Introduction to the Julia and Mandelbrot Sets

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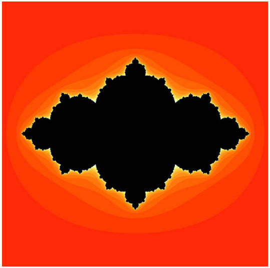
Julia set

Definition

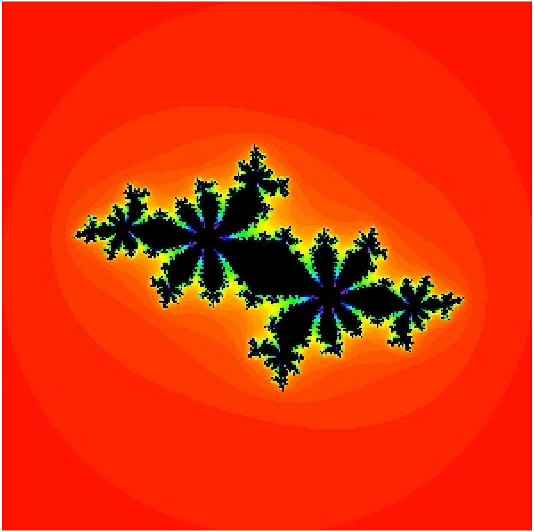
Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. The **filled-in Julia Set** K_p and the **Julia Set** J_p are

$$K_p = \{z \in \mathbb{C} | p^n(z) \nrightarrow \infty\}, \quad J_p = \partial K_p$$

