Final Exam

Math1110 (Summer 2013)
August 5

Name: ______________________________

☐ Cristina's Section
☐ Yao's Section

- This is 2h30m exam with **10** problems.
- Put away anything other than pencils.
- Adhere to the Honor Code.

Signature:

OFFICIAL USE ONLY	
1. ________	/23
2. ________	/20
3. ________	/20
4. ________	/20
5. ________	/25
6. ________	/20
7. ________	/12
8. ________	/20
9. ________	/20
10. ________	/20
Total: ________	/200

Question 1. (23 points) Find the following derivatives

a) $\dfrac{d}{dx}\left(x^3 - 3x^2 + \cos x\right) =$

b) $\dfrac{d}{dx}\ln(\ln(2x)) =$

c) $\dfrac{d^2}{dx^2}\left(x^2 \sin x\right) =$

d) $\dfrac{d}{dx}(x^2 + 1)^{1/x} =$

Question 2. (20 points)

a) $\lim\limits_{x \to 1} \dfrac{x}{\cos(\pi x)} =$

b) $\lim\limits_{x \to \infty} \dfrac{\ln(x^2 + 1)}{x^2 + x^3} =$

c) $\lim\limits_{t \to 0} \dfrac{\ln(t^2 + 1)}{\sin t} =$

d) $\lim\limits_{x \to \infty} \dfrac{e^{1/x}}{x} =$

Question 3. (20 points)

a) $\displaystyle\int (3x^2 - 6x - \sin x)dx =$

b) $\displaystyle\int \frac{x\sqrt{x} + x}{\sqrt{x}} dx =$

c) $\displaystyle\int_0^{\pi/4} \tan\theta \, d\theta =$

d) $\displaystyle\int_0^1 \frac{2z}{\sqrt[3]{z^2+1}} dz =$

Question 4. (20 points) For this problem, make sure to specify what theorem(s) you're using.

a) Consider the function $f(x) = \int_0^{\sin^2 x} e^{\sin t} dt$.

i) Find $\frac{df}{dx}(x) =$

ii) Using the inequality $0 \leq e^{\sin t} \leq e$, deduce that $0 \leq f(x) \leq e \sin^2 x$.

iii) Find $\lim_{x \to 0} f(x) =$

b) Let $g(x) = \int_0^{x^2} \cos^2 t \, dt$.

i) Find $\frac{dg}{dx}(x) =$

ii) Using the inequality $-1 \leq \cos t \leq 1$, deduce that $0 \leq g(x) \leq x^2$.

iii) Find $\lim_{x \to 0} g(x) =$

c) Possibly using parts a) and b), compute

$$\lim_{x \to 0} \frac{\displaystyle\int_0^{\sin^2 x} e^{\sin t} dt}{\displaystyle\int_0^{x^2} \cos^2 t \, dt} =$$

Question 5. (25 points) In this problem we investigate the function $f(x) = \frac{1}{2} + \frac{x}{x^2+1}$. For your convenience, here are the first and second derivatives:

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} \qquad\qquad f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Determine the following information about the function f: (a) x- and y-intercepts (that is, where the graph intersects the x- and y-axes); (b) horizontal and vertical asymptotes; (c) critical poitns; (d) intervals where f is decreasing and increasing; (e) local extrema; (f) absolute extrema; (g) intervals of concavity; (h) inflection points. Use this information to sketch the graph of f.

In the workspace below, clearly show your calculations. Neatly organize your work. (It will be graded!) On the next page, report the results from your calculations and sketch the graph of f.

Question 5 - Answers

Note: If there is no information to report to a category, write N/A.

(a) x-intercept ______ y-intercept ______

(b) Horizontal asympototes $y =$______ Vertical asympototes $x =$______

(c) Critical points ________________________

(d) Intervals where f is decreasing __________________ and increasing __________________

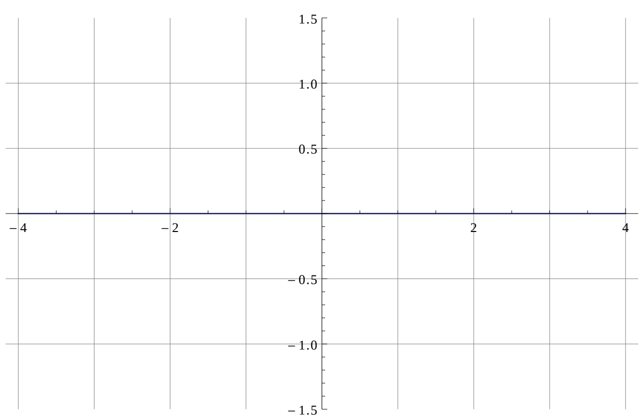
(e) Local maxima: __________ Local minima: __________

