1 Notations

We will write elements of $\mathbb{Z}^d$ as x, y, z. We will use superscripts to denote components, $x = (x^1, \ldots, x^d)$ and subscripts, $x_1, x_2, \ldots$ to enumerate elements of $\mathbb{Z}^d$. Let $e_j, j = 1, \ldots, d$ be the unit vector in $\mathbb{Z}^d$ with jth component equal to 1. We use $\xi, \xi_1, \xi_2, \ldots$ to represent complex numbers. If r_n, s_n are sequences of positive numbers, we say that r_n and s_n are *asymptotic*, written $r_n \sim s_n$ if $r_n/s_n \to 1$. We say that r_n and s_n are *comparable*, written $r_n \asymp s_n$, if there exist constants c_1, c_2 such that $c_1 r_n \leq s_n \leq c_2 r_n$.

We will call p an *increment distribution (for an irreducible random walk) on* $\mathbb{Z}^d$ if p is a probability distribution on $\mathbb{Z}^d$ such that every $x \in \mathbb{Z}^d$ can be written as $x = x_1 + \cdots + x_k$ with $x_1, \ldots, x_k \in \mathcal{G}_p := \{z \in \mathbb{Z}^d : p(z) > 0\}$. Given an increment distribution p on $\mathbb{Z}^d$, we have the *discrete time* and *continuous time* random walks. The discrete random walk is the discrete Markov chain with transition probability

$$P(x,y) = p(y-x)$$

while the continuous time walk is the continuous time chain with rate $p(y-x)$ for jumping from x to y. We let $\mathbf{L}^p$ denote the Laplacian associated to p,

$$\mathbf{L}^p f(x) = \sum_y p(y)\,[f(x+y) - f(x)].$$

We define $\mathbf{L}^p f(x)$ only if f is defined on $\{x\} \cup \{y : p(y-x) > 0\}$ and $\sum_y p(y)|f(x+y)| < \infty$. We will usually write just $\mathbf{L}$. If we have a function of more than one variable, say $f(x,z)$ we will sometimes use subscripts to denote which variable to take the Laplacian in, i.e.,

$$\mathbf{L}_x f(x,z) = \sum_y p(y)\,[f(x+y,z) - f(x,z)].$$

We use S_n to denote the discrete Markov chain and $\tilde{S}_t$ to denote the continuous chain. We let

$$P_n(x,y) = \mathbb{P}^x\{S_n = y\}, \qquad p_t(x,y) = \mathbb{P}^x\{\tilde{S}_t = y\}.$$

(We will use $\mathbb{P}^x, \mathbb{E}^x$ to denote expectations assuming $S_0 = x$ or $\tilde{S}_0 = x$. We will write just $\mathbb{P}, \mathbb{E}$ for $\mathbb{P}^0, \mathbb{E}^0$.) It is easy to check that $P_0(x,y) = \delta(y-x) = p_0(x,y)$, and

$$P_{n+1}(x,y) - P_n(x,y) = \sum_{z \in \mathbb{Z}^d} p(z-x)\,[P_n(z,y) - P_n(x,y)] = \mathbf{L}_x P_n(x,y), \tag{1}$$

and similarly,

$$\dot{p}_t(x,y) = \mathbf{L}_x p_t(x,y).$$

We will write $P_n(x) = P_n(0,x), p_t(x) = p_t(0,x)$, and note that $P_n(x,y) = P_n(y-x), p_t(x,y) = p_t(y-x)$. We will write $X_1, X_2, \ldots$ for i.i.d. random variables with distribution p. Then $P_n(\cdot)$ is the distribution of $X_1 + \cdots + X_n$ and $p_t(\cdot)$ is the distribution of

$$X_1 + \cdots + X_{N_t},$$

where N_t is a Poisson process with mean 1 independent of $X_1, X_2, \ldots$. In fact, we can set

$$\tilde{S}_t = S_{N_t}.$$

There are a number of assumptions we can put on the increment distribution p.

- *(Discrete or continuous time) simple random walk in* $\mathbb{Z}^d$ is random walk with increment distribution $p(x) = 1/2d$ for all $|x| = 1$. Its Laplacian is given by

$$\begin{aligned} \mathbf{L} f(x) &= \frac{1}{2d} \sum_{|y|=1} [f(x+y) - f(x)] \\ &= \frac{1}{2d} \sum_{j=1}^{d} \left([f(x+e_j) - f(x)] - [f(x) - f(x-e_j)] \right). \end{aligned}$$

  This is a discrete analogue of the continuous Laplacian defined for $g : \mathbb{R}^d \to \mathbb{R}$:

$$\Delta g(x) = \sum_{j=1}^{d} \frac{\partial^2}{\partial x^j \partial x^j} g(x).$$

- If $\alpha > 0$, p is L^α if

$$\mathbb{E}[|X_1|^\alpha] = \sum_x p(x)\,|x|^\alpha < \infty.$$

  We write $L^{\alpha+} = \cap_{\beta > \alpha} L^\beta$ and $L^{\alpha-} = \cup_{\beta < \alpha} L^\beta$.

- p is L^∞ or has *finite range* if $\mathcal{G}_p$ is finite. We say that the range is bounded by R if $p(x) = 0$ for $|x| > R$.

- We write $\mathcal{E}_\alpha$ for the collection of p for which

$$\mathbb{E}[e^{\alpha|X_1|}] = \sum_x p(x)\,e^{\alpha|x|} < \infty.$$

  We also write

$$\mathcal{E}_\infty = \bigcap_{\alpha > 0} \mathcal{E}_\alpha, \quad \mathcal{E} = \mathcal{E}_{0+} = \bigcup_{\alpha > 0} \mathcal{E}_\alpha.$$

  We call $\mathcal{E}$ the set of increment distributions for which an *exponential moment* exists.

- If $\alpha \geq 1$, we let L_0^α be the set of p in L^α such that

$$\mathbb{E}[X_1] = \sum_x p(x)\,x = 0.$$

- If p is in L_0^2, we let $\Gamma = \Gamma_p = [\Gamma_{jk}]$ be the *covariance matrix*

$$\Gamma_{jk} = \sum_x p(x)\, x^j\, x^k.$$

Note that Γ is a strictly positive definite $d \times d$ matrix. We can write $\Gamma = O^T D O$, where O is orthogonal and D is a diagonal matrix with strictly positive entries. If $\sqrt{D}$ denotes the matrix whose entries are the nonnegative square roots of the entries of D and $\Lambda = \sqrt{D} O$, then $\Gamma = \Lambda^T \Lambda$. Note that Λ is an invertible $d \times d$ matrix. We let $Q = Q_p$ denote the corresponding quadratic form,

$$Q(\theta) = \theta \cdot \Gamma\theta = (\Lambda\theta)\cdot(\Lambda\theta) = \sum_{1 \leq j,k \leq d} \Gamma_{jk}\theta^j\theta^k = \mathbb{E}[(\theta \cdot X_1)^2].$$

Note that

$$\lambda_{\min}|\theta|^2 \leq Q(\theta) \leq \lambda_{\max}|\theta|^2,$$

where $0 < \lambda_{\min} \leq \lambda_{\max} < \infty$ are the smallest and largest eigenvalues of Γ. For simple random walk in $\mathbb{Z}^d$, $\Gamma = d^{-1} I$. We will also use $\mathcal{J} = \mathcal{J}_p$ to denote the norm

$$\mathcal{J}(x) = |x \cdot \Gamma^{-1} x|^{1/2} = |\Lambda^{-1} x|.$$

- p is *symmetric* if $p(x) = p(-x)$ for all x. If p is symmetric and L^1, then $p \in L_0^1$. However, it is possible for p to be symmetric but not in L^1. We say that p is *symmetric in each component* if $p(x^1,\ldots,x^j,\ldots,x^d) = (x^1,\ldots,-x^j,\ldots,x^d)$ for each $x = (x^1,\ldots,x^j)$ and each d. We say that p is *isotropic* if p is symmetric in each component and is invariant under permutation of the coordinates, i.e., for each $x = (x^1,\ldots,x^d)$ and each permutation σ of $\{1,\ldots,d\}$,

$$p(x^1,\ldots,x^d) = p(x^{\sigma(1)},\ldots,x^{\sigma(d)}).$$

If p is symmetric in each component and in L_0^2, then $\mathbb{E}[(X_1^1)^{j_1}(X_1^2)^{j_2}\cdots(X_1^d)^{j_d}]$ is zero provided the moment exists and at least one of the j_k is an odd integer, and hence, Γ is a diagonal matrix. If p is also isotropic, then Γ is a multiple of the identity. Note that simple random walk is isotropic.

- If p is an increment distribution, we let $p^*(x) = p(-x)$ be the increment distribution of the *reversed random walk*. We write $\mathbf{L}^*$ for the Laplacian of the reversed walk

$$\mathbf{L}^* f(x) = \sum_{y\in\mathbb{Z}^d} p^*(y)\,[f(x+y) - f(y)] = \sum_{y\in\mathbb{Z}^d} p(y)\,[f(x-y) - f(y)].$$

Note that p is symmetric if and only if $\mathbf{L} = \mathbf{L}^*$. Let $P_n^*(x,y)$ be the analogue of $P_n(x,y)$ for the reversed walk. Then it is easy to see that $P_n^*(x,y) = P_n(y,x)$. In particular,

$$P_{n+1}(x,y) - P_n(x,y) = P_{n+1}^*(y,x) - P_n^*(y,x) = \mathbf{L}_y^* P_n^*(y,x) = \mathbf{L}_y^* P_n(x,y). \qquad (2)$$

Remark. If d is a fixed positive integer and we wish to consider walks in $\mathbb{Z}^d$, we will sometime write $p \in L^\alpha(\mathbb{Z}^d)$ as a shorthand for "p is an increment distribution of a random walk in $\mathbb{Z}^d$ that is in L^α".

Remark. The condition $\sum p(x) = 1$ is not necessary for a continuous time random walk; however, there is no loss of generality in assuming it. Suppose p is a nonnegative function on $\mathbb{Z}^d$ with $\sum p(x) = M < \infty$, and $\tilde{S}_t$ is a continuous time Markov chain with rate $p(y)$ to jump from x to $x+y$. Then $\tilde{S}_{t/M}$ is a continuous time random walk with increment distribution p/M. If $\tilde{S}^{(1)},\ldots,\tilde{S}^{(d)}$ are independent one-dimensional continuous simple random walks and

$$Y_t = (\tilde{S}_t^{(1)},\ldots,\tilde{S}_t^{(d)}),$$

then Y_t is a continuous time Markov chain with rate $1/2$ of going from x to $x \pm e_j$ for each j. Hence $Y_{t/d}$ is a d-dimensional continuous time simple random walk.

Sometimes one wishes to consider random walks taking values in other lattices. A *(discrete) lattice* in $\mathbb{R}^d$ is an additive subgroup $\mathbb{L}$ such that for some $\epsilon > 0$, $\{0\} = \{x \in \mathbb{L} : |x| \leq \epsilon\}$.

