1. INTRODUCTION AND SUMMARY.

The earliest example of percolation was discussed in Broadbent (1954) and Broadbent and Hammersley (1957) as a model for the spread of fluid or gas through a random medium. The fluid, say, spreads through channels; fluid will move through a channel if and only if the channel is wide enough. There is therefore no randomness in the motion of the fluid itself, such as in a diffusion process, but only in the medium, i.e., in the system of channels. Broadbent and Hammersley modeled this as follows. The channels are the edges or bonds between adjacent sites on the integer lattice in the plane, $\mathbb{Z}^2$. Each bond is passable (blocked) with probability $p(q = 1 - p)$, and all bonds are independent of each other. Let P_p denote the corresponding probability measure for the total configuration of all the bonds. One is now interested in probabilistic properties of the configuration of passable bonds, and, especially in the dependence on the basic parameter p of these properties. Broadbent and Hammersley began with the question whether fluid from outside a large region, say outside $|x| < N$, can reach the origin. This is of course equivalent to asking for the probability of a passable path[1] from the origin to $|x| \geq N$. For $v \in \mathbb{Z}^2$, let $W(v)$ be the union of all edges which belong to a passable path starting at v. This is the set of all points which can be reached by fluid from v. It is called the open component or cluster of v. $W(v)$ is empty iff the four edges incident on v are blocked. If we write W for $W(0)$, then the above question asks for the behavior for large N of

$$(1.1) \qquad P_p\{W \cap \{|x| \geq N\} \neq \emptyset\} .$$

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[1] This is a path made up of passable edges between neighbors of $\mathbb{Z}^2$. Two successive edges of the path must have a vertex of $\mathbb{Z}^2$ in common. Precise definitions are given in later chapters.

The limit of (1.1) as $N \to \infty$ equals

$$(1.2) \qquad \theta(p) := P_p\{\# W = \infty\} ,$$

where $\#W$ denotes the number of edges in W. It is immediate that $\theta(0) = 0$, $\theta(1) = 1$ and that $p \to \theta(p)$ is non-decreasing. Therefore the so called critical probability

$$(1.3) \qquad p_H := \sup\{p : \theta(p) = 0\}$$

is such that $\theta(p) = 0$ for $p < p_H$, and $\theta(p) > 0$ for $p > p_H$. Broadbent and Hammersley (1957), Hammersley (1957), and Hammersley (1959) made the remarkable discovery that p_H lies strictly between 0 and 1, and in fact

$$(1.4) \qquad \frac{1}{\lambda} \le p_H \le 1 - \frac{1}{\lambda} ,$$

where λ is the so-called connectivity constant of $\mathbb{Z}^2$. The exact value of λ is unknown, but one trivially has $\lambda \le 3$. Thus, there are two regions for the parameter p with drastically different behavior of the system. For $p < p_H$ no infinite clusters are formed. For $p > p_H$ there is a positive probability of an infinite cluster. In fact, Harris (1960) proved that with probability one there is a unique infinite cluster, when $p > p_H$ (see also Fisher (1961)). The existence of a threshold value such as p_H was well known in many models of statistical mechanics, in particular in the Ising model for magnetism. Moreover the proof of (1.4) involved a Peierls argument - i.e., a counting of certain paths and contours - quite familiar to students of critical phenomena. As a consequence much of the work on percolation theory has and is being done by people in statistical mechanics, in the hope that the percolation model is simple enough to allow explicit computation of many quantities which are hard to deal with in various other models for critical phenomena. Despite the considerable activity in the field, as witnessed by the recent survey articles of Stauffer (1979), Essam (1980), Hammersley and Welsh (1980), Wierman (1982a), few mathematically rigorous results have been obtained. In this monograph we want to present those results which can be proved rigorously. As our title indicates we expect that the stress on rigour will appeal more to the mathematician than the physicist. Even mathematicians will surely become impatient with the unpleasant details of many a proof; it often happened to the author while writing the proofs. What is worse though, there are quite a few phenomena about which we cannot (yet?) say much, if anything, rigorously. This monograph will therefore not go as far as physicists would like.

We stressed above the interest for statistical mechanics of percolation theory, because that seems the most important area of application for percolation theory. There are, however, other interpretations and applications of percolation theory such as the spread of disease in an orchard; the reader is referred to Frisch and Hammersley (1963) for a list of some of these.