(f) Absolute maxima: __________ Absolute minima: __________

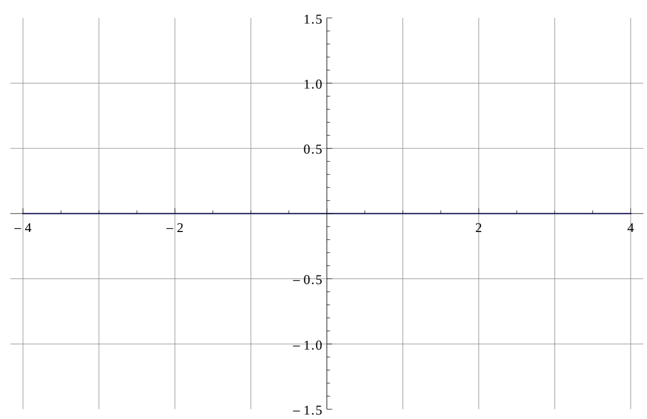
(g) Invervals of concave-up: __________ Invervals of concave down: __________

(h) Inflection points: __________

Sketch the graph of $f(x)$



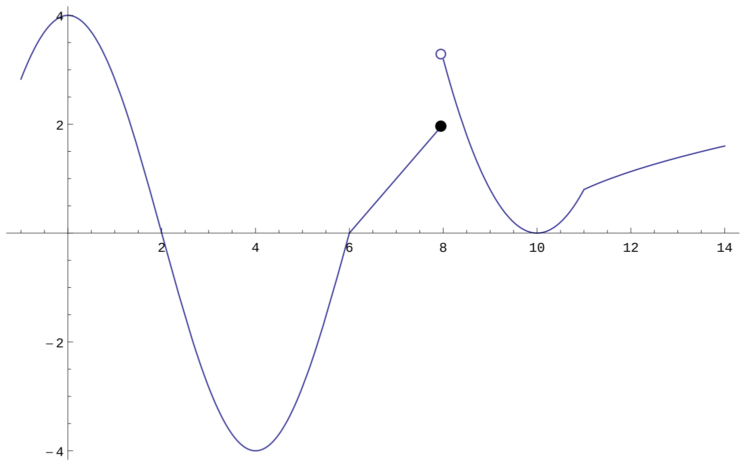
FOR PRACTICE



FINAL ANSWER

Question 6. (20 points)

Let $f(x)$ be the function on the interval $(-1, 14)$ as shown in the graph



and let

$$g(x) = \int_0^x f(t)dt, \qquad -1 < x < 14$$

a) Find all the critical points of $g(x)$.

b) Find the local maxima and local minima of $g(x)$.

c) Find the inflection points of $g(x)$.

Question 7. (12 points) True/False (No explanation needed.)

a) If $f(x)$ is an increasing function, and is concave up on the interval (a, b), then the inverse f^{-1} is concave down on the interval $(f(a), f(b))$.

☐ True ☐ False

b) If $x = a$ is an absolute maximum point of the function $f(x)$, then it is also an absolute maximum point of $f(x)^2$.

☐ True ☐ False

c) The linearization of $f(x) = \sin^{-1} x$ at $\frac{\sqrt{2}}{2}$ is $L(x) = 1 + \sqrt{2}(x - \frac{\sqrt{2}}{2})$

☐ True ☐ False

d) The equation $x^3 + x = 15$ has exactly one solution on $\mathbb{R}$.

☐ True ☐ False

Question 8. (20 points)

Let $f(x) = (x-1)^2 e^{-x}$.

a) Show that $F(x) = (-x^2 - 1)e^{-x}$ is an antiderivative of $f(x)$.

b) Find $\int_0^1 f(x)dx$

c) Find $\lim_{a \to \infty} \int_0^a f(x)dx$

Question 9. (20 points)

a) Let $f(x)$ be twice differentiable, and $f''(x) > 0$ for all x in $\mathbb{R}$. Prove the following inequality, also known as Jensen's inequality:

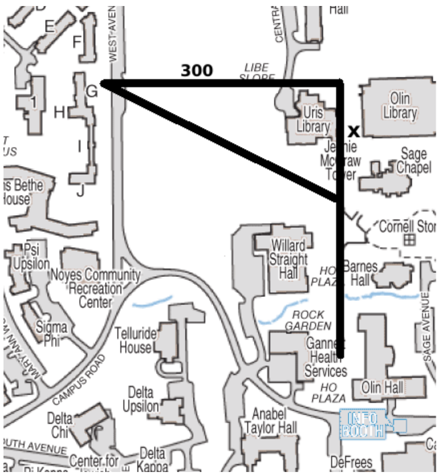
$$f(\frac{a+b}{2}) < \frac{f(a)+f(b)}{2}, \quad \text{for all } a < b.$$

(*Hint: start by applying Mean Value Theorem on the intervals* $[a, \frac{a+b}{2}]$ *and* $[\frac{a+b}{2}, b]$.)

b) Show that $e^x + e^{-x} > 2$ for all $x > 0$.

Question 10. (20 points)

Thomas gets a bad cold and wants to go from West campus to Gannett as fast as possible. Assume, for the sake of simplicity, that he walks up the slope at the speed of 3 feet/second, no matter which direction he takes. Walking from the Tower to Gannett is faster, at 5 feet/second. The distance from West campus to the Tower is exactly 300 feet, and the distance from Tower to Gannett is 400 feet. What is the shortest time that he needs to get to Gannett?



FORMULAS YOU MAY NEED

$$e^{i\pi} = -1$$

$$\sin(2x) = 2\sin x \cos x$$

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\int a^x dx = \frac{1}{\ln a} a^x + C$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$