}\colon S^1\hookrightarrow T=(S^1)^2;\ s\mapsto(s^m,s^n)$$

induces a projection on the duals of the Lie algebras $\mathbb{R}^2\to\mathbb{R}$ defined by $(m,n)\in\mathbb{Z}^2$, explicitly $\mathbb{R}^2\ni(x_1,x_2)\mapsto mx_1+nx_2$. Composing this projection on the moment map of the torus action yields the moment map of the circle action: $\frac{m|z_1|^2}{2}+\frac{n|z_2|^2}{2}$.

1.1. Let $T=(S^1)^r$ act symplectically and linearly on a symplectic vector space (V,ω) with $\dim V=2n$. As we saw before, since T is compact, there is a T-invariant positive definite scalar product on V, which together with the symplectic form, determine an invariant complex structure J on V and a T-invariant Hermitian structure whose imaginary part is the given symplectic form. In other words, T is a sub-torus of a maximal torus in $U(n)$. We can then find an orthonormal basis consisting of simultaneous eigen-vectors (weight vectors), corresponding to simultaneous (real) eigen-values (weights) $\alpha_1,\dots,\alpha_n$ of the elements of $\mathfrak{t}$. Here the α_is are linear functions on $\mathfrak{t}$. We get a symplectic linear isomorphism of V with $\mathbb{C}^n$ such that the linear symplectic action of $\mathfrak{t}$ on V is given by the homomorphism $\rho\colon\mathfrak{t}\to\mathfrak{t}_n\cong\mathbb{R}^n$:

$$\rho(\eta)=(\alpha_1(\eta),\dots,\alpha_n(\eta)).$$

The moment map of the T-action on V is the composition of the standard moment map (1.1) with the adjoint $\rho^*\colon\mathbb{R}^n\to\mathfrak{t}^*$, and so is given by

$$\Phi(z_1,\dots,z_n)=\alpha_1\frac{|z_1|^2}{2}+\cdots+\alpha_n\frac{|z_n|^2}{2}+\text{constant}. \tag{1.2}$$

The moment map image is the convex region in $\mathfrak{t}^*$

$$S(\alpha_1,\dots,\alpha_n)=\{\sum_{i=1}^n s_i\alpha_i\,|\,s_1,\dots,s_n\geq 0\};$$

the α_is (as vectors) are called the *weights of representations* of T on V.

1.2. The local behavior of a moment map is understood by the equivariant version of the Darboux theorem. Let $T={S^1}^n$ act on (M,ω) with moment map

$$\Phi\colon M\to\mathfrak{t}^*.$$

Let x_0 be a fixed point of the action of T.

Equip M with a T-invariant Riemannian metric and let

$$\exp\colon T_{x_0}M \to M$$

be the exponential map defined by this metric. This map intertwines the action of T on M and a linear action of T on $T_{x_0}M$ by the maps $\{d_{x_0}\tau_a\}_{a\in G}$, and it maps a T-invariant neighbourhood of zero in $T_{x_0}M$ diffeomorphically onto a neighbourhood of x_0 in M. Applying the equivariant version of Darboux theorem to $0 \in T_{x_0}M$ with the linearized T-action, and the forms $\omega_1 = \omega|_{x_0}$ and $\omega_0 = \exp^*\omega$, gives a T-invariant neighbourhood V_0 of 0 and a T-equivariant map $\Psi\colon (V_0, 0) \to (T_{x_0}M, 0)$ such that $\Psi^*\omega_1 = \omega_0$. The linear action of T on $(T_{x_0}M, \omega_1)$ defines, up to translation, a moment map $\Phi_0\colon T_{x_0}M \to \mathfrak{t}^*$ as in §1.1. The maps $\Phi \circ \exp$ and $\Phi_0 \circ \Psi$ are moment maps on (U_0, ω_0) defined by the linear action of T. Therefore, they differ by a translation.

As a result of §1.2 and §1.1 we get the *local convexity theorem*:

Theorem 1.1. *Let $T = {S^1}^n$ act on (M, ω) with moment map $\Phi\colon M \to \mathfrak{t}^*$, and let x_0 be a fixed point of the action of T let $p = \Phi(x_0)$.*

Then there exists a neighbourhood U of x_0 and a neighbourhood U' of p in $\mathfrak{t}^*$ such that

$$\Phi(U) = U' \cap (p + \{\sum_{i=1}^n s_i\alpha_i\ ,\ \ s_i \geq 0\}),$$

where $\alpha_i, \dots, \alpha_n$ are the weights associated with the linear representation of T on the tangent space to M at x_0.