Proposition 1.1 *If $\mathbb{L}$ is a lattice in $\mathbb{R}^d$, then there exists an integer $k \leq d$ and elements $x_1,\ldots,x_k \in \mathbb{L}$ that are linearly independent as vectors in $\mathbb{R}^d$ such that*

$$\mathbb{L} = \{j_1\,x_1 + \cdots + j_k\,x_k, \quad j_1,\ldots,j_k \in \mathbb{Z}\}.$$

Proof. Suppose first that $\mathbb{L}$ is contained in a one-dimensional subspace of $\mathbb{R}^d$. Choose $x_1 \in \mathbb{L}\setminus\{0\}$ with minimal distance from the origin. Clearly $\{jx_1 : j \in \mathbb{Z}\} \subset \mathbb{L}$. Also, if $x \in \mathbb{L}$, then $jx_1 \leq x < (j+1)x_1$ for some $j \in \mathbb{Z}$, but if $x > jx_1$, then $x - jx_1$ would be closer to the origin than x_1. Hence $\mathbb{L} = \{jx_1 : j \in \mathbb{Z}\}$.

More generally, suppose we have chosen linearly independent $x_1,\ldots,x_j$ such that the following holds: if $\mathbb{L}_j$ is the subgroup generated by $x_1,\ldots,x_j$, and V_j is the real subspace of $\mathbb{R}^d$ generated by the vectors $x_1,\ldots,x_j$, then $\mathbb{L}\cap V_j = \mathbb{L}_j$. If $\mathbb{L} = \mathbb{L}_j$, we stop. Otherwise, let $y \in \mathbb{L}\setminus\mathbb{L}_j$ and let

$$\begin{aligned} U &= \{ty : ty + t_1x_1 + \cdots + t_jx_j \in \mathbb{L} \text{ for some } t_1,\ldots,t_j \in \mathbb{R}\} \\ &= \{ty : ty + t_1x_1 + \cdots + t_jx_j \in \mathbb{L} \text{ for some } t_1,\ldots,t_j \in [0,1]\}. \end{aligned}$$

The second equality uses the fact that $\mathbb{L}$ is a subgroup. Using the first description, we can see that U is a subgroup, and using the second condition we see that U is a one-dimensional lattice, and hence we can choose $x_{j+1} = t_0 y \in U\setminus\{0\}$ so that $|x_{j+1}|$ is minimal. Let $\mathbb{L}_{j+1}, V_{j+1}$ be as above. We claim that $\mathbb{L}\cap V_{j+1} = \mathbb{L}_{j+1}$. Indeed, suppose that $z \in \mathbb{L}\cap V_{j+1}$, and write $z = s_0y + s_1x_1 + \cdots + s_jx_j$. Then $s_0y \in U$, and hence $s_0y = lx_{j+1}$ for some integer l. But, $z - ly_{j+1} \in V_j \cap \mathbb{L} = \mathbb{L}_j$. Hence $z \in \mathbb{L}_{j+1}$. □

Remark. The map $x_j \mapsto e_j$ gives an isomorphism between $\mathbb{L}$ and $\mathbb{Z}^k$. This can also be considered as a linear transformation from the real vector space generated by $\{x_1,\ldots,x_k\}$

to $\mathbb{R}^k$ taking basis vector x_j to basis vector e_j. The upshot of this is that random walks on the lattice $\mathbb{L}$ can be considered as random walks on $\mathbb{Z}^k$ by means of a linear transformation.

Remark. Conversely, if $\Lambda : \mathbb{R}^d \to \mathbb{R}^d$ is an invertible linear transformation, then $\mathbb{L} = \Lambda(\mathbb{Z}^d)$ is a d-dimensional lattice in $\mathbb{R}^d$. Sometimes it can be more useful to consider the random walk on the lattice $\mathbb{L}$. For example, if $p \in L_0^2(\mathbb{Z}^d)$ with covariance matrix Γ, by appropriate choice of Λ we can make the increment distribution mean zero with covariance matrix I.

Definition. The *period* of p is the greatest common divisor of the set $\{n : P_n(0) = 0\}$. Our assumptions imply that this set is infinite. We call p *aperiodic* if p has period one. An easy exercise shows that if p is aperiodic, then for every $x \in \mathbb{Z}^d$ there is an N_x such that $P_n(x) > 0$ for all $n \geq N_x$.

Note that if p is periodic of period $k > 1$, then $Y_n := S_{kn}$ is an aperiodic random walk on a sublattice. In the case of simple random walk this sublattice is

$$\{(x^1, \ldots, x^d) \in \mathbb{Z}^d : x^1 + \cdots + x^d \text{ is even}\}.$$

Hence, it really suffices to consider aperiodic random walks only. However, since the most common random walk, the simple random walk, is not aperiodic, it is useful not to restrict ourselves to this case. For continuous time random walks, periodicity is not an issue. This is one of the reasons tha people sometimes prefer continuous time simple random walk over discrete time walk. If p is an increment distribution and $\alpha \in (0,1)$ define p_α by

$$p_\alpha(x) = \alpha\, p(x) + (1-\alpha)\,\delta(0). \tag{3}$$

In other words, p_α is the increment distribution of the random walk with probability $1-\alpha$ that stays in place with probability $1-\alpha$ and with probability α moves according to the distribution p. Even if p is periodic, p_α is aperiodic.

Remark. Suppose $V = \{v_1, \ldots, v_j\} \subset \mathbb{Z}^d$ is a finite set that generates $\mathbb{Z}^d$ in the sense that every $x \in \mathbb{Z}^d$ can be written as $x = k_1 v_1 + \cdots + k_j v_j$, with $k_1, \ldots, k_j \in \{0, 1, 2, \ldots\}$. Then by writing $0 = v_1 - v_1 + v_2 - v_2 + \cdots + v_j - v_j$, we see that we can write

$$0 = k_1\, v_1 + \cdots + k_j v_j, \qquad k_1, \ldots, k_j \in \{1, 2, \ldots\}.$$

Therefore, we can find an increment distribution p such that $V = \{x : p(x) > 0\}$ and $\sum x p(x) = 0$; indeed, we just choose $p(v_l) = k_l/(k_1 + \cdots + k_j)$. As a consequence, if p is any increment distribution in $\mathbb{Z}^d$, we can find increment distributions p_1, p_2 and $\alpha \in (0,1)$ such that

$$p = \alpha\, p_1 + (1-\alpha)\, p_2, \tag{4}$$

where $p_1 \in L_0^\infty$ and $\{x : p_2(x) > 0\} = \{x : p(x) > 0\}$. Note also that if p is aperiodic, then we can choose a p_1 that is aperiodic. If $P_{n,1}, P_{n,2}$ denote the analogues of P_n using p_1, p_2, respectively, and M_n denotes a binomial random variable with parameters n and α, then

$$\begin{aligned} P_n(x) &= \sum_{j=0}^{n} \mathbb{P}\{M_n = j\} \sum_{y \in \mathbb{Z}^d} P_{j,1}(y) P_{n-j,2}(x-y) \qquad (5) \\ &= \sum_{j=0}^{n} \binom{n}{j} \alpha^j (1-\alpha)^{n-j} \sum_{y \in \mathbb{Z}^d} P_{j,1}(x-y) P_{n-j,2}(y), \end{aligned}$$

Remark. If $S_n = (S_n^{(1)}, \ldots, S_n^{(d)})$ is a d-dimensional random walk with increment distribution p, then $S_n^{(j)}$ is a random walk in $\mathbb{Z}$ with increment distribution

$$p^{(j)}(x^j) = \sum_{x^1, \ldots, x^{j-1}, x^{j+1}, \ldots x^d} p(x^1, \ldots, x^{j_1}, x^j, x^{j+1}, \ldots, x^d).$$

The "marginal" random walks $S_n^{(1)}, \ldots, S_n^{(d)}$ might not be independent; in fact, for (discrete time) simple random walk the marginals are dependent. Similarly, if $\mathbb{L}$ is a k-dimensional lattice in $\mathbb{R}^d$, the projection of S_n onto $\mathbb{L}$ gives a k-dimensional random walk on $\mathbb{L}$.

2 Strong Markov property

A *filtration* for the random walk S is any increasing sequence of σ-algebras $\{\mathcal{F}_n\}$ such that S_n is $\mathcal{F}_n$-measurable and $S_{n+1} - S_n, S_{n+2} - S_{n+1}, \ldots$ are independent of $\mathcal{F}_n$. Similarly a filtration for the random walk $\tilde{S}$ will be a right-continuous increasing family of σ-algebras $\mathcal{F}_t$ such that S_t is $\mathcal{F}_t$ measurable and $\{\tilde{S}_s - \tilde{S}_t : s \geq t\}$ is independent of $\mathcal{F}_t$. It is convenient to allow filtrations to contain "extra information", i.e., $\mathcal{F}_n$ can be larger than the σ-algebra generated by $S_0, \ldots, S_n$. A common example is to take a geometric random variable $\bar{T}$ independent of $S_0, S_1, \ldots$ and assume that the event $\{\bar{T} \leq n\}$ is in $\mathcal{F}_n$.

A *stopping time (with respect to the filtration $\{\mathcal{F}_n\}$)* is a random variable T taking values in $\{0, 1, \ldots\} \cup \{\infty\}$ such that $\{T = n\}$ is $\mathcal{F}_n$-measurable. Similarly a stopping time with respect to $\{\mathcal{F}_t\}$ is a $[0,\infty]$-valued random variable with $\{T \leq t\} \in \mathcal{F}_t$ for all t. (Since we are assuming $\mathcal{F}_t$ is right-continuous, this is the same as assuming $\{T < t\} \in \mathcal{F}_t$ for all t.) Let $\mathcal{F}_T$ denote the σ-algebra consisting of all events E such that $E \cap \{T \leq n\} \in \mathcal{F}_n$ for all n. We omit the proof of the following standard result. We state it for discrete walks, but the analogous result holds for continuous time walks. We will use this proposition throughout the book, often without making explicit reference to it.

Proposition 2.1 (Strong Markov Property) *If T is a stopping time with respect to the filtration $\{\mathcal{F}_n\}$, then, conditioned on the event $\{T < \infty\}$,*

$$Y_m := S_{m+T} - S_T$$

is a random walk with increment distribution p starting at the origin independent of $\mathcal{F}_T$.