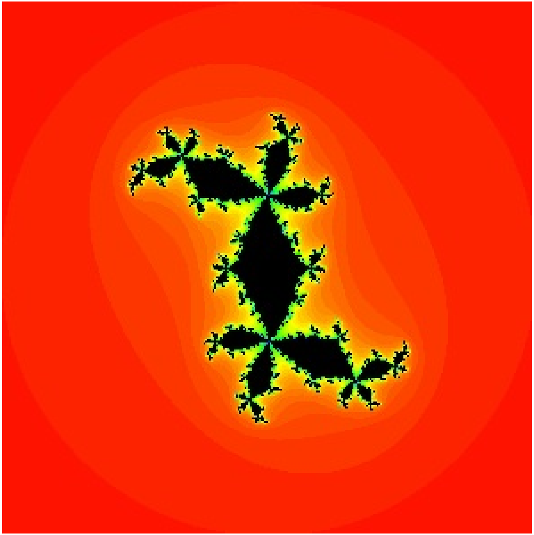


$p(z) = z^2 + \frac{1-\sqrt{5}}{2}$

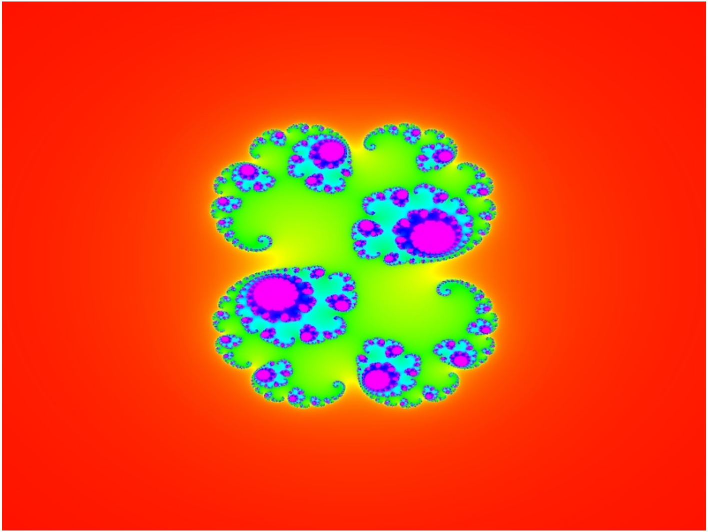
K_p for $p(z) = z^2 - 0.624 + .435i$



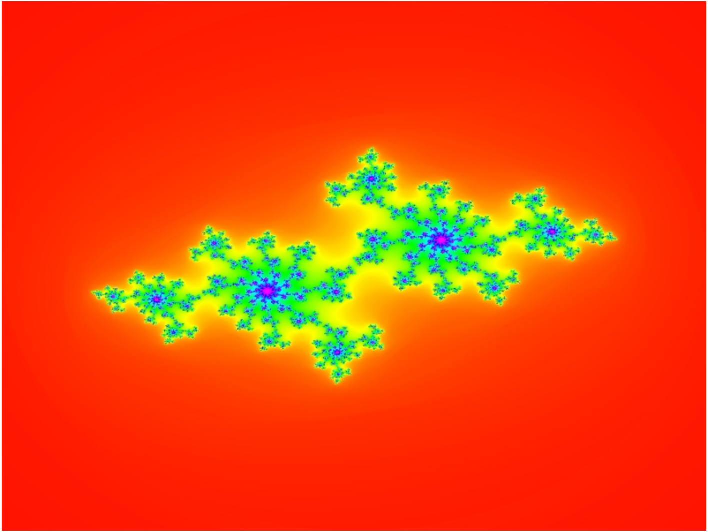
K_p for $p(z) = z^2 + .295 + .55i$



K_p for $p(z) = z^2 + 0.285 + 0.01i$



K_p for $p(z) = z^2 - 0.70176 - 0.3842i$



Definition

z_0 is a **critical point** of p if p is not a local homeomorphism at z_0. We will use Ω_p to denote the critical points of p.

Theorem

Let p be a polynomial of degree $d \geq 2$. If $\Omega_p \subset K_p$, then K_p is connected. If $\Omega \cap K_p = \emptyset$, then K_p is a Cantor set.

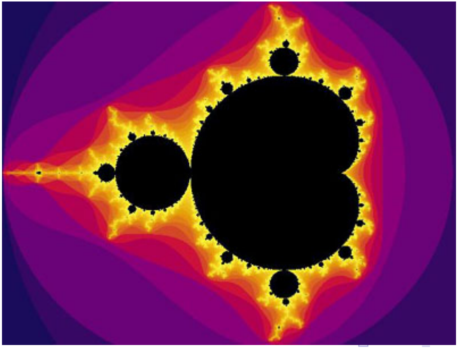
If p only has one critical point, then this theorem gives a dichotomy.

Mandelbrot Set

Definition

Let K_c be the filled Julia set corresponding to the polynomial $p_c(z) = z^2 + c$. The **Mandelbrot set** M is the set

$$M = \{c \in \mathbb{C} : 0 \in K_c\} = \{c \in \mathbb{C} : K_c \text{ is connected}\}.$$



Youtube video zoom in on Mandelbrot Set

Mandelbrot explorer applet

Wikipedia

From Wikipedia's "Mandelbrot Set:"

"Mandelbrot studied the parameter space of quadratic polynomials in an article that appeared in 1980. The mathematical study of the Mandelbrot set really began with work by the mathematicians Adrien Douady and John Hubbard, who established many of its fundamental properties and named the set in honor of Mandelbrot....The work of Douady and Hubbard coincided with a huge increase in interest in complex dynamics and abstract mathematics, and the study of the Mandelbrot set has been a centerpiece of this field ever since"

Uniformization Theorem

Theorem

Any simply connected Riemann surface is conformally isomorphic to exactly one of $\mathbb{C}$, the open disk $\mathbb{D}$, or the Riemann sphere $\hat{\mathbb{C}}$.

Theorem

Every Riemann surface S is conformally isomorphic to a quotient of the form $\tilde{S}/\Gamma$, where $\tilde{S}$ is a simply connected Riemann surface and $\Gamma \cong \pi_1(S)$ is a group of conformal automorphisms which acts freely and properly discontinuously on $\tilde{S}$.

Definition

A **hyperbolic** Riemann surface is one whose universal cover is conformally isomorphic to $\mathbb{D}$

Poincaré metric on hyperbolic surfaces

Poincaré metric on $\mathbb{D}$
is the unique Riemannian metric (up to multiplication by a constant) on $\mathbb{D}$ that is invariant under all conformal isomorphisms of $\mathbb{D}$. The Poincaré metric on $\mathbb{D}$ is complete and has the property that any two points are connected by a unique geodesic.

Poincaré metric on a hyperbolic surface S The universal cover $\tilde{S}$ is conformally isomorphic to $\mathbb{D}$, so $\tilde{S}$ has a Poincaré metric, which is invariant under deck transformations. The Poincaré metric on S is the unique Riemannian metric on S so that the projection $\tilde{S} \to S$ is a local isometry.