When the acting torus is of half the dimension of the manifold, the weights are independent (as vectors in $\mathbb{R}^n$), so the T-fixed point is isolated.

We will also need the following generalization: Theorem [3, Theorem 32.3].

Theorem 1.2. *Consider a Hamiltonian action of a torus T on a compact symplectic manifold X with moment map Φ. Let T_1 be the stabilizer group of $x \in M$.*

The inclusion of T_1 in T dualizes to give a linear mapping, $\pi\colon \mathfrak{t}^* \to {t_1}^*$. Let $\alpha_1, \dots, \alpha_n$ be the weights of the representation of T_1 on the tangent space of $x \in M$. Let $S(\alpha_1, \dots, \alpha_n)$ be the convex region

$$S(\alpha_1, \dots, \alpha_n) = \{\sum_{i=1}^n s_i\alpha_i\ ,\ \ s_i \geq 0\}$$

in $\mathfrak{t}_1{}^*$*, and let*

$$S'(\alpha_1, \ldots, \alpha_n) = \pi^{-1} S(\alpha_1, \ldots, \alpha_n)$$

in $\mathfrak{t}^*$*.*

Then there exists a neighbourhood U *of* x *in* M *and a neighbourhood* U' *of* $p = \Phi(x)$ *in* $\mathfrak{t}^*$ *such that*

$$\Phi(U) = U' \cap (p + S'(\alpha_1, \ldots, \alpha_n)). \tag{1.3}$$

To prove the convexity theorem, we will need two global assertions for a moment map $\Phi \colon M \to \mathfrak{t}^*$ associated with a Hamiltonian torus action on a connected compact symplectic manifold (M, ω).

Theorem 1.3. *For every* $\eta \in \mathfrak{t}$*, the* η*th component* $\Phi^\eta = \langle \Phi, \eta \rangle$ *has a unique local maximum.*

Theorem 1.4. *The image* $\Phi(M)$ *of the moment map is a finite union of convex sets.*

To prove the first theorem we will use Morse-Bott theory. To prove the second theorem we will look closer at stabilizers and orbits.

2. The components of the moment map are Morse-Bott functions

For the required definitions and results from Morse-Bott theory see Kimoi Kemboi's lecture on Morse-Bott theory.

The arguments used in the proof of the local convexity theorem also show the following lemma.

Lemma 2.1. *Consider a Hamiltonian action of a torus* T *on a connected compact symplectic manifold* (M, ω)*, with moment map* $\Phi \colon M \to \mathfrak{t}^*$*. Then for each* $\xi \in \mathfrak{t}$*, the component* $\Phi^\xi = \langle \Phi, \xi \rangle$ *is a Morse-Bott function, and the indices and dimensions of its critical submanifolds are all even. When the* T*-fixed points are isolated, then for almost all* $\xi \in \mathfrak{t}$ *the function* ϕ^ξ *is a Morse function.*

Proof. For $0 \neq \xi \in \mathfrak{t}$, consider the flow $\rho_t \colon M \to M$ defined by $m \mapsto \exp(t\xi).m$ and its generating vector field ξ_M. By Hamilton's equation $d\Phi^\xi = -\iota(\xi_M)\omega$ (and the fact that ω is non-degenerate), the critical set

$$\begin{aligned} C_{\Phi^\xi} : &= \{m \in M \,|\, d\Phi^\xi(m) = 0\} \\ &= \{m \in M \,|\, \iota_{\xi_M}(m) = 0\} \\ &= \{m \in M \,|\, \xi_M(m) = 0\} \\ &= \{m \in M \,|\, a.m = m \;\forall a \in T_\xi\} =: M^{T_\xi}, \end{aligned}$$

where T_ξ is the closure of $\{\exp(t\xi), t \in \mathbb{R}\}$ (the image of the group homomorphism $t \mapsto \exp(t\xi)\colon \mathbb{R} \to T$ with $\frac{d}{dt}|_{t=0} \exp(t\xi) = \xi$). Let $p \in C_{\Phi^\xi}$. Set $V = T_pM$ and $L\colon V \to V$ be the infinitesimal generator of the linearized flow $(d\rho_t)_p\colon V \to V$. Set W to be the subspace of V on which $L = 0$.