An important class of stopping times are those associated to the first visit to a set $A \subset \mathbb{Z}^d$. Throughout this book, we will use the notation

$$\tau_A = \min\{n \geq 1 : S_n \in A\},$$

$$\bar\tau_A = \min\{n \geq 0 : S_n \in A\},$$
$$\tilde\tau_A = \inf\{t \geq 0 : \tilde S_t \in A\}.$$

If $A = \{y\}$, we write just $\tau_y, \bar\tau_y, \tilde\tau_y$. Note that τ_A and $\bar\tau_A$ agree if $S_0 \notin A$, but differ if $S_0 \in A$. For continuous time random walks, we do not need to make the distinction since, even if $\tilde S_0 \in A$, w.p.1[1]

$$\inf\{t \geq 0 : \tilde S_t \in A\} = \inf\{t > 0 : \tilde S_t \in A\}.$$

3 Green's function and generating functions

If p is an increment distribution of a random walk in $\mathbb{Z}^d$, $x, y \in \mathbb{Z}^d$ and $\xi \in \mathbb{C}$, we define the *Green's generating function* by

$$G(x,y;\xi) = \sum_{n=0}^{\infty} P_n(x,y)\,\xi^n.$$

We write just $G(x;\xi)$ for $G(0,x;\xi)$ and note that $G(x,y;\xi) = G(y-x;\xi)$ The radius of convergence of this series is greater than or equal to 1; in fact, the series converges absolutely for all $|\xi| \leq 1$. For $\xi \in (0,1)$, we can interpret $G(x,y;\xi)$ as the expected number of visits of a random walk starting at x to y (strictly) before time T_ξ, where T_ξ is a geometric random variable, independent of the random walk, with distribution $\mathbb{P}\{T_\xi > k\} = \xi^k, k = 0,1,2,\ldots$. To see this, let V denote the number of such visits. Then

$$\mathbb{E}^x[V] = \mathbb{E}^x[\sum_{n=0}^{T_\xi - 1} 1\{S_n = y\}] =$$

$$\mathbb{E}^x[\sum_{n=0}^{\infty} 1\{S_n = y; T > n\}] = \sum_{n=0}^{\infty} \mathbb{P}^x\{S_n = y; T > n\} = \sum_{n=0}^{\infty} P_n(y-x)\,\xi^n.$$

The *Green's function* is defined by

$$G(x,y) = G(y-x) = G(y-x;1) = \sum_{n=0}^{\infty} P_n(x,y).$$

This is the expected number of visits to y of a random walk starting at x. This can be finite or infinite. We will write G^* for the corresponding quantities for the reversed random walk, i.e., $G^*(x,y;\xi) = G(y,x;\xi)$. Similarly, we can define

$$\tilde G(x,y;\xi) = \int_0^\infty \xi^t\, p_t(x,y)\,dt.$$

[1] with probability one

For $\xi \in (0,1)$ this is the expected number of visits to y before an exponential random variable with rate $-\log(1-\xi)$. If we set $\xi = 1$, we get the same Green's function as the discrete walk, i.e., $\tilde G(x,y,1) = G(x,y)$. This is because the expected number of distinct visits to y is the same for the continuous time walk and the expected time in each of these distinct visits is one.

The strong Markov property implies that

$$G(x,y) = \mathbb{P}^x\{\bar\tau_y < \infty\}\,G(0) = \mathbb{P}^x\{\tilde\tau_y < \infty\}\,G(0).$$

Irreducibility of the random walk can be used to see that either $G(x,y)$ is infinite for all x,y or $G(x,y)$ is finite for all x,y. In the first case, we call the random walk *recurrent* and the second case the random walk is *transient*. Also,

$$G(0) = 1 + \mathbb{P}^x\{\tau_0 < \infty\}\,G(0),$$

and, hence,

$$\mathbb{P}\{\tau_0 = \infty\} = \frac{1}{G(0)}. \tag{6}$$

The left-hand side is positive if and only if the walk is transient. The quantity $\mathbb{P}\{\tau_0 = \infty\}$ is often call the *escape probability* and $\mathbb{P}\{\tau_0 < \infty\}$ is called the *return probability*.

If $n \geq 1$, define $f_n(x,y)$ to be the probability that a random walk starting at x first visits y at time n (not counting time $n = 0$), i.e.,

$$f_n(x,y) = \mathbb{P}^x\{S_n = y; S_1 \neq y, \ldots, S_{n-1} \neq y\} = \mathbb{P}^x\{\tau_y = n\}.$$

Let $f_n(x) = f_n(0,x)$ and note that

$$\mathbb{P}^x\{\tau_y < \infty\} = \sum_{n=1}^{\infty} f_n(x,y) = \sum_{n=1}^{\infty} f_n(y-x) \leq 1.$$

Define the *first visit generating function* by

$$F(x,y;\xi) = F(y-x;\xi) = \sum_{n=1}^{\infty} \xi^n f(y-x;\xi).$$

If $\xi \in (0,1)$, then

$$F(x,y;\xi) = \mathbb{P}^x\{\tau_y < T_\xi\}.$$

Proposition 3.1 *If $n \geq 1$,*

$$P_n(y) = \sum_{j=1}^{n} f_j(y)\,P_{n-j}(0).$$

If $\xi \in \mathbb{C}$,

$$G(y;\xi) = \delta(y) + F(y;\xi)\,G(0;\xi). \tag{7}$$

In particular, if $|F(y,\xi)| < 1$,

$$G(0;\xi) = \frac{1}{1 - F(0;\xi)}.$$

Proof. The first inequality follows from

$$\mathbb{P}^x\{S_n = y\} = \sum_{j=1}^n \mathbb{P}\{S_1 \neq y, \ldots, S_{j-1} \neq y, S_j = y, S_n - S_j = 0\}.$$

The second follows from the equality

$$\sum_{n=1}^\infty P_n(x)\,\xi^n = [\sum_{n=1}^\infty f_n(x)\,\xi^n]\,[\sum_{m=0}^\infty P_m(0)\,\xi^m],$$

which follows from the first equality. For $\xi \in (0,1]$, there is a probabilistic interpretation of (7): if $y \neq 0$, the expected number of visits to y is the probability of reaching y times the expected number of visits to y given that we reach y. If $y = 0$, we have to add an extra 1 to account for $P_0(y)$. □

Proposition 3.2 *If $d = 1$ and $\mathbb{E}[X_1] = 0$, then p is recurrent.*

Proof. Let

$$G_n(x) = \sum_{j=0}^n P_j(x).$$

It suffices to show that $G_n(0) \to \infty$ as $n \to \infty$. Note that if $x \neq 0$.

$$G_n(x) = \sum_{j=1}^n f_j(x)\,G_{n-j}(0) \leq G(0).$$

The strong law of large numbers tells us that $S_n/n \to 0$ w.p.1. Hence for every $\epsilon > 0$, for all n sufficiently large, with probability at least $1/2$, $|S_j| < \epsilon n$ for all $|j| \leq n$. This implies

$$\sum_{|x|<\epsilon n} G_n(x) \geq n/2,$$

which in term implies that $(2\epsilon n + 1)G_n(0) \geq n/2$. □

Remark. This result does not hold for $d > 1$. In fact, as we will show, all irreducible random walks in $\mathbb{Z}^d, d \geq 3$ are transient. For $d = 2$, there exist transient mean zero random walks although such walks are not in L^2.

3.1 A Tauberian Theorem

For random walks it is often easier to prove a result about the generating function of a sequence rather than the sequence itself. In this section we will prove a Tauberian theorem that relates a sequence to its generating function. A measurable function $L : [0,\infty) \to (0,\infty)$ is called *slowly varying (at infinity)* if for every $b > 0$,

$$L(bx) \sim L(x), \quad x \to \infty.$$

Lemma 3.3 *Suppose $L : [0,\infty) \to (0,\infty)$ is a slowly varying function.*

- *For every $b > 1$,*

$$\inf\{L(y) : x \leq y \leq bx\} \sim \sup\{L(y) : x \leq y \leq bx\}, \quad x \to \infty.$$

- *For every $\alpha > 0$, there is a $c = c_\alpha$ such that for all positive integers m, n,*

$$c^{-1}(m/n)^{-\alpha}\,L(n) \leq L(m) \leq c\,(m/n)^\alpha\,L(n).$$

Proof. It suffices to prove the first assertion for $b = 2$. Suppose $r > 1$ and let

$$I_N = \{t \in [1,4] : r^{-1}L(x) \leq L(tx) \leq rL(x) \text{ for all } x \geq N\}.$$

Since $\cup_N I_N = [1,4]$, we can find an $N = N_r$ such that the length of $[1,4] \setminus I_N$ is at most $1/4$. Suppose $x \geq N$ and $1 \leq s \leq 2$. Note that the set of $t \in [1,2]$ such that $st \notin I_N$ has length at most $1/4$. Hence there is a $t \in [1,2] \cap I_N$ such that $st \in I_N$ (in fact, the length of the set of such t is at least $1/2$). Therefore,

$$L(x) \leq r\,L(stx) \leq r^2\,L(sx),$$

and similarly $L(sx) \leq r^2\,L(x)$. Therefore,

$$r^{-2}\,L(x) \leq L(y) \leq r^2\,L(x), \quad x \geq N, x \leq y \leq 2x.$$

This proves the first assertion.

For the second assertion, note that there is an $N = N_\alpha$ such that for all $x \geq 2^N$ and $t \in [1,2]$,

$$2^{-\alpha}\,L(x) \leq L(tx) \leq 2^\alpha\,L(x).$$

It follows that there exist c_1, c_2 such that for all positive *integers* n, m with $n \leq m \leq 2n$,

$$c_1\,L(n) \leq L(m) \leq c_2\,L(n).$$

(For n sufficiently large, this follows from the previous assertion and there are only a finite number of other cases to consider. We can change the constant if necessary so it holds for all n, m with $n \leq m \leq 2n$). Let $r_k = 1 \vee [L(2^k)/L(2^{k-1})], q_k = 1 \wedge [L(2^k)/L(2^{k-1})]$, $c_3 = r_1 \cdots r_N, c_4 = q_1 \cdots q_N$. If $t_1, t_2 \in [1,2]$, $m = t_1 2^k, n = t_2 2^j, n \leq m$, then

$$L(m) = L(t_1\,2^k) \leq c\,L(2^k) \leq c\,c_3\,2^{(k-j)\alpha}\,L(2^j) \leq c\,2^{(k-j)\alpha}\,L(t_2 2^j) \leq c\,(m/n)^\alpha\,L(n).$$

Similarly, $L(n) \leq c\,(m/n)^\alpha L(m)$. □

Note that if $f(x) = x^\alpha\,L(x)$ where L is slowly varying and $\alpha \geq 0$, then for all $b > 0$,

$$\lim_{x\to\infty} \frac{f(bx)}{f(x)} = b^\alpha.$$

Lemma 3.4 *Suppose $\alpha > 0$, L is a slowly varying function, and u_n is a sequence of positive numbers such that*

$$u_n \sim n^{\alpha-1} L(n), \qquad n \to \infty.$$

Then,

$$\sum_{n=0}^{\infty} \xi^n u_n \sim \frac{\mathbf{\Gamma}(\alpha)}{(1-\xi)^\alpha} L\left(\frac{1}{1-\xi}\right), \quad \xi \to 1-.$$

Here Γ denotes the Gamma function.

