Solution to Midterm Exam

July 15, 2015

1. Compute the following limits, justifying your steps. L'Hopital's Rule is NOT allowed.

(a) $\lim\limits_{x\to\pi} e^{\frac{1}{x}+\cos(x)}$

Sol: Since e^x is continuous in $\mathbb{R}$ and $\frac{1}{x}+cos(x)$ is continuous at $x=\pi$,
$\lim\limits_{x\to\pi} e^{\frac{1}{x}+\cos(x)} = e^{\lim\limits_{x\to\pi}(\frac{1}{x}+\cos(x))} = e^{(\frac{1}{\pi}-1)}$.

(b) $\lim\limits_{x\to-\infty} \frac{x+1}{\sqrt{x^2+1}}$

Sol: Let $y=-x$, so $x=-y$, and $x\to-\infty$ is *equivalent* to $y\to\infty$.
$\lim\limits_{x\to-\infty} \frac{x+1}{\sqrt{x^2+1}} = \lim\limits_{y\to\infty} \frac{-y+1}{\sqrt{(-y)^2+1}} = \lim\limits_{y\to\infty} \frac{-y+1}{\sqrt{y^2+1}} = \lim\limits_{y\to\infty} \frac{-1+\frac{1}{y}}{\sqrt{1+\frac{1}{y^2}}} = \frac{-1}{1} = -1.$

(c) $\lim\limits_{x\to\pi} \frac{\sin(x)-\sin(\pi)}{\pi-x}$

Sol: Let $h=x-\pi$, then $x=\pi+h$, and $x\to\pi$ is equivalent to $h\to 0$.
$\lim\limits_{x\to\pi} \frac{\sin(x)-\sin(\pi)}{\pi-x} = \lim\limits_{h\to0} \frac{\sin(\pi+h)-\sin(\pi)}{-h}$
$=-\lim\limits_{h\to0} \frac{\sin(\pi+h)-\sin(\pi)}{h} = -\frac{d}{dx}|_{x=\pi} sin(x) = -cos(\pi) = 1.$

2. (20 pts) Compute the following derivatives:

(a) $\dfrac{d}{dx}\left(x^3e^x\right) = 3x^2e^x + x^3e^x$, using the Product Rule.

(b) $\dfrac{d}{d\theta}\left(\tan(2\theta+1) - 3\theta + e^2\right) = 2\sec^2(2\theta+1) - 3$

(c) $\dfrac{d}{dy}\left(\dfrac{\sin(y)+1}{y+1}\right) = \dfrac{\cos(y)(y+1) - (\sin(y)+1)}{(y+1)^2}$, using the Quotient Rule.

(d) $\dfrac{d}{dx}\left(x^{\cos(x)} + \ln(x)\right) = \dfrac{d}{dx}\left(x^{\cos(x)}\right) + \dfrac{1}{x}$

To find $\frac{d}{dx}\left(x^{\cos(x)}\right)$ we use logarithmic differentiation. Let $y = x^{\cos(x)}$. Then

$$\ln(y) = \ln(x^{\cos(x)}) = \cos(x)\ln(x)$$

Differentiate both sides:

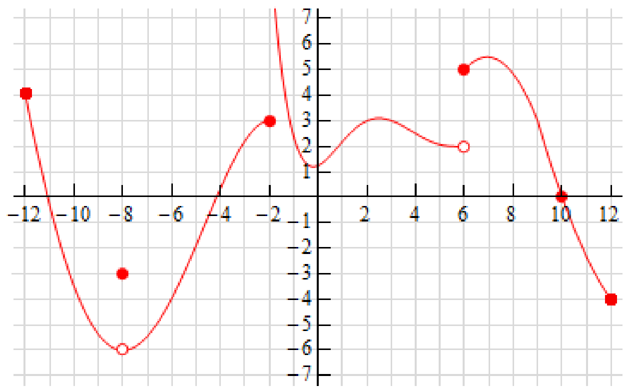
$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos(x)}{x} - \sin(x)\ln(x)$$

and therefore

$$\frac{dy}{dx} = y\left(\frac{\cos(x)}{x} - \sin(x)\ln(x)\right) = x^{\cos(x)}\left(\frac{\cos(x)}{x} - \sin(x)\ln(x)\right)$$

Therefore $\dfrac{d}{dx}\left(x^{\cos(x)} + \ln(x)\right) = x^{\cos(x)}\left(\dfrac{\cos(x)}{x} - \sin(x)\ln(x)\right) + \dfrac{1}{x}$

3. (20 pts) Consider the following graph of the function $f(t)$:



(a) $\lim_{t\to -2-} f(t) = 3$

(b) $\lim_{t\to -6} f(t) = -4$

(c) $\lim_{t\to -8} f(t) = -6$

(d) Estimate the instantaneous rate of change of f at $t = 10$ (don't worry too much about getting it perfect).

Either draw in the tangent line at $t = 10$ or the secant line on $[10, 11]$ or $[9, 10]$. This gives us an estimate between -2 and -3.