Let M be equipped with a T-invariant Riemannian metric and let

$$\exp\colon V \to M$$

be the exponential map. This map intertwines the flow ρ_t on M and the linearized flow $(d\rho_t)_p$ on V. It maps a neighbourhood of the origin in W diffeomorphically onto a neighbourhood of p in M^{T_ξ}. Therefore the connected components of the critical set of Φ^ξ $(=M^{T_\xi})$ are manifolds. It is easy to check that the Hessian $d^2(\Phi^\xi)_p$ at p is the quadratic form

$$v \mapsto -\omega_p(Lv, v).$$

Therefore the Hessian is non-degenerate on V/W, so Φ^ξ is Morse-Bott.

The restriction of the T-action to the subgroup T_ξ fixes C_f. (Recall that as a closed subgroup of a Lie group, T_ξ is a Lie group, it is compact, connected and Abelian hence a torus.) Moreover, it is Hamiltonian with moment map

$$\pi \circ \Phi\colon M \to \mathfrak{t}_\xi^*,$$

where $\mathfrak{t}_\xi$ is the Lie algebra of T_ξ and π is the map $\mathfrak{t}^* \to \mathfrak{t}_\xi^*$ dual to the inclusion of the torus $T_\xi \to T$. By the local convexity theorem, near p,

$$\pi \circ \Phi(x, z) = \frac{1}{2} \sum_{i=1}^{k} |z_i|^2 \alpha_i,$$

so

$$\Phi^\xi(x, z) = \frac{1}{2} \sum_{i=1}^{k} |z_i|^2 \langle \alpha_i, \xi \rangle.$$

(Here x are the coordinates on the connected component of C_f containing p, and z are the coordinates on the complement.) Therefore the dimension of the connected component of C_f containing p is even, and the index of the Hessian (the dimension of the maximal subspace of the tangent space on which the Hessian $d^2(\Phi^\xi)$ is negative definite) is even.

□

We will need the following result from Morse-Bott theory.

Theorem 2.1. *If $f\colon M \to \mathbb{R}$ is Morse-Bott and for all the critical manifolds C_i the dimension and index of C_i are even, then f has a unique local maximum.*

We deduce Theorem 1.3, using also Lemma 2.1.

Corollary 2.1. *If a compact connected symplectic M admits a Hamiltonian action whose fixed points are isolated, then M is simply connected.*

See Theorem 11 and Corollary 12 in the talk on Morse-Bott theory. On the other hand, when $H^1(M;\mathbb{R}) = 0$, (in particular, when the manifold M is simply connected), for any action of a torus $T \cong (S^1)^r$ on M there is a moment map that satisfies Hamilton's equation. In Lecture 4, Remark 5.3, we saw an example of an action on a manifold M with $H^1(M;\mathbb{R}) \neq 0$ that does not admit a moment map.

3. Fixed point sets

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Assume that a compact Lie group G acts on a manifold M. Let $H \subset G$ be a non-trivial subgroup. Denote its fixed point set

$$M^H = \{m \in M \mid a \cdot m = m \text{ for all } a \in H\},$$

and the set of points whose stabilizer is H

$$M_H = \{m \in M \mid G_m = H\}.$$

3.1. Notice that (as in the proof of Lemma 2.1) for a compact H (e.g., a closed subgroup of a compact G), the exponential map for an H-invariant metric intertwines the H-action on M and the linearized action, hence gives an H-equivariant diffeomorphism of a neighbourhood U of m in M with an open subset V of the vector space $W = T_m M$, that carries $U \cap M^H$ to $V \cap W^H$, where W^H is the linear subspace consisting of vectors that are fixed by H.

Corollary 3.1. *The set M^H is a disjoint union of closed connected submanifolds of M.*

Note that the fixed point set of H coincides with that of the closure of H (by continuity), so we can assume that H is closed. The corollary follows then from §3.1.

Claim 3.1. *M_H is an open subset of M^H.*

Proof. M_H is the complement in M^H of the set of ms that are fixed by a subgroup F such that H is strictly contained in H. □

Corollary 3.2. *M_H is a disjoint union of connected submanifolds of M,*

Claim 3.2. *If M is symplectic, and G is a compact Lie group that acts by symplectomorphisms, the connected components of M_H are symplectic submanifolds of M.*

Proof. For this, we observe that the tangent space to M_H at any point $m \in M_H$ consists of the space of H-fixed vectors in $T_m M$, (as follows from the fact that the exponential map above sends $U \cap M^H$ to $V \cap W^H$), and apply the following claim. $\square$

Claim 3.3. *Let (V, Ω) be a symplectic vector space and H a compact subgroup of $\mathrm{Sp}(V)$. Then the space W of H-fixed vectors is a symplectic subspace.*