Proof. Since $\sum u_n = \infty$, it suffices to prove that

$$\sum_{n=0}^{\infty} \xi^n n^{\alpha-1} L(n) \sim \frac{\mathbf{\Gamma}(\alpha)}{(1-\xi)^\alpha} L\left(\frac{1}{1-\xi}\right), \quad \xi \to 1-.$$

Let $\delta = 1-\xi$. For any integer R we can divide the sum on the left into three pieces: $n\delta < 1/R, 1/R \le n\delta \le R, n\delta > R$. Note that for fixed R, as $n \to \infty$,

$$\begin{aligned}\sum_{1/R \le n\delta \le R} \xi^n n^{\alpha-1} L(n) &\sim L(1/\delta)\,\delta^{-\alpha} \sum_{1/R\le n\delta\le R} [(1-\delta)^{1/\delta}]^{n\delta} (n\delta)^{\alpha-1}\,\delta \\ &\sim L(1/\delta)\,\delta^{-\alpha}\int_{1/R}^{R} e^{-x} x^{1-\alpha}\,dx.\end{aligned}$$

Let $0 < \beta < \alpha < 1$. By Lemma 3.3, there exist c_1, c_2 such that for all positive integers n, m,

$$c_1 (m/n)^{\beta-\alpha} \le L(m)/L(n) \le c_2 (m/n)^{\alpha-\beta}.$$

Therefore,

$$\sum_{n\delta<1/R} \xi^n n^{\alpha-1} L(n) \le c\,\delta^{-\alpha}\, L(1/\delta) \sum_{n\delta\le 1/R} (n\delta)^{\beta-1}\,\delta \le c\,\delta^{\alpha}\, L(1/\delta)\, R^{-\beta},$$

$$\sum_{n\delta>R} \xi^n n^{\alpha-1} L(n) \le c\,\delta^{-\alpha}\, L(1/\delta) \sum_{n\delta>R} e^{-n\delta}\,(n\delta)^{2\alpha-\beta}\,\delta \le c\,\delta^{\alpha}\, L(1/\delta)\, e^{-R/2}$$

Combining all of these estimates, we get

$$\sum_{n=0}^{\infty} \xi^n u_n \sim \delta^{-\alpha} L(1/\delta) \int_0^\infty e^{-x} x^{\alpha-1}\,dx = \mathbf{\Gamma}(\alpha)\,\delta^{-\alpha} L(1/\delta).$$

□

Theorem 3.5 *Suppose $\alpha > 0$, $L \ge 0$ is a slowly varying function, and u_n is a sequence of nonnegative real numbers. Then*

$$\sum_{n=0}^{\infty} \xi^n u_n \sim \frac{L((1-\xi)^{-1})}{(1-\xi)^\alpha}, \quad \xi \to 1-, \tag{8}$$

if and only if

$$\sum_{j=0}^{n-1} u_j \sim \frac{n^\alpha L(n)}{\mathbf{\Gamma}(\alpha+1)}, \quad n \to \infty. \tag{9}$$

If

$$u_n \sim \frac{n^{\alpha-1} L(n)}{\mathbf{\Gamma}(\alpha)}, \quad n \to \infty, \tag{10}$$

then (8) holds. If the sequence u_n is monotone, then (8) implies (10).

Remark. By approximating the sum by an integral, it is easy to check that if $u_n \sim n^{\alpha-1} L(n)$, then

$$\sum_{j=0}^{n-1} u_j \sim \alpha^{-1} n^\alpha L(n).$$

However, if $v_{2n} = u_{2n} + u_{2n+1}, v_{2n+1} = 0$, it is also true that

$$\sum_{j=0}^{n-1} v_j \sim \alpha^{-1} n^\alpha L(n).$$

Hence, (8) does not imply (10) in general.

Proof. [2] We have already remarked that (10) implies (9). Let $U_n = \sum_{j\le n} u_j$ with $U_{-1} = 0$. Summation by parts gives

$$\begin{aligned}\sum_{n=0}^{\infty} \xi^n u_n &= \sum_{n=0}^{\infty} \xi^n [U_n - U_{n-1}] \\ &= \sum_{n=0}^{\infty} \xi^n U_n - \sum_{n=0}^{\infty} \xi^{n+1} U_n = (1-\xi)\sum_{n=0}^{\infty} \xi^n U_n.\end{aligned}$$

Hence Lemma 3.4 shows that (9) implies (8).

We need to show that (8) implies (9), so we will assume that (8) holds. If μ is a positive measure on $[0,\infty)$, Let $F(r;\mu)$ denote the Laplace transform,

$$F(r;\mu) = \int_0^\infty e^{-rx}\,\mu(dx).$$

[2] This proof is adapted from Feller, Vol. II, Chap. XIII.

If $F(r;\mu) < \infty$, we let μ_r denote the probability measure whose density with respect to μ is $e^{-rx}/F(r;\mu)$. Note that $F(r';\mu_r) = F(r+r';\mu)/F(r;\mu)$.

If μ^1, μ^2 are two measures such that for some $r_0 < \infty$, $F(r;\mu^1) = F(r;\mu^2) < \infty$ for all $r > r_0$, then $\mu^1 = \mu^2$. To see this, note that if suffices to show that $\mu^1_r = \mu^2_r$ for some $r > r_0$. But these are probability measures for which the Laplace transform exists in a neighborhood of the origin and it is well known that the transform determines a probability measure if the transform exists in a neighborhood fo the origin. Similarly, if $\{\mu^k\}$ is a sequence of measures such that $F(r;\mu^k) \to F(r)$ for all $r > r_0$, then $F(r) = F(r;\mu)$ for some μ and $\mu^k \Longrightarrow \mu$; again, this can be proved by considering μ^k_r for some $r > r_0$.

We now suppose that $\mu(dx) = g(x)\,dx$ where g is a nonnegative function with

$$F(r;\mu) = \int_0^\infty e^{-rx}\, g(x)\, dx \sim r^{-\alpha}\, L(1/r), \quad r \to 0+. \tag{11}$$

Let $\mu_b(dx) = g(x/b)/[bF(b;\mu)]\ dx.$, and note that $F(r;\mu_b) = F(br;\mu)/F(b;\mu)$. Therefore, for $r > 0$,

$$\lim_{b\to 0+} F(r;\mu_b) = r^{-\alpha} = \int_0^\infty \frac{x^{\alpha-1}}{\mathbf{\Gamma}(\alpha)} e^{-rx}\, dx.$$

This implies that as $b \to 0+$, μ_b converges to the measure with density $\mathbf{\Gamma}(\alpha)^{-1}\, x^{\alpha-1}$. In particular,

$$\lim_{b\to\infty} \mu_b[0,1] = \int_0^1 \frac{x^{\alpha-1}}{\mathbf{\Gamma}(\alpha)}\, dx = \frac{1}{\mathbf{\Gamma}(\alpha+1)}.$$

But,

$$\mu_b[0,1] = F(b;\mu)^{-1} \int_0^{1/b} u(x)\, dx.$$

Hence (11) implies

$$\int_0^T g(x)\, dx \sim \frac{T^\alpha\, L(T)}{\mathbf{\Gamma}(\alpha+1)}, \quad T \to \infty. \tag{12}$$

If u_n is a sequence of non-negative numbers, we associate the function

$$g(x) = \sum_{n=0}^\infty u_n\, 1_{[n,n+1)}(x).$$

Then if $\xi = e^{-r}$,

$$\sum_{n=0}^\infty \xi^n\, u_n = \frac{1-e^{-r}}{r} F(\mu;r), \qquad \sum_{j=0}^{n-1} u_n = \int_0^n g(x)\, dx.$$

Hence the implication (11) $\Longrightarrow$ (12) yields the implication (8) $\Longrightarrow$ (9).

If the u_n are monotonically increasing, then for all n, k,

$$(k+1)\, u_n \le \sum_{j=n}^{n+k} u_j \le (k+1)\, u_{n+k}.$$

and the reversed inequalities hold if u_n are monotonically decreasing. For fixed $\epsilon > 0$, (9) implies

$$\sum_{n\le j\le(1+\epsilon)n} u_j \sim \frac{n^\alpha\, L(n)}{\mathbf{\Gamma}(\alpha+1)}\, [(1+\epsilon)^\alpha - 1], \quad n \to \infty.$$

By letting $n \to \infty$ and then letting $\epsilon \to 0+$, we get (10) (recall that $\mathbf{\Gamma}(\alpha+1) = \alpha\mathbf{\Gamma}(\alpha)$). $\square$

4 Stirling's Formula

Because we will need these later, we include some standard expansions. Suppose $t \ge 1$, $r \in (0,1)$ and and ξ is a complex number with $|\xi| \le rt$. If log denotes the branch of the logarithm with $\log 1 = 0$, we can write

$$\log(1+\frac{\xi}{t})^t = \xi - \frac{\xi^2}{2t} + \frac{\xi^3}{3t^2} + \cdots + (-1)^{k+1}\frac{\xi^k}{kt^{k-1}} + O\left(\frac{|\xi|^{k+1}}{t^k}\right),$$

$$(1+\frac{\xi}{t})^t = e^\xi\ \exp\left\{-\frac{\xi^2}{2t} + \frac{\xi^3}{3t^2} + \cdots + (-1)^{k+1}\frac{\xi^k}{kt^{k-1}} + O\left(\frac{|\xi|^{k+1}}{t^k}\right)\right\}. \tag{13}$$

The implicit constant in the $O(\cdot)$ term depend on r and k but not on ξ, t. Therefore, if $r \in (0,1), R < \infty, |\xi| \le rt, |\xi|^2 \le Rt$, we can write

$$(1+\frac{\xi}{t})^t = e^\xi \left[1 - \frac{\xi^2}{2t} + \frac{8\xi^3 + 3\xi^4}{24t^2} + \cdots + \frac{f_k(\xi)}{t^{k-1}} + O\left(\frac{|\xi|^{2k+2}}{t^k}\right)\right], \tag{14}$$

where f_k is a polynomial of degree at most $2k$ and the constant in the $O(\cdot)$ term depends only on r, R and k.