In the remaining part of this introduction we want to summarize the results and questions which are treated here. Much of the early work dealt with the determination of the critical probability p_H defined above. Harris (1960) improved the lower bound in (1.4) to $p_H \geq \frac{1}{2}$. Sykes and Essam (1964) gave an ingenious, but incomplete, argument that p_H should equal $\frac{1}{2}$. No progress on this problem seems to have been made between 1964 and the two independent articles Seymour and Welsh (1978) and Russo (1978). These articles introduced two further critical probabilities:

(1.5) $$p_T = \sup \{p : E_p\{\#W\} < \infty \}$$

and [1)]

(1.6) $$p_S = \sup \{p : \lim \ \tau_0((n,n),1,p) = 0\} \quad ,$$

where

(1.7) $$\tau_0((n,n),1,p) = P_p \{ \exists \text{ passable path in } [o,n] \times [o,3n] \text{ connecting the left and right edge of this rectangle}\} .$$

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1) p_S appears explicitly only in Seymour and Welsh (1978) and is defined slightly differently from (1.6). However, it turns out that under certain symmetry conditions the two definitions lead to the same p_S. For our treatment (1.6) is the more useful definition.

p_T separates the p-values where the expected size of the cluster of the origin is finite and infinite, respectively. p_S concerns the possibility of crossing large rectangles. For $p < p_S$ the probability of a passable horizontal crosssing of a rectangle of size $n \times 3n$ tends to zero, while such crossings occur with a probability bounded away from zero - at least along a subsequence of n's - if $p > p_S$. Seymour and Welsh and Russo proved various relations between p_H, p_T and p_S which finally enabled Kesten (1980a) to prove $p_H = p_T = p_S = \frac{1}{2}$. This will be a special case of the main theorem in Ch. 3 .

The problem discussed so far is called the bond-percolation problem for $\mathbb{Z}^2$. To describe the contents of Theorems 3.1 and 3.2 somewhat more, we observe first that one can easily replace $\mathbb{Z}^2$ by any infinite graph $\mathcal{G}$. One obtains bond-percolation on $\mathcal{G}$, by choosing the edges of $\mathcal{G}$ independently of each other as passable or blocked. In this case the clusters W(v) can be defined as above. Another variant of the model is site-percolation on $\mathcal{G}$. One now divides the sites, rather than the bonds into two classes. Usually these are named "occupied" and "vacant". Again the classifications of the sites are random and independent of each other. The occupied cluster of v, W(v), will now be defined as the union of all edges and vertices, which can be reached from v by a path on $\mathcal{G}$ which passes only through occupied sites. In both bond- and site-percolation one often allows different probabilities for different bonds to be passable, or sites to be occupied. Normally, one only allows a finite number of parameters; the graph and the pattern of probabilities assigned to the bonds or edges are taken periodic. For instance, Sykes and Essam (1964) considered the bond-problem on $\mathbb{Z}^2$ in which each horizontal (vertical) bond is passable with probability p_{hor} ($p_{vert.}$). In all cases the first question is when percolation occurs, i.e., for which values of the parameters do infinite clusters arise? In only very few cases has one been able to determine this "percolative region" in the parameter space explicitly. These were all derived heuristically already by Sykes and Essam (1964). Theorem 3.1 contains a rigourous confirmation of most of the Sykes and Essam work. This includes the triangular site problem (with one parameter; the critical probability equals $\frac{1}{2}$), the two parameter bond-problem on $\mathbb{Z}^2$ mentioned above (percolation occurs iff $p_{hor} + p_{vert} > 1$) and a two parameter bond-problem on the triangular and hexagonal lattice. (Wierman (1981) gave the first rigorous treatment of the one-parameter case; Sykes and Essam allow three different probabilities for different bonds, but our method requires too much symmetry to be applicable to the original three parameter problem).

Theorems 3.1 and 3.2 only apply to so called matching pairs of graphs $\mathcal{G}$ and $\mathcal{G}^*$ in the plane with one of the coordinate-axes as symmetry axis and certain relations between $\mathcal{G}$ and $\mathcal{G}^*$. Moreover these theorems only yield explicit answers in the few examples mentioned above. In other cases the best one can obtain from the theorems is some generalized form of the result

$$(1.8) \qquad p_H = p_T = p_S \quad ,$$

and even this often requires extra work. We generalize Russo (1981) in demonstrating (1.8) for a two-parameter site-problem on $\mathbb{Z}^2$, and a one-parameter site-problem on the diced lattice.