Proof. First find (as in the past, using an invariant positive definite scalar product $b(\cdot, \cdot)$ on V and the polar decomposition) an H-invariant Ω-compatible almost complex structure J on V such that $\omega(u, Jv) = b(u, v)$. Deduce $JW \subset W$. Therefore $\Omega|_W$ cannot be degenerate: suppose that $0 \neq u \in W$ satisfies $\omega(u, v) = 0$ for all $v \in W$, then, since $JW \subset W$, we have that $b(u, v) = \omega(u, Jv) = 0$ for all $v \in W$ in particular $b(u, u) = 0$ which implies that $u = 0$ since b is positive definite. $\square$

Now, assume that the action of the compact Lie group G on (M, ω) is Hamiltonian with moment map $\Phi \colon M \to \mathfrak{g}^*$.

3.2. By definition of a moment map, for $\xi \in \mathfrak{g}$

$$\langle d\Phi_m(v), \xi \rangle = \omega_m(v, \xi_M(m)) \text{ for all } v \in TmM. \tag{3.1}$$

It follows that

$$(\mathrm{im}\, d\Phi_m)^0 = \{\xi \in \mathfrak{g} \text{ s.t. } \xi_M(m) = 0\} = \text{ the Lie algebra of } G_m, \tag{3.2}$$

where $(\mathrm{im}\, d\Phi_m)^0$ denotes the space of vectors in $\mathfrak{g}$ that vanish when evaluated on any element of $\mathrm{im}\, d\Phi_m$.

In particular, $d\Phi_m$ is surjective if and only if the stabilizer group of m is discrete.

3.3. Let H be a closed subgroup of G. By Claim 3.2, a connected component of M_H is a symplectic submanifold. Let $m \in M_H$. By (3.2), $d\Phi(T_m M) = \mathfrak{h}^0 (= \{\alpha \in \mathfrak{g}^* \mid \langle \alpha, \xi \rangle = 0 \, \forall \xi \in \mathfrak{h}\})$, so Φ maps each connected component of M_H into an affine subspace of the form $l + \mathfrak{h}^0$.

If H is a normal subgroup of G, i.e., $gHg^{-1} = H$ for all $g \in G$, then M_H is G-invariant and the restriction of Φ to M_H is the moment map for the G-action on M_H. Applying (3.2) to the restriction of the moment map to each component shows that it is a submersion onto an open subset of the affine space. If $H = G$, then M_G is the set of fixed points, and Φ maps each component of M_G to a point. If M is compact, there will be finitely many components. The corresponding points in $\mathfrak{g}^*$ are called the vertices of Φ.

4. How many conjugacy classes of subgroups of G can arise as stabilizers?

The following Proposition is of Mostow, 1957.

Proposition 4.1. *Let the compact Lie group G act smoothly on the manifold X. Then each point $x \in X$ has a neighbourhood U, so that, up to conjugacy, only a finite number of subgroups of G arise as stabilizers of points of U.*

In particular, if X is compact, only a finite number of conjugacy classes of subgroups of G can arise as conjugacy classes of stabilizers of points of X.

The proof if by induction on $\dim X$, and applies the slice theorem. See [3, Proposition 27.4].

5. The image of the moment map is a finite union of convex sets

Consider a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω), with moment map $\Phi \colon M \to \mathfrak{g}^*$.

Assume that M is compact and $G = T$ is a torus. By Proposition 4.1 there is a finite list of connected components $M_1, \dots, M_N$ and subgroups $T_1, \dots, T_N$ such that

$$M = \sqcup_{i=1}^{N} M_i, \tag{5.1}$$

and

$$M_i \text{ is a component of } M_{T_i}.$$

The T_i-s are labeled with possible repetition. We denote the codimension of T_i in T by m_i.

Theorem 5.1 (Theorem 27.3 [3]). *For every $1 \le i \le N$, the image $\Phi(M_i)$ is the union of a finite number of open convex sets.*

Sketch of proof. By §3.3, each M_i is a T-invariant symplectic submanifold of M, and for each i there is a vector $l_i \in \mathfrak{t}^*$ such that $\Phi(M_i)$ is an open subset of the m_i-dimensional affine subspace $l_i + \mathfrak{t}_i{}^0$, and $\Phi|_{M_i}$ is a submersion onto its image.