Lemma 4.1 (Stirling's Formula with estimate) *If n is a positive integer,*

$$n! = \sqrt{2\pi}\, n^{n+(1/2)}\, e^{-n}\ \left[1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right].$$

Proof. Let $b_n = n^{n+(1/2)}e^{-n}/n!$. Then,

$$\begin{aligned} b_{n+1}/b_n &= e^{-1}\,[1+\frac{1}{n}]^n\,[1+\frac{1}{n}]^{1/2} \\ &= \left[1 - \frac{1}{2n} + \frac{11}{24n^2} + O\left(\frac{1}{n^3}\right)\right]\ \left[1 + \frac{1}{2n} - \frac{1}{8n^2} + O\left(\frac{1}{n^3}\right)\right] \\ &= 1 + \frac{1}{12n^2} + O\left(\frac{1}{n^3}\right). \end{aligned}$$

Therefore,

$$\lim_{m\to\infty} \frac{b_m}{b_n} = \prod_{l=n}^\infty \left[1 + \frac{1}{12l^2} + O\left(\frac{1}{l^3}\right)\right] = 1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right).$$

This establishes that

$$b_n = C\, n^{n+(1/2)}\, e^{-n} \left[1 - \frac{1}{12n} + O\left(\frac{1}{n^2}\right)\right],$$

for some positive constant C. There are a number of ways to determine the constant. For example, the probability that a simple one-dimensional random walk is at point $2k$ at time $2n$ is

$$4^{-n}\binom{2n}{n+k},$$

and hence we can compute the limit as $n \to \infty$ of the probability that the walk takes value between 0 and $2\sqrt{n}$ in terms of C. However, we can also obtain the asymptotic value of this probability by the usual central limit theorem, so comparing these results gives C. $\square$

As an application, we prove the local central limit theorem for one-dimensional simple random walk. Although the result is stated for $|k| \le n$, it is most useful for $|k| \le n^{3/4}$ where

$$\exp\left\{O\left(\frac{1}{n}\right) + O\left(\frac{k^4}{n^3}\right)\right\} = 1 + O\left(\frac{1}{n}\right) + O\left(\frac{k^4}{n^3}\right).$$

Proposition 4.2 *For simple random walk in $\mathbb{Z}$, if n is a positive integer and k is an integer with $|k| \le n$,*

$$\mathbb{P}\{S_{2n} = 2k\} = \frac{1}{\sqrt{\pi n}}\, e^{-k^2/n} \exp\left\{O\left(\frac{1}{n}\right) + O\left(\frac{k^4}{n^3}\right)\right\}.$$

Proof. If $n/8 \le |k| \le n$, the result is immediate using only the estimate $2^{-2n} \le \mathbb{P}\{S_{2n} = 2k\} \le 1$. Hence, we may assume that $|k| \le n/8$. Note that

$$\mathbb{P}\{S_{2n} = 2k\} = 2^{-2n}\binom{2n}{n+k} = \frac{(2n)!}{2^{2n}(n+k)!(n-k)!}.$$

If we restrict to $|k| \le n/2$, we can use Stirling's formula (Lemma 4.1) to see that

$$\mathbb{P}\{S_{2n} = 2k\} = \left[1 + O\left(\frac{1}{n}\right)\right] \frac{1}{\sqrt{\pi n}} \left(1 - \frac{k^2}{n^2}\right)^{-1/2} \left(1 - \frac{k^2}{n^2}\right)^{-n} \left(1 - \frac{2k}{n+k}\right)^{k}.$$

Using (13) we get,

$$\left(1 - \frac{k^2}{n^2}\right)^{n} = e^{-k^2/n} \exp\left\{O\left(\frac{k^4}{n^3}\right)\right\}.$$

$$\begin{aligned}\left(1 - \frac{2k}{n+k}\right)^{k} &= e^{-2k^2/(n+k)} \exp\left\{\frac{2k^3}{(n+k)^2} + O\left(\frac{k^4}{n^3}\right)\right\}\\ &= e^{-2k^2/(n+k)} \exp\left\{-\frac{2k^3}{n^2} + O\left(\frac{k^4}{n^3}\right)\right\},\end{aligned}$$

$$e^{-2k^2/(n+k)} = e^{-2k^2/n} \exp\left\{\frac{2k^3}{n^2} + O\left(\frac{k^4}{n^3}\right)\right\}.$$

Combining all of this gives the lemma. $\square$

5 Characteristic function

If X is a random variable taking values in $\mathbb{Z}^d$, its *characteristic function* $\phi_X : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\phi_X(\theta) = \mathbb{E}[e^{iX\cdot\theta}] = \sum_{x\in\mathbb{Z}^d} \mathbb{P}\{X = x\}\, e^{ix\cdot\theta}.$$

Note that ϕ_X is periodic of period 2π in each component, i.e.,

$$\phi_X(\theta^1, \ldots, \theta^j, \ldots, \theta^d) = \phi_X(\theta^1, \ldots, \theta^j + 2\pi, \ldots, \theta^d).$$

Hence we can consider ϕ_X as a function on the torus $\mathbb{T}^d := [-\pi, \pi]^d$, with boundaries identified appropriately. We will write $\theta = (\theta^1, \ldots, \theta^d)$ for elements of $\mathbb{T}^d$ and we will write

$$d\theta = d\theta^1\, d\theta^2 \cdots d\theta^d.$$

Lemma 5.1 *Let $X = (X^1, \ldots, X^d)$ be a random variable in $\mathbb{Z}^d$, and let $\phi = \phi_X$.*

- *ϕ is uniformly continuous, $|\phi| \le 1$, $\phi(0) = 1$.*
- *For every $x \in \mathbb{Z}^d$,*
$$\mathbb{P}\{X = x\} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-ix\cdot\theta}\, \phi(\theta)\, d\theta. \tag{15}$$
- *If k is a positive integer and $\mathbb{E}[|X|^k] < \infty$, then ϕ is C^k and the derivatives at zero are given by the moments of X, i.e., if $l \le k$,*
$$\partial_{x^{j_1}} \partial_{x^{j_2}} \cdots \partial_{x^{j_l}} \phi(0) = i^l\, \mathbb{E}[X^{j_1} X^{j_2} \cdots X^{j_l}].$$
Moreover, if h_k denotes the kth order Taylor approximation of ϕ about the origin and $\mathbb{E}[|X|^{k+r}] < \infty$ for some $r \in [0, 1)$.
$$\phi(\theta) = h_k(\theta) + o(|\theta|^{k+r}), \quad \theta \to 0.$$

Proof. The first and third assertions can be found in any book on probability (and hold for random variables taking values in $\mathbb{R}^d$). To establish (15), note that

$$\begin{aligned}\int_{\mathbb{T}^d} e^{-ix\cdot\theta}\, \phi(\theta)\, d\theta &= \int_{\mathbb{T}^d} e^{-ix\cdot\theta}\, [\sum_{y\in\mathbb{Z}^d} \mathbb{P}\{X = y\} e^{iy\cdot\theta}]\, d\theta\\ &= \sum_{y\in\mathbb{Z}^d} \mathbb{P}\{X = y\} \int_{\mathbb{T}^d} e^{-i(y-x)\cdot\theta}\, d\theta = (2\pi)^d\, \mathbb{P}\{X = x\}.\end{aligned}$$

The interchange of the sum and integral is justified because the function $(\theta, y) \mapsto \mathbb{P}\{X = y\} e^{i(y-x)\cdot\theta}$ on $\mathbb{T}^d \times \mathbb{Z}^d$, is bounded by the L^1 function $(\theta, y) \mapsto \mathbb{P}\{X = y\}$. $\square$

If p is an increment distribution of a random walk in $\mathbb{Z}^d$, we let $\phi = \phi_p$ denote the characteristic function of X_1, i.e.,

$$\phi(\theta) = \sum_{x \in \mathbb{Z}^d} p(x)\, e^{ix\cdot\theta}.$$

If p is simple random walk in $\mathbb{Z}^d$, then

$$\phi(\theta^1, \ldots, \theta^d) = \frac{1}{d}\sum_{j=1}^{d} \cos\theta^j.$$

If $S_0 = \tilde{S}_0 = 0$, then the characteristic function of S_n is $\phi(\theta)^n$ and the characteristic function of $\tilde{S}_t$ is

$$\mathbb{E}[e^{i\tilde{S}_t\cdot\theta}] = \sum_{n=0}^{\infty} \frac{t^n\, e^{-t}}{n!} \mathbb{E}[e^{iS_n\cdot\theta}] = \tilde{\phi}(\theta)^n,$$

where $\tilde{\phi}(\theta) = \exp\{\phi(\theta) - 1\}$ is the characteristic function for $\tilde{S}_1$. Therefore

$$P_n(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-ix\cdot\theta}\, \phi(\theta)^n\, d\theta, \tag{16}$$

$$p_t(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-ix\cdot\theta}\, \tilde{\phi}(\theta)^t\, d\theta = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-ix\cdot\theta}\, \exp\{t[\phi(\theta)-1]\}\, d\theta. \tag{17}$$

By summing over n, we get

$$G(x;\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} e^{-ix\cdot\theta}\,\frac{1}{1-\xi\phi(\theta)}\, d\theta, \quad |\xi| < 1.$$

If $x = 0$, we get

$$G(0) = \lim_{\xi\to 1-} G(0;\xi) = \lim_{\xi\to 1-} \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\xi\phi(\theta)}\right] d\theta.$$

(Since $G(0;\xi)$ is nonnegative for $\xi \in (0,1)$, it suffices to consider the integral of the real part of the integrand.) Since the escape probability of the random walk is $1/G(0)$, we immediately get the following.

Proposition 5.2 *The random walk is transient if and only if*

$$\lim_{\xi\to 1-}\int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\xi\,\phi(\theta)}\right] d\theta < \infty,$$

in which case the escape probabiilty is given by

$$(2\pi)^d\left[\lim_{\xi\to 1-}\int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\xi\phi(\theta)}\right] d\theta\right]^{-1}.$$

Remark. It is also true that the random walk is transient if and only if $\mathrm{Re}[1/(1-\phi)]$ is Lebesgue integrable. One direction is straightfoward using Fatou's lemma,

$$\begin{aligned}\int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\phi(\theta)}\right] d\theta &\leq \lim_{\xi\to 1-}\int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\xi\,\phi(\theta)}\right] d\theta \\ &= \lim_{\xi\to 1-} (2\pi)^d\, G(0;\xi) = (2\pi)^d\, G(0).\end{aligned}$$

The converse is more difficult and we will not do it here. It is not always true that the inequality above can be replaced with an equality. We define a constant associated to the random walk,

$$\hat{k} := \frac{1}{(2\pi)^d}\left[\lim_{\xi\to 1-}\int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\xi\,\phi(\theta)}\right] d\theta - \int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\phi(\theta)}\right] d\theta\right] \in [0, G(0)]. \tag{18}$$

If $d \geq 2$, $\hat{k} = 0$, see Proposition 6.16.

It is important that we are considering $\mathrm{Re}[1/(1-\phi)]$ rather than $1/(1-\phi)$. For example, suppose $p \in L^2(\mathbb{Z})$ with mean $\mu \neq 0$. Then $\phi(\theta) = 1 + i\mu\theta + O(\theta^2)$, $\theta \to 0$, and hence $1/(1-\phi)$ is not Lebesgue integrable.

Proposition 5.3 *Suppose $p \in L_0^{k+r}$ with covariance matrix Γ where $k \geq 2$ is an integer and $r \in [0,1)$. Then,*

$$\phi(\theta) = 1 - \frac{1}{2}(\theta\cdot\Gamma\theta) + g_3(\theta) + \cdots + g_k(\theta) + o(|\theta|^{k+r}), \quad \theta \to 0,$$

$$\tilde{\phi}(\theta) = 1 - \frac{1}{2}(\theta\cdot\Gamma\theta) + \tilde{g}_3(\theta) + \cdots + \tilde{g}_k(\theta) + o(|\theta|^{k+r}), \quad \theta \to 0,$$

where $g_j, \tilde{g}_j, j = 3, \ldots, k$ are homogeneous polynomials[3] *of degree j. The polynomials $g_j, \tilde{g}_j$ are determined by the moments of p of degree no more than j. If p is symmetric, then $g_j \equiv \tilde{g}_j \equiv 0$ for odd j.*

Proof. This is essentially a corollary of Lemma 5.1. Note that if p is symmetric in each of the components, then all of the odd-order coefficients in the expansion of p about 0 vanish. □

Corollary 5.4 *Suppose $p \in L_0^{k+r}$ with covariance matrix Γ where $k \geq 2$ is an integer and $r \in [0,1)$. Then if n is a positive integer and $|\theta| \leq n^{1/4}$,*

$$\phi\left(\frac{\theta}{\sqrt{n}}\right)^n = e^{-\theta\cdot\Gamma\theta/2}\left[1 + \frac{f_3(\theta)}{\sqrt{n}} + \frac{f_4(\theta)}{n} + \cdots + \frac{f_k(\theta)}{n^{(k-2)/2}} + o\left(\frac{|\theta|^{k+r}}{n^{(k+r-2)/2}}\right)\right],$$

[3] That is, $g_j(r\theta) = r^j g_j(\theta)$ for all $r > 0$, or, in other words, g_j is a nonempty sum of monomials of degree j.

$$\tilde{\phi}\left(\frac{\theta}{\sqrt{n}}\right)^n = e^{-\theta\cdot\Gamma\theta/2}\left[1+\frac{\tilde{f}_3(\theta)}{\sqrt{n}}+\frac{\tilde{f}_4(\theta)}{n}+\cdots+\frac{\tilde{f}_k(\theta)}{n^{(k-2)/2}}+o\left(\frac{|\theta|^{k+r}}{n^{(k+r-2)/2}}\right)\right],$$

where $f_j, \tilde{f}_j$ are polynomials of degree at most $2j$ that are determined by the moments of p of degree less than or equal to j. The polynomials $f_3, \tilde{f}_3$ are odd. If p is symmetric, then $f_j \equiv \tilde{f}_j \equiv 0$ for odd j.