(e) Estimate the average rate of change of f on the interval [2, 6]. $\frac{f(6)-f(2)}{6-2} = \frac{5-3}{6-2} = \frac{2}{4} = \frac{1}{2}$

(f) Does the graph f have any vertical asymptotes? If so, what are the equations? Yes, there's a vertical asymptote at $t = -2$. We get a vertical asymptote at $t = c$ if either $\lim_{t\to c-} f(t) = \pm\infty$ or $\lim_{t\to c+} f(t) = \pm\infty$.

(g) An interval on which f is decreasing is: $[-12, -8)$ or $(-2, 0]$ or $[2.5, 6)$ or $[7, 12]$ or some interval subset of one of these intervals.

(h) Is f a 1-to 1 function, and why?
No it isn't 1-1 function. It doesn't satisfy the horizontal line test. For example, $f(-11) = f(-4) = 0$.

(i) The range of f on the interval [-12, 12] is: $(-6, \infty)$

(j) Put an x on the t-axis at every point in the domain of f where f is not continuous.
Not continuous at $t = -12, -8, -2, 6, 12$.

(k) Put a circle $\circ$ on the t-axis at every point in the domain of f where f is continuous but not differentiable.
The function f happens to be differentiable whenever it is continuous (though this is not true in general).

4. Find the equation of the tangent line to the curve

$$x + \sqrt{xy} = 6$$

at the point $(4, 1)$.

<u>*Sol*</u>: Since $x + \sqrt{xy} = 6$, differentiate both sides with respect to x we get

$$1 + \frac{1}{2\sqrt{xy}}(y + xy') = 0$$

$$\Rightarrow \frac{x}{2\sqrt{xy}}y' = -1 - \frac{y}{2\sqrt{xy}}$$

$$\Rightarrow y' = \frac{2\sqrt{xy}}{x}(-1 - \frac{y}{2\sqrt{xy}})$$

$$= -\frac{2\sqrt{xy}}{x} - \frac{y}{x}$$

Therefore at the point $(4, 1)$ $y' = -\frac{4}{4} - \frac{1}{4} = -\frac{5}{4}$.
So the equation of the tangent line at the point $(4, 1)$ is

$$y - 1 = -\frac{5}{4}(x - 4)$$

i.e.

$$y = -\frac{5}{4}x + 6$$

5. Suppose $f(x)$ is a continuous function with domain $[-5, 5]$. Let $g(x) = x$. If $f(-5) = 6$ and $f(5) = -6$, show that there is $c \in [-5, 5]$ such that $f(c) = g(c)$.

Pf: Let $h(x) = f(x) - g(x)$. Since $f(x)$ and $g(x)$ are continuous functions, $h(x)$ is also continuous on $[-5, 5]$.

$$h(-5) = f(-5) - g(-5) = 6 - (-5) = 11 > 0$$

$$h(5) = f(5) - g(5) = -6 - 5 = -11 < 0$$

By *Intermediate Value Theorem*, there exists a $c \in (-5, 5)$ such that $h(c) = 0$.
i.e., $f(c) - g(c) = 0$, *i.e.*, $f(c) = g(c)$.

6. Consider the function

$$f(x) = \begin{Bmatrix} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{Bmatrix}$$

(a) Is f differentiable at $x = 3$, and why or why not?

<u>*Sol*</u>: Since $\frac{1}{x}$ is differentiable at $x = 3$ and $sin(x)$ is differentiable at $x = \frac{1}{3}$, then $sin(\frac{1}{x})$ is differentiable at $x = 3$. x^2 is also differentiable at $x = 3$, so by product rule, $x^2 sin(\frac{1}{x})$ is differentiable at $x = 3$, *i.e.*, $f(x)$ is differentiable at $x = 3$.

(b) Calculate the derivative of f at $x = 0$.

<u>*Sol*</u>: By definition $f'(0) = \lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \frac{h^2 sin(\frac{1}{h})-0}{h} = \lim\limits_{h\to 0} hsin(\frac{1}{h})$

Define $q(h) = hsin(\frac{1}{h})$, $p(h) = -|h|$, $r(h) = |h|$, since $-1 \leq sin(\frac{1}{h}) \leq 1$, we have

$$p(h) \leq q(h) \leq r(h)$$

Since $\lim\limits_{h\to 0} p(h) = \lim\limits_{h\to 0} r(h) = 0$, by <u>*Squeeze Theorem*</u>, $\lim\limits_{h\to 0} q(x) = 0$, *i.e.*,

$$\lim_{h\to 0} hsin(\frac{1}{h}) = 0$$

Therefore $f'(0) = 0$.

(c) Decide if $f(x)$ is even, odd, both, or neither.

<u>*Sol*</u>: If $x \neq 0$, $f(-x) = (-x)^2 sin(\frac{1}{-x}) = x^2 sin(-\frac{1}{x}) = -x^2 sin(\frac{1}{x}) = -f(x)$.
If $x = 0$, $f(-0) = f(0) = 0 = -f(0)$.
So $f(x)$ is an odd function.