- In particular, if $m_i = 0$ then M_i and $\Phi(M_i)$ are points.
- A point of the boundary of $\Phi(M_i)$ is the image of a point of M where $d\Phi$ has lower rank than on M_i. Thus the boundary of $\Phi(M_i)$ is a union of sets of the form $\Phi(M_j)$, where T_j strictly contains T_i hence $m_j < m_i$. Now apply induction.

□

Theorem 1.4 follows from Theorem 5.1 and (5.1).

6. Proof of the convexity theorem

Since M is connected, so is $\Phi(M)$. We saw in the previous section that $\Phi(M)$ is a finite union of convex sets. In particular, it has finitely many boundary components. We show now that $\Phi(M)$ is an intersection of finitely many half spaces (i.e., a convex polytope) near each of its boundary points, and conclude that $\Phi(M)$ is a convex polytope.

Let p be a point in the boundary of $\Phi(M)$, and $m \in \Phi^{-1}(p)$; let T_1 be the stabilizer group of m. By the generalized local convexity theorem, there exist neighbourhoods U of m and U' of p such that $\Phi(U) = U' \cap (p + S'(\alpha_1, \dots, \alpha_n))$, where $\alpha_i \in \mathfrak{t}^*$ are the weights of the representation of T_1 on $T_m M$. Since $\Phi(M)$ has finitely many boundary components, it is enough to show that

$$\Phi(M) \subseteq p + S'(\alpha_1, \dots, \alpha_n). \tag{6.1}$$

To see that, let S_i be a boundary component of $S'(\alpha_1, \dots, \alpha_n)$. Choose $\xi \in \mathfrak{t}$ such that $e_\xi = 0$ on S_i and e_ξ is negative on the interior of $S'(\alpha_1, \dots, \alpha_n)$, where $e_\xi \colon \mathfrak{t}^* \to \mathbb{R}$ is the evaluation $f \mapsto f(\xi)$. (Choosing ξ is choosing 'directions' that are outward normal to S_i; $\mathfrak{t}^{**} = \mathfrak{t}$) Then, if $e_\xi(p) = a$,

$$\Phi^\xi(x) = (e_\xi \circ \Phi)(x) \leq a$$

for $x \in U$, so a is a local maximum of Φ^ξ. Recall that Φ^ξ has a unique local maximum (as a Morse-Bott function with even indices and dimensions of critical manifolds). So $\Phi^\xi(M) \leq a$. Hence, applying the above argument to all faces S_i of S', we get (6.1).

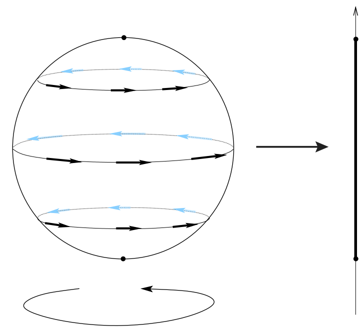
Remark 6.1. The proof of the convexity theorem also yields that $\Phi(M)$ is the convex hull of the T-fixed points: the *vertices*. This will be used to determine the moment map image in examples.

Example. S^1 acts on $(S^2, dh \wedge d\theta)$ by rotations about the h-axis. The generating vector field is $\frac{\partial}{\partial\theta}$. The map $\Phi \colon S^2 \to \mathfrak{t}^* \sim \mathbb{R}$ satisfying

$$d\Phi(v) = -dh \wedge d\theta(\frac{\partial}{\partial\theta}, v) = dh(v)$$

is $\Phi = h +$ constant. Its image is an interval. The vertices are the images of the fixed points: the poles.

Example. The standard action of $T = (S^1)^2$ on $\mathbb{CP}^2 = \mathbb{C}^2 - \{0\}/\mathbb{C} - \{0\}$ is $(a, b) \cdot [z_0 : z_1 : z_2] = [z_0 : az_1 : bz_2]$. It preserves the Kähler form that is the imaginary part of the Hermitian inner product $\langle z, w \rangle =$



The S^1 action on S^2 by rotations.

$z_0\bar{w}_0+z_1\bar{w}_1+z_2\bar{w}_2$. Each of the one parameter subgroups $(e^{i\theta},0)$, $(0,e^{i\theta})$ is globally Hamiltonian, and in homogeneous coordinates, its moment map is $\frac{1}{2}|z_r|^2/|z|^2$. So the action of T is globally Hamiltonian with moment map

$$\Phi([z_0:z_1:z_2]) = \frac{1}{2}(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}).$$

The fixed points are $[1:0:0],[0:1:0],[0:0:1]$, their images are $(0,0),(\frac{1}{2},0),(0,\frac{1}{2})$, and the moment map image is their convex hull, which is the triangle

$$\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\geq 0, x_2\geq 0, x_1+x_2\leq \frac{1}{2}\}. \tag{6.2}$$

7. The Delzant theorem

If $\dim T = \frac{1}{2}\dim M$, the triple (M,ω,Φ) is a *symplectic toric manifold*, and the T-action is called *toric*. By *Delzant's uniqueness theorem*, (M,ω,Φ) is determined by Δ up to an equivariant symplectomorphism [2].