Proof. This follows immediately from Proposition 5.3 and (14).

Proposition 5.5 *If p is aperiodic, then there is a $b > 0$ such that*

$$|\phi(\theta)| \le 1 - b|\theta|^2, \qquad \theta \in \mathbb{T}^d. \tag{19}$$

Proof. We wiil first prove that $|\phi(\theta)| < 1$ for all $\theta \in \mathbb{T}^d \setminus \{0\}$. To see this, suppose $|\phi(\theta)| = 1$. then $|\phi(\theta)|^n = 1$ for all n. Since

$$\phi(\theta)^n = \sum_{x\in\mathbb{Z}^d} P_n(x)\, e^{ix\cdot\theta} = P_n(0) + \sum_{x\neq 0} P_n(x)\, e^{ix\cdot\theta},$$

and for fixed x, $P_n(x) > 0$ for all n sufficiently large, this implies $e^{ie_j\cdot\theta} = e^{i\theta^j} > 0$ for $j = 1, \ldots, d$. This is true only for $\theta = 0$.

It follows that for every $\epsilon > 0$, there is a $\delta > 0$ such that $|\phi(\theta)| \le 1 - \delta$ if $\theta \in \mathbb{T}^d, \mathrm{dist}(\theta, 0) \ge \epsilon$. Hence, it suffices to prove (19) in a neighborhood of the origin to conclude (19) for all θ, perhaps with a different value of b. If $p \in L_0^\infty$, we can use the expansion of the characteristic function about the origin. For general p, write $p = \alpha p_1 + (1-\alpha)p_2$ as in (4), where p_1 is aperiodic, mean zero, with finite range. If ϕ_1, ϕ_2 denote the characteristic functions of p_1, p_2, respectively, then $\phi(\theta) = \alpha\phi_1(\theta) + (1-\alpha)p_2(\theta)$. If $|\phi_1(\theta)| \le 1 - b|\theta|^2$, then $|\phi(\theta)| \le 1 - \alpha b|\theta|^2$. □

Remark. For simple random walk with $\phi(\theta) = d^{-1}\sum\cos\theta^j$ we have $\phi(\pi, \pi, \ldots, \pi) = -1$. In general, if p has period k, then there will be a $\theta \in \mathbb{T}^d$ with $\phi(\theta) = e^{2\pi i/k}$. For any increment distribution p, $\tilde{\phi}$ is the characteristic function of an aperiodic increment distribution and hence $|\tilde{\phi}(\theta)| < 1$ for $\theta \in \mathbb{T}^d \setminus \{0\}$.

6 Local central limit theorem

The most fundamental result about random walks with increment distributions in L_0^2 is the local central limit theorem. If $X_1, X_2, \ldots$ are i.i.d. random variables with distribution $p \in L_0^2$, then the standard central limit theorem states that $[X_1 + \cdots + X_n]/\sqrt{n}$ converges in distribution to a normal random variable with mean zero and covariance matrix Γ, i.e., to a random variable with density

$$f(s;\Gamma) := \frac{1}{(2\pi)^{d/2}(\det\Gamma)^{1/2}}\exp\left\{-\frac{1}{2}(s\cdot\Gamma^{-1}s)\right\}, \qquad s \in \mathbb{R}^d.$$

Let

$$\bar{P}_t(x) = \frac{1}{t^{d/2}}\,\frac{1}{(2\pi)^{d/2}(\det\Gamma)^{1/2}}\,e^{-\mathcal{J}(x)^2/2t},$$

where, as before,

$$\mathcal{J}(x) = |x\cdot\Gamma^{-1}x|^{1/2}.$$

Then the local central limit theorem roughly states, if p is aperiodic and L_0^2, then as $n \to \infty$, $\bar{P}_n(x)$ is a good approximation of $P_n(x)$ and $p_n(x)$. In this section we will give some precise versions of this result. The exact nature of the statement and the bounds we can give on the error will depend on the assumptions on p.

Lemma 6.1 *If $p \in L_0^2$ with covariance matrix $\Gamma = \Lambda^T\Lambda$ and $x \in \mathbb{Z}^d$,*

$$\int_{\mathbb{R}^d} e^{-(\theta\cdot\Gamma\theta)/2}\, e^{-i\theta\cdot x}\, d\theta = \frac{(2\pi)^{d/2}}{(\det\Gamma)^{1/2}}\, e^{-\mathcal{J}(x)^2/2}. \tag{20}$$

Moreover, if f is a polynomial of degree k, there is a $c = c(f, \Gamma) < \infty$ such that for all x,

$$\left|\int_{\mathbb{R}^d} f(\theta)\, e^{-(\theta\cdot\Gamma\theta)/2}\, e^{i\theta\cdot x}\, d\theta\right| \le c\,(|x|^k + 1)\, e^{-\mathcal{J}(x)^2/2}. \tag{21}$$

Proof. All the integrals in this proof will be over $\mathbb{R}^d$. Let f be a polynomial of degree k. The substitution $\theta_1 = \Lambda\theta$ gives

$$\int f(\theta)\, e^{-(\theta\cdot\Gamma\theta)/2}\, e^{i\theta\cdot x}\, d\theta = [\det\Lambda]^{-1}\int g(\theta_1)\, e^{-|\theta_1|^2/2}\, e^{i(\theta_1\cdot\Lambda^{-1}\theta_1)}\, d\theta_1,$$

whre g is the k^{th} degree polynomial $g(\theta_1) = f(\Lambda^{-1}\theta_1)$. By completing the square we see that the integral on the right equals

$$e^{-(x\cdot\Gamma^{-1}x)/2}\int g(\theta_1)\,\exp\left\{\frac{1}{2}(i\theta_1 + \Lambda^{-1}x)\cdot(i\theta_1 + \Lambda^{-1}x)\right\}\, d\theta_1.$$

The substitution $\theta_2 = \theta_1 - i\Lambda^{-1}x$ converts this to

$$e^{-\mathcal{J}(x)^2/2}\int g(\theta_2 + i\Lambda^{-1}x)\, e^{-|\theta_2|^2/2}\, d\theta_2.$$

If $f \equiv 1$, then $g \equiv 1$ and the last integral equals $(2\pi)^{d/2}$. This gives (20). The inequality

$$|g(\theta_2 + i\Lambda^{-1}x)| \le c\,(|x|^k + 1)\,(|\theta_2|^k + 1)$$

yields (21). □

Theorem 6.2 *Suppose $p \in L_0^k(\mathbb{Z}^d)$ is aperiodic where $k \geq 2$ is an integer and $0 \leq r < 1$. Let Γ denote the covariance matrix and define $\epsilon_{n,k}(x)$ by*

$$P_n(x) = \overline{P}_n(x) \left[1 + \frac{u_3(x/\sqrt{n})}{\sqrt{n}} + \frac{u_4(x/\sqrt{n})}{n} + \cdots + \frac{u_k(x/\sqrt{n})}{n^{(k-2)/2}}\right] + \epsilon_{n,k}(x),$$

where

$$\overline{P}_n(x) = \frac{1}{(\det\Gamma)^{1/2}\,(2\pi n)^{d/2}} \exp\left\{-\frac{\mathcal{J}(x)^2}{2n}\right\}, \tag{22}$$

$$u_j(z) = \frac{e^{z\cdot\Gamma^{-1}z/2}}{(2\pi)^{d/2}\sqrt{\det\Gamma}} \int_{\mathbb{R}^d} f_k(\theta)\, e^{-\theta\cdot\Gamma\theta/2}\, e^{-iz\cdot\theta}\, d\theta,$$

and $f_3, \ldots, f_k$ are as in Corollary 5.4. Then,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}^d} n^{(d/2)+((k+r-2)/2)}\, |\epsilon_{n,k}(x)| = 0.$$

Moreover, there is a $c < \infty$ such that

$$|u_j(z)| \leq c\, \left(|z|^j + 1\right),$$

and if p is symmetric, then $u_j \equiv 0$ for odd j.

Proof. By (16),

$$P_n(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \phi(\theta)^n\, e^{-ix\cdot\theta}.$$

Let b be as in (19). Then,

$$|\phi(\theta)|^n \leq (1 - b|\theta|^2)^n \leq e^{-b|\theta|^2 n}.$$

Hence ,

$$P_n(x) = O(\exp\{-bn^{1/4}\}) + \frac{1}{(2\pi)^d} \int_{|\theta|\leq n^{-3/8}} \phi(\theta)^n\, e^{-ix\cdot\theta}\, d\theta. \tag{23}$$

We will include the $O(\exp\{-bn^{1/4}\})$ term in the $\epsilon_{n,k}(x)$ error term. We now consider the integral on the right hand side above which, after a change of variables, becomes

$$\frac{1}{n^{d/2}} \int_{|\theta|\leq n^{1/8}} \phi\left(\frac{\theta}{\sqrt{n}}\right)^n\, e^{-i\tilde{x}\cdot\theta}\, d\theta,$$

where $\tilde{x} = x/\sqrt{n}$. We now expand $\phi(\theta/\sqrt{n})^n$ as in Corollary 5.4. The part of the integral with the $o(\cdot)$ can be estimated:

$$\left|\int_{|\theta|\leq n^{1/8}} o\left(\frac{|\theta|^{k+r}}{n^{(k+r-2)/2}}\right) e^{-\theta\cdot\Gamma\theta/2}\, e^{-i\tilde{x}\cdot\theta}\, d\theta\right| \leq \frac{o(1)}{n^{(k+r-2)/2}} \int_{\mathbb{R}^d} [|\theta|^{k+1} + 1]\, e^{-\theta\cdot\Gamma\theta/2} = \frac{o(1)}{n^{(k+r-2)/2}}.$$

We put this error into the $\epsilon_{n,k}(x)$ term. The remaining part of the integral is

$$\int_{|\theta|\leq n^{1/8}} e^{-\theta\cdot\Gamma\theta/2} \left[1 + \frac{f_3(\theta)}{\sqrt{n}} + \cdots + \frac{f_k(\theta)}{n^{(k-2)/2}}\right] e^{-i\tilde{x}\cdot\theta}\, d\theta.$$

If we change the integrals to all of $\mathbb{R}^d$ we will add a term of order $o(e^{-n^{1/8}})$. Let us do that and include the error in the $\epsilon_{k,n}(x)$ term. This gives the expansion and the remaining assertions follow from Lemma 6.1. □

The continuous time version of the local central limit theorem has an identical proof so we just state the result. Note that we do not need an aperiodicity assumption.