The Schwarz lemma says that if $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map that fixes the origin, then $|f(z)| \leq |z|$ for all z, and if there exists $z_0 \neq 0$ such that $|f(z_0)| = |z_0|$ then f is a rotation.

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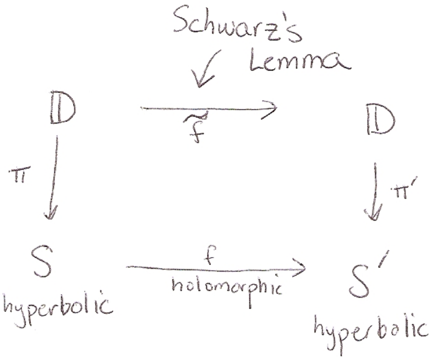


Theorem

(Pick Theorem) If $f : S \to S'$ is a holomorphic map between hyperbolic surfaces, then exactly one of the following three statements holds:

1. f is a conformal isomorphism; f is an isometry.

2. f is a covering map but is not one-to-one; f is locally but not globally a Poincare isometry.

3. f strictly decreases all nonzero distances.

covering maps have constant degree

Theorem

Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a proper non-constant holomorphic map. Then there exists $n \in \mathbb{N}$ such that f takes every value $c \in Y$, counting multiplicies, n times.

A proper non-constant holomorphic maps is called an n-sheeted holomorphic covering map, where n is as above.

the idea of Bottcher coordinates

Question: for a polynomial p, how can we understand the structure of K_p?

Idea: Find a neighborhood U_R of ∞ on which there is an analytic homeomorphism

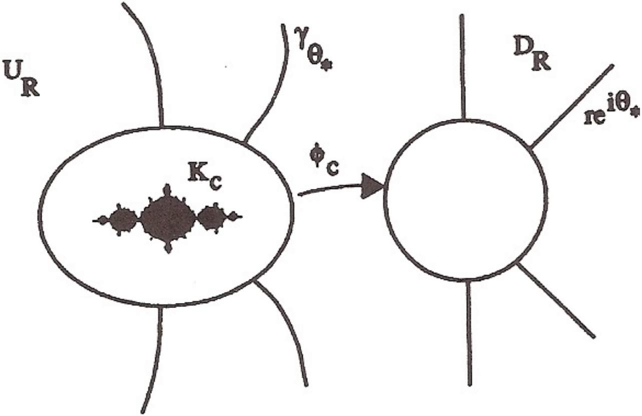
$$\varphi : U_R \to D_R = \{z : |z| > R\}$$

which conjugates f and the map $z \mapsto z^k$, i.e. for $z \in U_R$,

$$\varphi(f(z)) = (\varphi(z))^k.$$

Then try to extend this to $\varphi : \mathbb{C} - K_p \to \mathbb{C} - \mathbb{D}$.

the idea of Bottcher coordinates



Via φ, we may assign "polar" coordinates on U_R.

Bottcher coordinates on a neighborhood of ∞

Theorem

Let $f(z) = z^k(1 + g(z))$ be an analytic mapping on $\hat{\mathbb{C}}$ with $f(\infty) = \infty$, $k \geq 2$ and $|g(z)| \in O(1/|z|)$. Then there exists a neighborhood U of ∞ and an analytic mapping $\varphi : U \to \mathbb{C}$ with $(\varphi(z))^k = \varphi(f(z))$.

The idea is to "define" $\varphi(z) = \lim_{n\to\infty} (f^n(z))^{1/k^n}$, BUT the problem is we have to specify which k^nth root is being considered.

$$(\varphi(z))^k = \left(\lim_{n\to\infty} (f^n(z))^{1/k^n}\right)^k = \lim_{n\to\infty} \left((f^{n+1}(z)^{1/k^{n+1}})^k\right)$$

$$= \lim_{n\to\infty} \left(f^{n+1}(z)\right)^{\frac{k}{k^{n+1}}} = \lim_{n\to\infty} \left(f^{n+1}(z)\right)^{\frac{1}{k^n}} = \varphi(f(z))$$

Proof of Bottcher coordinates on nbhd of ∞

We will write $(f^n(z))^{1/k^n}$ as a telescoping product in such a way that in each factor we can use the principal branch of the root (i.e. no factor lies on the negative real axis).