The emphasis in the recent physics literature has shifted to power laws or scaling laws. In one-parameter problems, many quantities show some kind of singular behavior in their dependence on the basic parameter p, as p approaches p_H. Many people believe, and numerical evidence supports them, that the singular behavior will be like that of powers of $|p-p_H|$ (see Stauffer (1979), Essam (1980)). More specifically, one expects that (see p.422 for E_p)

$$(1.9) \qquad \theta(p) \sim C(p-p_H)^{\beta} \qquad , p \downarrow p_H ,$$

$$(1.10) \qquad E_p\{\#W; \#W < \infty\} \sim C_{\pm} |p-p_H|^{-\gamma\pm} , p \to p_H \quad ,$$

for some positive constants $C, C_{\pm}, \beta, \gamma\pm$, where the plus (minus) in (1.10) refers to the approach of $p-p_H$ to zero from the positive (negative) side. The meaning of (1.9) or (1.10) is still somewhat vague. It may mean that the ratio of the left and right hand side tends to one, but its meaning may be as weak as convergence of the ratio of the logarithms of the left and right hand side to one. In addition there is the belief that the exponents β and γ are universal, that is, that their values depend only on the dimension of the graph $\mathcal{G}$, but are (practically) the same for a large class of graphs. The numerical evidence presently available does not seem to rule this out (see Essam (1980), Appendix 1). We still seem to be far removed from proving any power law. The best results known to us are presented in Ch. 8.

There we prove power estimates of the form

$$(1.11) \qquad C_1(p-p_H)^{\beta_1} \le \theta(p) \le C_2(p-p_H)^{\beta_2} \quad , \quad p \ge p_H,$$

$$(1.12) \qquad C_3 \, |p-p_H|^{-\gamma_1} \le E_p\{\#W; \#W < \infty\} \le C_4 \, |p-p_H|^{-\gamma_2}$$

for some positive C_i, β_i and γ_i , for bond- or site-percolation on $\mathbb{Z}^2$.

Another function which is expected to have a power law is the second derivative of $\Delta(p)$, where $\Delta(p)$ is the average number of clusters per site under P_p (see Ch. 9 for a precise definition). The arguments of Sykes and Essam (1964) for determining p_H referred to above were based on this function $\Delta(p)$. On the basis of analogy with statistical mechanics they assumed that $p \to \Delta(p)$ has only one singularity and that it is located at $p = p_H$. In Sect. 9.3 we show that for bond- and site-percolation $\Delta(.)$ is twice continuously differentiable on $[0,1]$, including at p_H and that it is analytic for $p \neq p_H$. However, we have been unable to show that $\Delta(.)$ has a singularity at p_H.

The values for β_1 and β_2 , or γ_1 and γ_2, in (1.11) and (1.12) obtained from our proof are still very far apart. The difficulty lies in part in finding good estimates for

$$(1.13) \qquad P_p\{n \le \#W < \infty\}$$

for large n and $p < p_H$, but close to p_H. This problem is treated (in a multiparameter setting) in Ch. 5, where it is shown that for $p < p_T$ (1.13) decreases exponentially in n, i.e., that (1.13) is bounded by

$$(1.14) \qquad C_1(p) \exp - C_2(p)\, n \quad .$$

This estimate works for all graphs, but unfortunately only up to p_T . Only for those graphs for which we know that $p_T = p_H$ can this estimate be used for deriving power laws, and even then, it leads to poor estimates of β_i and γ_i , because the estimate for $C_2(p)$ in Ch. 5 is a very rough one.

Some results on the behavior of (1.13) for $p > p_H$ and the continuity of $\theta(.)$ are in Sect. 5.2 and 5.3.

In Ch. 10 we prove that

$$(1.15) \qquad p_H(\mathcal{H}) > p_H(\mathcal{G})$$

for certain graphs $\mathcal{G}$, $\mathcal{H}$, with $\mathcal{H}$ a subgraph of $\mathcal{G}$. Here $p_H(\mathcal{G})$ is the critical probability p_H for site-percolation on $\mathcal{G}$.