Theorem 6.3 *Suppose $p \in L_0^{k+r}(\mathbb{Z}^d)$ where $k \geq 2$ is an integer and $r \in [0, 1)$. Let Γ denote the covariance matrix and define $\epsilon_{t,k}(x)$ by*

$$p_t(x) = \overline{P}_t(x) \left[1 + \frac{\tilde{u}_3(x/\sqrt{t})}{\sqrt{t}} + \frac{\tilde{u}_4(x/\sqrt{t})}{t} + \cdots + \frac{\tilde{u}_k(x/\sqrt{t})}{t^{(k-2)/2}}\right] + \epsilon_{t,k}(x),$$

where

$$\overline{P}_t(x) = \frac{1}{(\det\Gamma)^{1/2}\,(2\pi t)^{d/2}} \exp\left\{-\frac{\mathcal{J}(x)^2}{2t}\right\},$$

$$\tilde{u}_j(z) = (2\pi)^{d/2}\, e^{(\tilde{x}\cdot\Gamma^{-1}x)/2} \int_{\mathbb{R}^d} \tilde{f}_k(\theta)\, e^{-(\theta\cdot\Gamma\theta)/2}\, e^{-i\theta\cdot z}\, d\theta,$$

and $\tilde{f}_3, \ldots, \tilde{f}_k$ are as in Corollary 5.4. Then,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{Z}^d} t^{(d/2)+((k+r-2)/2)}\, |\epsilon_{t,k}(x)| = 0.$$

Moreover, there is a $c < \infty$ such that

$$|\tilde{u}_j(z)| \leq c\, \left(|z|^j + 1\right),$$

and if p is symmetric, then $\tilde{u}_j \equiv 0$ for odd j.

Remark. If $p \in L_0^k$ define $Q_n(x; k), \tilde{Q}_n(x; k)$ by

$$Q_n(x;k) = \frac{1}{(\det\Gamma)^{1/2}\,(2\pi n)^{d/2}}\, e^{-\mathcal{J}(\tilde{x})^2/2} \left[1 + \frac{u_3(\tilde{x})}{\sqrt{n}} + \frac{u_4(\tilde{x})}{n} + \cdots + \frac{u_k(\tilde{x})}{n^{(k-2)/2}}\right],$$

$$\tilde{Q}_n(x;k) = \frac{1}{(\det\Gamma)^{1/2}\,(2\pi t)^{d/2}}\, e^{-\mathcal{J}(\tilde{x})^2/2} \left[1 + \frac{\tilde{u}_3(\tilde{x})}{\sqrt{t}} + \frac{\tilde{u}_4(\tilde{x})}{t} + \cdots + \frac{\tilde{u}_k(\tilde{x})}{t^{(k-2)/2}}\right],$$

where $\tilde{x} = x/\sqrt{n}$. Using $o(\cdot)$ notation, we can write the conclusion of Theorem 6.2 as

$$P_n(x) = Q_n(x;k) + o\left(\frac{1}{n^{(d+k+r-2)/2}}\right), \tag{24}$$

$$\tilde{P}_t(x) = \tilde{Q}_t(x;k) + o\left(\frac{1}{t^{(d+k+r-2)/2}}\right). \tag{25}$$

The implicit constant in the $o(\cdot)$ notation depends on k, r.

For periodic walks in discrete time, one can similarly prove a local central limit theorem. Either one can adapt the proof of Theorem 6.2 or one can consider the aperiodic random walk on the sublattice and use the statement of Theorem 6.2.

Theorem 6.4 *Suppose p is periodic with period m and in L_0^{k+r} where $k \geq 2$ is an integer and $r \in [0,1)$. Let Γ denote the covariance matrix. Then we can write*

$$P_n(x) = m\,\overline{P}_n(x)\,\left[1 + \frac{\hat{u}_3(\tilde{x})}{\sqrt{n}} + \frac{\hat{u}_4(\tilde{x})}{n} + \cdots + \frac{\hat{u}_k(\tilde{x})}{n^{(k-2)/2}}\right] + \epsilon_{n,k}(x),$$

where $\overline{P}_n(x)$ is as in (22), $\hat{u}_3, \hat{u}_4, \ldots$ are functions satisfying

$$|u_j(\tilde{x})| \leq c\,\left(|\tilde{x}|^j + 1\right),$$

and if p is symmetric, then $u_j \equiv 0$ for odd j, so that if m divides n and $x \in \{\mathbb{Z}^d : P_{lm}(x) > 0$ for some integer $l \geq 0\}$, then

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}^d} n^{(d/2)+((k+r-2)/2)}\,|\epsilon_{n,k}(x)| = 0.$$

Proposition 6.5 *If S_n is a d-dimensional random walk, then there is a $c < \infty$ such that for all n, x, $P_n(x) \leq c\,n^{-d/2}$.*

Proof. If $p \in L_0^2$, this follows from Theorems 6.2 and 6.4. We noted earlier that we can find increment distributions p_1, p_2 and an $\alpha \in (0,1)$ such that $p = \alpha p_1 + (1-\alpha)p_2$ and $p_1 \in L_0^\infty$. Using the notation of (5), we have

$$P_n(x) = \sum_{j=0}^{n} \mathbb{P}\{M_n = j\} \sum_{y\in\mathbb{Z}^d} P_{n-j,2}(y)P_{j,1}(x-y) \leq \sum_{j=0}^{n} \mathbb{P}\{M_n = j\}\frac{c}{\sqrt{j+1}} = O(n^{-1/2}).$$

The last equality is uses an easy estimate on the binomial distribution. □

Theorem 6.6 *If $d \geq 3$, every random walk in $\mathbb{Z}^d$ is transient. If $d = 2$ and $p \in L_0^2$, then the random walk is recurrent.*

Proof. Since $P_n(0) = O(n^{-d/2})$, $G(0) < \infty$ if $d \geq 3$. For $d = 2, p \in L_0^2$, we have $P_n(0) \asymp n^{-1}$ and hence $G(0) = \infty$. □

Proposition 6.7 *If p is aperiodic in L_0^{2+r} where $r \in [0,2)$,*

$$P_n(0) = \frac{1}{(\det\Gamma)^{1/2}\,(2\pi n)^{d/2}} + + o\left(\frac{1}{n^{(d+r)/2}}\right).$$

If p is aperiodic in L_0^4, then

$$P_n(0) = \frac{1}{(\det\Gamma)^{1/2}\,(2\pi n)^{d/2}} + O\left(\frac{1}{n^{(d+2)/2}}\right).$$

Proof. Since f_3 is odd, $u_3(0) = 0$. □

Proposition 6.8 *Suppose $p \in L_0^2(\mathbb{Z}^1)$ with $\Gamma = \mathbb{E}[X_1^2]$, and*

$$G_r = \sum_{j\leq r} P_j(0).$$

Then,

$$G_r \sim \sqrt{\frac{2}{\pi\,\Gamma}}\,r^{1/2}, \quad r\to\infty.$$

Proof. This follows immediately from $P_n(0) \sim (2\,\pi\,\Gamma\,n)^{-1/2}$. □

Proposition 6.9 *Suppose $p \in L_0^2(\mathbb{Z}^2)$, and*

$$G_s = \sum_{j\leq s} P_j(0).$$

Then $G_s \sim C_2 \log s$, where $C_2 = [2\pi\sqrt{\det\Gamma}]^{-1}$. If $p \in L^{2+r}$ for some $r \in (0,2]$, then there is a constant C_0 such that

$$G_s = C_2 \log s + C_0 + O(s^{-r/2}), \quad s\to\infty. \tag{26}$$

Proof. For ease we assume that p is aperiodic and that $s = n$ is an integer. The first assertion follows immediately from $P_n(0) \sim C_2/n$. If $p \in L_0^{2+r}$,

$$P_n(0) = \frac{C_2}{n} + O\left(\frac{1}{n^{1+(r/2)}}\right). \tag{27}$$

(If $r < 2$, the $O(\cdot)$ can be replaced with $o(\cdot)$.) It is standard that

$$\sum_{j=1}^{n}\frac{1}{j} = \log n + \gamma + O\left(\frac{1}{n}\right).$$

where γ is Euler's constant. Let

$$c_0 = \sum_{j=1}^{\infty} \left[P_j(0) - \frac{C_2}{j} \right].$$

By (27) the sum converges and

$$\sum_{j=1}^{n} \left[P_j(0) - \frac{C_2}{j} \right] = c_0 + O\left(\frac{1}{n^{r/2}} \right).$$

Then (26) holds with $C = 1 + c_0 + C_2\,\gamma$. □

Proposition 6.10

- *If p is an increment distribution for a random walk in $\mathbb{Z}$, then there exist a c_1 such that for all n, $\mathbb{P}^0\{\tau_0 > n\} \geq c_1\, n^{-1/2}$. If $p \in L_0^2(\mathbb{Z}^1)$, then*

$$\mathbb{P}\{\tau_0 > n\} \sim \sqrt{\frac{\Gamma}{\pi\, n}}.$$

- *If p is an increment distribution for a random walk in $\mathbb{Z}^2$, then there exist a c_1 such that for all n, $\mathbb{P}^0\{\tau_0 > n\} \geq c_1 / \log n$. If $p \in L_0^2(\mathbb{Z}^2)$, then*

$$\mathbb{P}^0\{\tau_0 > n\} \sim \frac{2\,\pi\,\sqrt{\det \Gamma}}{\log n}.$$

Proof. We first assume $p \in L_0^2$. Let $\xi \in (0,1)$ and consider

$$q(\xi) = \mathbb{P}\{\tau_0 > T_\xi\}; \quad G(0;\xi) = \sum_{n=0}^{\infty} P_n(0)\,\xi^n.$$

. Let $\sigma_\xi = \max\{n \geq 0 : S_n = 0; n < T_\xi\}$. Then

$$1 = \sum_{n=0}^{\infty} \mathbb{P}\{\sigma_\xi = n\} = \sum_{n=0}^{\infty} \xi^n\, P_n(0)\, q(\xi) = G(\xi)\; q(\xi).$$

By the local central limit theorem and Theorem 3.5, we can see that as $\xi \to 1-$,

$$G(\xi) \sim \frac{1}{\sqrt{\Gamma}\,(1-\xi)^{1/2}}, \qquad d = 1,$$

$$G(\xi) \sim \frac{1}{2\,\pi\,\sqrt{\det \Gamma}}\; \log \frac{1}{1-\xi}, \qquad d = 2.$$

