MATH1110 Summer 2013 Prelim 1 Solution

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(a) No. It fails the horizontal line test. For example, the line $y = 1$ intersect the graph in 5 points.

(b) $[-2, 0], (1, 2)$

(c) Since $\lim_{x\to 1^-} f(x) = 1$ while $\lim_{x\to 1^-} f(x) = 0.4$. They do not agree and hence the limit does not exist.

(d) They are at $x = 1$ and $x = 2$.

(e) At $x = 2$.

2

(a) $\lim_{x\to 1} \log_2(x^2 + 3) = \log_2(1 + 3) = \log_2(4) = 2$

(b) $\lim_{x\to \frac{\pi}{2}^+} \tan x = \lim_{x\to \frac{\pi}{2}^+} \frac{\sin x}{\cos x} = \frac{1^+}{0^-} = -\infty$

(c) $\lim_{x\to \pi} \frac{\sin x+1}{\cos x} = \frac{0+1}{-1} = -1$

(d) $\lim_{x\to\infty} \frac{x\sqrt{x^2+2x}}{3x^2-5x+1} = \lim_{x\to\infty} \frac{x\sqrt{x^2+2x}/x^2}{(3x^2-5x+1)/x^2} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{2}{x}}}{3-\frac{5}{x}+\frac{1}{x^2}} = \frac{\sqrt{1}}{3} = \frac{1}{3}$

3

(a) True

(b) True

(c) False

(d) True

(e) False

4

(a) Both the functions $g_1(x) = x$ and $g_2(x) = \ln x$ are continuous functions. Hence so is their product $g(x) = g_1(x)g_2(x) = x \ln x$. The domain of g_1 is the entire real line, while the domain of g_2 is the half real line $(0, \infty)$, hence the domain of g is $(0, \infty)$.

(b) The function $h(x) = e^x$ is continuous. Hence $h(g(x)) = e^{x \ln x}$ is also continuous in x. Moreover, $e^{x \ln x} = (e^{\ln x})^x = x^x$.

(c) Note that $f(1) = 1^1 = 1$ and $f(2) = 2^2 = 4$. For this part we use the Intermediate Value Theorem: a continuous function (and we just proved that x^x is continuous) that goes from $f(1) = 1$ to $f(2) = 4$ must sweep through all the values in between, in particular, it must sweep through the value $y = 3$.This means there is some x, $1 < x < 2$ such that $f(x) = 3$.

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(a) It takes 1 hour to reach the top. The height at that point is

$$H(1) = 75 - 60\cos(\pi) = 75 + 60 = 135\text{m}$$

(b) The average rate of change is

$$\frac{H(\frac{2}{3}) - H(\frac{1}{2})}{\frac{2}{3} - \frac{1}{2}} = \frac{(75 - 60\cos(2\pi/3)) - (75 - 60\cos(\pi/2))}{\frac{1}{6}} =$$
$$= 360(-\cos 2\pi/3) = 360\sin(\pi/6) = 180$$

(c) First we simplify

$$\frac{H(\frac{1}{2} + h) - H(\frac{1}{2})}{h} = \frac{(75 - 60\cos(\frac{\pi}{2} + \pi h)) - (75 - 60\cos(\frac{\pi}{2}))}{h} =$$
$$= \frac{60(\cos(\frac{\pi}{2}) - \cos(\frac{\pi}{2} + \pi h))}{h} =$$
$$= \frac{60\sin(\pi h)}{h} = \frac{60\sin(\pi h)}{\pi h} \times \pi$$

Hence $\lim_{h\to 0} \frac{H(\frac{1}{2}+h)-H(\frac{1}{2})}{h} = \lim_{h\to 0} \frac{60\sin(\pi h)}{\pi h} \times \pi = 60\pi$.

6

(a)

$$f(1) = \lim_{t\to\infty} \frac{1}{1+t} = \frac{1}{\infty} = 0$$
$$f(2) = \lim_{t\to\infty} \frac{1}{1+4t} = \frac{1}{\infty} = 0$$
$$f(0) = \lim_{t\to\infty} \frac{1}{1} = 1$$

(b) For any $a \neq 0$, we have $a^2 t \to \infty$ as $t \to \infty$. Hence:

$$f(a) = \lim_{t\to\infty} \frac{1}{1+a^2 t} = \frac{1}{\infty} = 0$$

(c) By (b) and by calculation of $f(0)$ at (a), $y = f(x)$ always stays at $y = 0$, except when $x = 0$ where it has a jump up to $y = 1$. Hence, it has a jump discontinuity at $x = 0$.