A different class of problems is treated in Ch. 11, which deals with random electrical networks. For simplicity we restrict discussion here to a bond-problem on $\mathbb{Z}^d$. Assume each bond between two vertices of $\mathbb{Z}^d$ is a resistor of 1 ohm with probability p, and removed from the graph with probability $q = 1 - p$ (equivalently we can make it an insulator with infinite resistance with probability q). Let B_n be the cube

$$B_n = \{x = (x(1),\dots,x(d)): 0 \le x(i) \le n,\ 1 \le i \le d\}$$

and

$$A_n^0 = \{x : x \in B_n,\ x(1) = 0\} \quad \text{and}$$

$$A_n^1 = \{x: x \in B_n, x(1) = n\}$$

its left and right face respectively. Finally, let R_n be the electrical resistance of the random network in B_n, between A_n^0 and A_n^1. We are interested in the behavior of $n^{d-2}R_n$ as $n \to \infty$ for various p. (see Ch. 11 for the motivation of the power of n). This leads to the introduction of a further critical probability p_R. Various definitions are possible; we expect that they all lead to the same value of p_R. Here we only mention

$$(1.16) \qquad p_R = \inf\{p : P_p\{\limsup n^{d-2}R_n < \infty\} = 1\} \quad .$$

It is immediate from the definitions that $R_n = \infty$ infinitely often a.s. $[P_p]$ when $p < p_S$ (and in fact for all d we show that $R_n = \infty$ eventually a.s. $[P_p]$ when $p < p_S$). We also show $p_R \le \frac{1}{2}$ so that

$$(1.17) \qquad p_S \le p_R \le \frac{1}{2} \quad \text{for all} \quad d \ge 2 \ .$$

For $d = 2$ we show that actually

$$p_H = p_T = p_S = p_R = \frac{1}{2} , \tag{1.18}$$

so that there still is only one critical probability. The last equality in (1.18) is obtained from the existence of a constant $C > 0$ such that for $p > \frac{1}{2}$ one has

(1.19) P_p {for all large n there exist Cn disjoint passable paths in $[0,n] \times [0,3n]$ connecting the left and right edge of this rectangle} = 1.

When we compare (1.19) with the definition (1.6) of p_S we see that for bond-percolation on $\mathbb{Z}^2$, not only is $\frac{1}{2}$ the separation point between the p-values for which there does not or does exist a single passable crossing of a large rectangle, but once p gets above this separation point, there are necessarily very many disjoint passable crossings.

We conclude this monograph with a list of some unsolved problems in Ch. 12.

For the expert we briefly point out which parts of this monograph have not appeared in print before; we also list some of the important topics which have been omitted. New are the determination in Ch. 3 of the percolative region in some multiparameter percolation problems, an improved lower bound for the cluster size distribution in the percolative region in Sect. 5.2, the strict inequalities in Sect. 10.2 (even though a special case of this has been proven recently by Higuchi (1982)) and the treatment of random electric networks in Ch. 11. As for restrictions and omissions, we are dealing only with Bernoulli percolation on the sites or bonds of an undirected graph. Thus we do not discuss mixed bond- site problems (about which little is known so far), but also omit "directed percolation problems" for which Durrett and Griffeath (1983) recently have obtained many new results, and we also omit any discussion of models in which the bonds or sites are not independent. Thus there is no mention of the Ising model, even though Kasteleyn and Fortuin (1969) proved an exact relation between the Ising model and percolation. As far as we are aware this relationship has enhanced people's intuitive understanding of both models, but it has not helped in proving new results about either model. Another area that is essentially untouched is percolation in dimension ≥ 3. Except in Ch. 5 the methods are strictly two-dimensional. Recently Aizenman (1982) and Aizenman and Fröhlich (1982) have dealt with random surfaces. These may be the proper dual for bond percolation on $\mathbb{Z}^3$ and may lead to a treatment of percolation problems in dimension three.

We have also left out the central limit theorems for various percolation-theoretical functions of Brånvalī (1980), Cox and Grimmett (1981), Newman (1980) and Newman and Wright (1981).

Finally, we mention a new and highly original proof of Russo (1982) for the equality $p_H = p_S$, which is known to imply $p_H = \frac{1}{2}$ for bond-percolation on $\mathbb{Z}^2$. Russo's proof uses less geometry than ours and may be useful for problems in higher dimensions. We stuck with the more geometric proof of Theorems 3.1 and 3.2 because so far it is the only method of proof which we know how to jack up to obtain the power estimates (1.11) and (1.12). Also the geometric method of proof seems to be the only suitable one for obtaining the strict inequalities of Sect. 10.2.