$$(f^n(z))^{1/k^n} = z \cdot \frac{(f(z))^{1/k}}{z} \cdot \frac{(f^2(z))^{1/k^2}}{(f(z))^{1/k}} \cdot \ldots \cdot \frac{(f^n(z))^{1/k^n}}{(f^{n-1}(z))^{1/k^{n-1}}}.$$

The general term of this product is

$$\frac{(f^m(z))^{1/k^m}}{(f^{m-1}(z))^{1/k^{m-1}}} = \frac{((f^{m-1}(z))^k(1+g(f^{m-1}(z))))^{1/k^m}}{(f^{m-1}(z))^{1/k^{m-1}}}$$

$$= (1+g(f^{m-1}(z)))^{1/k^m}$$

So we want to show that there exists $r > 0$ so that $|z| \geq r$ implies $|g(f^{m-1}(z))| < 1$ for all m.

Proof of Bottcher coordinates on nbhd of ∞

Pick $r_1 > 0$ and $C > 0$ so that $|g(z)| < \frac{C}{|z|}$ for $|z| \geq r_1$.
Let r_2 be the greatest positive root of $x^{k+1}(1 + Cx) = 1$.
Set $r = \max(r_1, r_2, \frac{1}{2C})$.

Then for $|z| \geq$r:

$$|f(z)| = |z|^k |1 + g(z)| \geq |z| r^{k-1} |1 + Cr| \geq |z|$$

$$\therefore \quad |f^m(z)| \geq |z| \quad \forall m$$

$$\therefore \quad |g(f^{m-1}(z)| \leq \frac{C}{|f^{m-1}(z)|} \leq \frac{C}{2C} = \frac{1}{2}$$

$\therefore$ the expression $(f^n(z))^{1/k^n}$ is well-defined

Proof of Bottcher coordinates on nbhd of ∞

It remains to show that the limit $\lim_{n\to\infty}(f^n(z))^{1/k^n}$ converges. The maximum value of $|\ln(1+w)|$ for $|w| \leq 1/2$ is $\ln(2)$ and is achieved at $w = -1/2$. Then

$$\left|\ln\left|1+g(f^{m-1}(z))\right|^{1/k^n}\right| = \frac{1}{k^m}\left|\ln\left|1+g(f^{m-1}(z))\right|\right| \leq \frac{\ln 2}{k^m}.$$

Green's function

Lemma

Let $f(z) = z^k(1 + g(z))$ be an analytic mapping on $\hat{\mathbb{C}}$ with $f(\infty) = \infty$, $k \geq 2$ and $|g(z)| \in O(1/|z|)$ near ∞. Let A_∞ be the basin of attraction of ∞ and let $Z = \{z : f^n(z) = \infty \text{ for some } n \in \mathbb{N}\}$. Then the sequence of functions $G_n : A_\infty \to [-\infty, \infty)$ defined by

$$G_n(z) = \frac{1}{k^n} \ln \left| \frac{1}{f^n(z)} \right|$$

converges uniformly on compact subset of A_∞, with poles on Z, and the limit G satisfies $G(f(z)) = kG(z)$.

Proof.

G converges on U because $|f^n(z)|^{1/k^n}$ converges $\Rightarrow |\frac{1}{f^n(z)}|^{1/k^n}$ converges $\Rightarrow k^{-n}\ln|\frac{1}{f^n(z)}| = G_n(z)$ converges on U.

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G converges on A_∞ because all points in A_∞ eventually enter U. $G_n(f(z)) = k^{-n}\ln|\frac{1}{f^{n+1}(z)}| =$

$$k \cdot k^{-(n+1)}\ln|\frac{1}{f^{n+1}(z)}| = kG_{n+1}(z).$$

□

Notation: For $0 < \rho \leq 1$, define U_ρ to be the connected component of $\{z \in U : G(z) < \ln \rho\}$ which contains 0, and let ρ_0 be the supremum of the ρ such that U_ρ contains no critical point of f other than 0.