A necessary and sufficient condition for Δ to occur as the moment map image of a symplectic toric manifold is that it be a *Delzant polytope*, meaning that the edges emanating from each vertex are generated by vectors $v_1,\ldots,v_n$ that span the lattice $\mathbb{Z}^n$.

Example. Let $\Delta=[0,a]$ for $a>0$, i.e.,

$$\Delta=\{x\in\mathbb{R} \mid \langle x,-1\rangle\leq 0,\ \langle x,1\rangle\leq a\},$$

where -1 and 1 are the outward-pointing normal vectors to the two facets of Δ. Let

$$\pi\colon \mathbb{R}^2\to\mathbb{R}$$

be the projection sending $(1,0) \mapsto -1$ and $(0,1) \mapsto 1$. The map π maps $\mathbb{Z}^2$ onto $\mathbb{Z}$, thus it induces a surjective map

$$\pi\colon \mathbb{R}^2/Z^2 \to \mathbb{R}/\mathbb{Z}.$$

The kernel of π is the diagonal Lie-subgroup

$$D = \{(e^{it}, e^{it})\}$$

of $T = \mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$. Denote by i the inclusion of D in T; it induces a map $i\colon \mathfrak{d} \to \mathbb{R}^2$ of the Lie algebras with dual $i^*\colon \mathbb{R}^2 \to \mathfrak{d}^*$:

$$i^*(x_1, x_2) = x_1 + x_2.$$

We have

$$0 \to D \xrightarrow{i} \mathbb{R}^2/\mathbb{Z}^2 \xrightarrow{\pi} \mathbb{Z}/\mathbb{R} \to 0$$

$$0 \to \mathfrak{d} \xrightarrow{i} \mathbb{R}^2 \xrightarrow{\pi} \mathbb{R} \to 0$$

$$0 \to \mathbb{R}^* \xrightarrow{\pi^*} (\mathbb{R}^2)^* \xrightarrow{i^*} \mathfrak{d}^* \to 0.$$

Equip $\mathbb{C}^2$ with the standard symplectic form $\omega_0 = \frac{i}{2}\sum_{j=1}^2 dz_j \wedge d\bar{z}_j$ and the standard $(S^1)^2$ Hamiltonian action, as in Example 1, with moment map

$$\Phi(z_1, z_2) = \frac{1}{2}(|z_1|^2, |z_2|^2) - \frac{1}{2}(0, a).$$

Then the restriction of the action to D is

$$(e^{it}, e^{it}).(z_1, z_2) = (e^{it}z_1, e^{it}z_2),$$

it is Hamiltonian with moment map

$$i^* \circ \Phi;\ (i^* \circ \Phi)(z_1, z_2) = \frac{1}{2}(|z_1|^2 + |z_2|^2) - \frac{1}{2}a.$$

Consider the zero-level set

$$Z := (i^* \circ \Phi)^{-1}(0) = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = a\} \overset{j}{\hookrightarrow} \mathbb{C}^2,$$

and its projection $p\colon Z \to Z/D$ to the orbit space. The orbit space $M_\Delta := Z/D$ coincides with $S^3/S^1 = \mathbb{CP}^1$; as in the (second) Example (after Corollary 1.3) in Lecture 2, it admits a symplectic form ω_Δ, a multiple of the Fubini-Study form, such that $j^*\omega = p^*\omega_\Delta$. (This is a special case of a Marsden-Weinstein-Meyer symplectic reduction, see e.g., [1, Chapter 23].) The toric S^1-action is through an isomorphism of S^1 with $(S^1 \times S^1)/D$. By Delzant's uniqueness theorem it is the action by rotations on $S^2 \sim \mathbb{CP}^1$ described in Example 6.

Moreover, since every Delzant polytope in $\mathbb{R}$ is a translation of an interval $[0, a]$ for $a \in \mathbb{R}$, Delzant's theorem implies that a compact connected symplectic toric manifold is equivariantly symplectomorphic to $\mathbb{CP}^1$ of symplectic area $2\pi a$ with the S^1-action by rotation.

References

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