Proposition

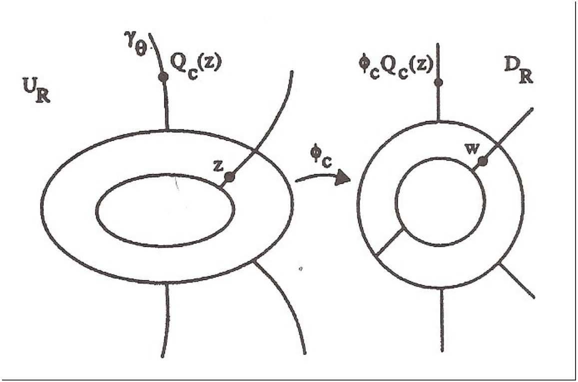
The map φ extends to an analytic isomorphism from U_{ρ_0} to $\{z : |z| > e^{\rho_0}\}$. In particular, if A_∞ contains no critical point of f other than ∞, the map φ is a conformal map from the immediate basin of 0 to $\mathbb{C} - \mathbb{D}$.

Proof.

There exists $n \in \mathbb{N}$ sufficiently large so that $U_{\rho_0^n}$ is contained in the domain of definition of φ. The restriction $f : U_{\rho_0^{n-1}} \to U_{\rho_0^n}$ is a covering map of degree k ramified only at ∞, and the map $z \mapsto z^k$ as a map from $\hat{\mathbb{C}} - D_{\rho_0^{n-1}}$ to $\hat{\mathbb{C}} - D_{\rho_0^n}$ is also a covering map ramified only at ∞ (because $U_{\rho_0^{n-1}} \cap \Omega_p = \emptyset$). There is only one such ramified covering space (up to automorphisms). Hence there are precisely k different maps $g_i : U_{\rho_0^{n-1}} \to D_{\rho_0^{n-1}}$ (the k lifts of φ) such that $\varphi(f(z)) = (g_i(z))^k$. These k lifts of φ different by postmultiplication by a *kth* root of unity. But precisely one of these g_i coincides with φ on $U_{\rho_0^n}$. This map is the analytic extension of φ to $U_{\rho_0^{n-1}}$. Iterating this process, we can extend φ successively to $U_{\rho_0^{n-1}} \subset U_{\rho_0^{n-2}} \subset \ldots \subset U_{\rho_0}$. □

The idea of the proof

The case in which A_∞ contains no critical point of f other than ∞ (ex. polynomials $z \mapsto z^2 + c$).



Can $\psi = \varphi^{-1}$ be extended to S^1?

Suppose you have extended φ to $\mathbb{C} - K_p$. Do external rays actually land on the Julia set? Define $\psi = \varphi^{-1}$, $\psi : \mathbb{C} - \mathbb{D} \to \mathbb{C} - K_p$. Can we extend ψ to S^1?

If we could extend ψ to S^1, one immediate consequence would be that K_p is locally connected (since the continuous image of a locally connected set is locally connected).

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Theorem

Let p be a polynomial of degree $d \geq 2$ such that every critical point of p is either

1. *attracted to an attracting cycle (not infinity),*
2. *has a finite orbit containing a repelling cycle, or*
3. *is attracted to a parabolic cycle.*

Then ψ_p extends to S^1.

proof external rays land in the hyperbolic case

Let Z be the set of attracting cycles. Let V_0 be a neighborhood of Z such that $p(V_0)$ is relatively compact in V_0. Define $V_n = p^{-n}(V_0)$ where $n \in \mathbb{N}$ is the smallest integer such that $\Omega_p \subset V_n$. Fix $R > 0$ sufficiently large that $p^{-1}(U)$ is a relatively compact subset of U, where

$$U = \mathbb{C} - (\bar{V}_n \cup \{z \in \mathbb{C} - K_p : |\varphi_p(z)| \geq R\}).$$

Let U' denote $p^{-1}(U)$. $p : U' \to U$ is a covering map, and so for $(z, w) \in \mathrm{T}U'$, $|(z, w)|_{U'} = |(p(z), p'(z) \cdot w)|_U$.

As U' is a proper subset of U, $|(z,w)|_U < |(z,w)|_{U'}$ for all $(z,w) \in TU'$. Since $\frac{|(z,w)|_U}{|(z,w)|_{U'}}$ is continuous on $\{(z,w) \in TU' : w \neq 0\}$ and U' is relatively compact in U, there exists $C < 1$ such that

$$|(z,w)|_U \leq C|(z,w)|_{U'}$$

for all $(z,w) \in TU'$, $w \neq 0$. Thus for all $(z,w) \in TU'$ with $w \neq 0$

$$|(z,w)|_U \leq C|(z,w)|_{U'} = C|p(z), p'(z)w|_U.$$

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Denote by $\alpha_{n,t}$ the arc that is the image of the map

$$\rho \to \psi_p(\rho e^{2\pi i t}), \quad R^{1/d^{n+1}} \leq \rho \leq R^{1/d^n}$$

and by $l_{n,t}$ the length of $\alpha_{n,t}$. Write $l_n = \sup_{t \in \mathbb{R}/\mathbb{Z}} l_{n,t}$. Since $\psi_p(z^d) = p(\psi_p(z))$, we have $p(\alpha_{n,t}) = \alpha_{n-1,td}$. Hence $l_{n,t} \leq Cl_{n-1,td}$ for all t, and so $l_n \leq Cl_{n-1}$. Thus the l_n form a convergent series (by comparison to the geometric series C^{-n}). It follows that the family of mappings $\beta_p : \mathbb{R}/\mathbb{Z} \to U$ given by $\beta_\rho(t) = \psi(\rho e^{2\pi i t})$ converges uniformly as $\rho \searrow 1$.

M is connected

For $c \in \mathbb{C}$, let φ_c be the map corresponding to the polynomial $P_c(z) = z^2 + c$, and let K_c be the corresponding filled-in Julia set. $\varphi_c : \mathbb{C} - K_c \to \mathbb{C} - \bar{\mathbb{D}}$. Define

$\Phi : \mathbb{C} - M \to \mathbb{C}$ by

$$\Phi(c) = \varphi_c(c).$$

Theorem

The map Φ is an analytic isomorphism $\mathbb{C} - M \to \mathbb{C} - \bar{\mathbb{D}}$. In particular, the Mandelbrot set M is connected.

Proof that M is connected

Φ is analytic:
Φ $(:= \varphi_c(c))$ the the composition of the analytic mappings $c \mapsto (c, c)$ and $(z, c) \mapsto \varphi_c(z)$, so Φ is analytic.

Φ is proper:

$$-G_c(z) = \lim_{n\to\infty} 2^{-n} \ln |P_c^n(z)| = \ln |z| + \sum_{n=1}^{\infty} \ln \left| 1 + \frac{c}{P_c^n(z)} \right| .$$

$$\ln |\Phi(c)| = -G_c(c) = \ln |c| + \sum_{n=1}^{\infty} 2^{-n} \ln \left| 1 + \frac{c}{P_c^n(c)} \right| .$$

If $|c| > 2$, then $\left|1 + \frac{c}{P_c^n(c)}\right| \leq 2$, so $-G_c(c) - \ln |c|$ is bounded, and $c \mapsto |G_c(c)|$ is a proper map.

$c \mapsto |G_c(c)|$ is proper implies immediately that $\Phi : C - M \to \mathbb{C} - \bar{\mathbb{D}}$ is proper, because for any compact set in $\mathbb{C} - \bar{\mathbb{D}}$ is a closed subset of an annulus $\{z \in C : r_1 \leq |z| \leq r_2\}$ for some $1 < r_1 < r_2 < \infty$, and Φ^{-1} of such an annulus is compact:

$$\{c \in \mathbb{C} : r_1 \leq |\Phi(c)| \leq r_2\} = \{c \in \mathbb{C} : \ln r_1 \leq |G_c(c)| \leq \ln r_2\},$$

which is compact.

Φ is surjective:
Φ proper implies its image is closed, and all non-constant analytic mappings are open, so the image is both closed and open in $\mathbb{C} - \bar{\mathbb{D}}$. Hence Φ is surjective.

Φ is a bijection:

$\ln|\Phi(c)| = -G_c(c) = \ln|c| + \sum_{n=1}^{\infty} 2^{-n} \ln\left|1 + \frac{c}{P_c^n(c)}\right|$, and $-G_(c) - \ln|c|$ is bounded, in particular bounded near ∞, so $\Phi(c)/c$ is bounded near ∞. Hence $\Phi(c)/c$ is bounded near ∞, which implies Φ has a simple pole at ∞. Thus Φ extends to an analytic map $\bar{\mathbb{C}} - M \to \bar{\mathbb{C}} - \bar{D}$, and the only inverse image of ∞ is ∞, which local degree 1. Hence Φ has degree 1, i.e. Φ is a bijection.