We therefore get the asymptotics for $q(\xi)$. But

$$q(\xi) = \sum_{n=0}^{\infty} \mathbb{P}\{T_\xi = n\}\,\mathbb{P}^0\{\tau_0 > n\} = (1-\xi) \sum_{n=0}^{\infty} \xi^n\,\mathbb{P}^0\{\tau_0 > n\}.$$

But, $\mathbb{P}^0\{\tau_0 > n\}$ is nonincreasing in n, and hence Theorem 3.5 gives

$$\mathbb{P}^0\{\tau_0 > n\} \sim \sqrt{\Gamma/\pi}\, n^{-1/2}, \qquad d = 1,$$

$$\mathbb{P}^0\{\tau_0 > n\} \sim 2\,\pi\,\sqrt{\det \Gamma}\,/\log n, \qquad d = 2.$$

For general p, we use the bound $G_n(0) \leq c\,n^{-1/2}$ to give a lower bound on $q(\eta)$ from which we conclude the lower bound on $\mathbb{P}^0\{\tau_0 > n\}$. □

Example. Let $d = 1$ and suppose $p(x) = e^{-1}/(x+1)!, x = -1, 0, 1, \ldots$, i.e., p is the distribution of $Y - 1$ where Y is Poisson with mean 1. Note that

$$\sum p(x) x^3 = \mathbb{E}[(Y-1)^3] > 0.$$

Since S_n has the distribution of $Y_n - n$ where Y_n is Poisson with mean n,

$$P_n(k) = e^{-n}\,\frac{n^{n+k}}{(n+k)!}, \qquad k = -n, -n+1, \ldots$$

By Stirling's Formula (Lemma 4.1)

$$P_n(0) = \frac{1}{\sqrt{2\pi n}}\left[1 + O\left(\frac{1}{n}\right)\right].$$

For $k > 0$, $P_n(k) = n\,P_n(k-1)/(n+k)$. Hence,

$$P_n(k) = P_n(0) \prod_{j=1}^{k} \left[1 - \frac{j}{n+j}\right].$$

Note that

$$\begin{aligned}
\log \prod_{j=1}^{k} \left[1 - \frac{j}{n+j}\right] &= \sum_{j=1}^{k} \log\left[1 - \frac{j}{n} + \frac{j^2}{n^2} + O(\frac{j^3}{n^3})\right] \\
&= \sum_{j=1}^{k} \left[-\frac{j}{n} + \frac{j^2}{2n^2} + O\left(\frac{j^3}{n^3}\right)\right] \\
&= -\frac{k^2}{2n} - \frac{k}{2n} + \frac{k^3}{6n^2} + O\left(\frac{k^4}{n^3}\right).
\end{aligned}$$

Therefore, if $k \leq n^{2/3}$,

$$P_n(k) = \frac{1}{\sqrt{2\pi n}} e^{-k^2/(2n)} \left[1 + \frac{k^3}{6n^2} - \frac{k}{2n} + O(\frac{1}{n}) + O(\frac{k^4}{n^3})\right].$$

In particular if $k = \lfloor \sqrt{n} \rfloor$, $|P_n(k) - \overline{P}_n(k)| \geq c\, n^{-1/2}$. Similarly, we can see that

$$\begin{aligned} P_n(-k) &= P_n(0) \prod_{j=1}^{k-1} \left[1 - \frac{j}{n}\right] \\ &= \frac{1}{\sqrt{2\pi n}} e^{-k^2/(2n)} \left[1 - \frac{k^3}{6n^2} + \frac{k}{2n} + O\left(\frac{1}{n} + \frac{k^4}{n^3}\right)\right]. \end{aligned}$$

Lemma 6.11 (Reflection estimate) *If $p \in L_0^2(\mathbb{Z}^d)$, there exist $\epsilon, c > 0$ such that for every $n > 1$ and every $\theta \in \mathbb{R}^d$,*

$$\mathbb{P}\{S_n \cdot \theta > 0\} \geq \epsilon, \qquad \mathbb{P}\{S_n \cdot \theta > -c\} \geq 1/4$$

Hence, for every n, θ, R,

$$\mathbb{P}\{\max_{0 \leq m \leq n} S_m \cdot \theta \geq R\} \leq (1/\epsilon)\, \mathbb{P}\{S_n \cdot \theta \geq R\},$$

$$\mathbb{P}\{\max_{0 \leq m \leq n} S_m \cdot \theta \geq R\} \leq 4\, \mathbb{P}\{S_n \cdot \theta \geq R - c\},$$

$$\mathbb{P}\{\max_{0 \leq m \leq n} |S_m| \geq R\} \leq (1/\epsilon)\, \mathbb{P}\{|S_n| \geq R\},$$

$$\mathbb{P}\{\max_{0 \leq m \leq n} |S_m| \geq R\} \leq 4\, \mathbb{P}\{|S_n| \geq R - c\}.$$

Proof. The local central limit theorem shows that

$$\lim_{n\to\infty} \inf_{\theta \in \mathbb{R}} \mathbb{P}\{S_n \cdot \theta > 0\} = \frac{1}{2}.$$

In particular, there is an N such that for $n \geq N$, $\mathbb{P}\{S_n \cdot \theta > 0\} \geq 1/4$. For $n \leq N$, we can prove the first estimate using a finite set of lattice points $x_1, \ldots, x_k$ with $p(x_j) > 0$ and such that for each $\theta \in \mathbb{R}^d \setminus \{0\}$, $x_j \cdot \theta > 0$ for some $j = 1, \ldots, k$. The second estimate is easily established by choosing c large enough to handle $n \leq N$. The remaining assertions are proved using the strong Markov property at the first time m such that $S_m \cdot \theta \geq R$ or $|S_m| \geq R$. . ☐

Lemma 6.12 *If $p \in L_0^2(\mathbb{Z}^d)$, there exists $R < \infty, \epsilon > 0$ such that for all n sufficiently large,*

$$\mathbb{P}\{|S_j| \leq Rn \text{ for all } j \leq n^2\} \geq 1/2, \tag{28}$$

$$\mathbb{P}\{Rn \leq |S_j| \text{ for all } n^2 \leq j \leq 2n^2\} \geq \epsilon. \tag{29}$$

Proof. The first follows from the previous lemma and the central limit theorem. The second follows from the fact that

$$\begin{aligned} &\mathbb{P}\{Rn \leq |S_j| \leq 2RN \text{ for all } n^2 \leq j \leq 2n^2\} \geq \\ &\qquad \mathbb{P}\{|S_{n^2}| \geq 3Rn\}\, \mathbb{P}\{|S_j| \leq Rn^2, j = 1, \ldots, n^2\}. \end{aligned}$$

☐

6.1 Difference estimates

If $q : \mathbb{Z}^d \to \mathbb{R}$ is a function with $\sum q(x) = 0$, then the *difference operator* associated to q is defined by

$$\nabla^q f(x) = \sum_y q(y)\, f(x+y).$$

The first difference operator in the direction y is

$$\nabla_y f(x) = f(x+y) - f(x),$$

which corresponds to $q(y) = 1, q(0) = -1$. The *order* of the difference operator is the minimum integer k such that there exist integers $j_1, \ldots, j_d$ with $j_1 + \cdots + j_d = k$,

$$\mathbf{L}^q[(x^1)^{j_1} \cdots (x^d)^{j_d}] \neq 0.$$

We say q has compact support if $\{x : q(x) \neq 0\}$ is finite.

If p is the increment distribution of a finite range random walk, then $q(x) = p(x) - \delta(x)$ gives a difference operator; in fact $\nabla^q = \mathbf{L}^p$. If $p \in L^1$ and $\mathbb{E}[X_1] \neq 0$, the operator has order 1; if $p \in L_0^2$, it has order 2.

By (24), we can see that if $p \in L_0^2 \cap L^k$ is aperiodic, then

$$\nabla^q P_n(x) = \nabla^q Q_n(x; k) + o\left(\frac{1}{n^{(d+k-2)/2}}\right),$$

where the constant in the $o(\cdot)$ depends on k and q. For example, if e is a unit vector and $|x + e| \geq |x|$,

$$|\nabla_e P_n(x)| = |P_n(x+e) - P_n(x)| \leq o\left(\frac{1}{n^{(d+k-2)/2}}\right) + \frac{c}{n^{(d+1)/2}} \left[\left(\frac{|x|}{\sqrt{n}}\right)^k + 1\right] e^{-(x \cdot \Gamma x)(2n)}. \tag{30}$$

Proposition 6.13 *If p is the increment distribution of an aperiodic random walk, there is a $c < \infty$ such that for all $y \in \mathbb{Z}^d$,*

$$\sum_{x \in \mathbb{Z}^d} |P_n(y+x) - P_n(x)| \leq c\, |y|\, n^{-1/2}.$$

Proof. Using the triangle inequality it is easy to see that it suffices to prove the result for $y = e$ where $|e| = 1$. Let us first assume that $p \in L_0^\infty$. Then there exists an R such that $P_n(x) = P_n(x+e) = 0$ for $|x| > Rn + 1$. Hence,

$$\sum_{x\in\mathbb{Z}^d} |P_n(x+e) - P_n(x)| = \sum_{|x|\le Rn+1} |P_n(x+e) - P_n(x)|.$$

Let $k = d+3$. Then the right hand side is bounded by

$$O(n^{-1/2}) + O(n^{-1/2}) \sum_{|x|\le Rn} n^{-d/2} \left[\left(\frac{|x|}{\sqrt{n}}\right)^k + 1 \right] e^{-\mathcal{J}(x)^2/(2n)} = O(n^{-1/2}).$$

For general p we write as before

$$p = \alpha p_1 + (1-\alpha) p_2,$$

where $p_1 \in L_0^2$ is aperiodic. Using the notation of (5), we get

$$\begin{aligned} |P_n(x+e) - P_n(x)| &= \left| \sum_{j=0}^n \mathbb{P}\{M_n = j\} \sum_{z\in\mathbb{Z}^d} [P_{j,1}(x+e-z) - P_{j,1}(x-z)|\, P_{n-j,2}(z) \right| \\ &\le \sum_{j=0}^n \mathbb{P}\{M_n = j\} \sum_{z\in\mathbb{Z}^d} P_{n-j,2}(z-x)\, |P_{j,1}(x+e-z) - P_{j,1}(x-z)|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{x\in\mathbb{Z}^d} |P_n(x+e) - P_n(x)| \\ &\qquad \le \sum_{j=0}^n \mathbb{P}\{M_n = j\} \sum_{z\in\mathbb{Z}^d} P_{n-j,2}(z-x) \sum_{y\in\mathbb{Z}^d} |P_{j,1}(y+e) - P_{j,1}(y)| \\ &\qquad \le \sum_{j=0}^n \mathbb{P}\{M_n = j\} \frac{c}{1+\sqrt{j}} = O(n^{-1/2}). \end{aligned}$$

□

Proposition 6.14 *If p is the increment distribution of an aperiodic random walk in $\mathbb{Z}^d$, there is a $c < \infty$ such that if $y \in \mathbb{Z}^d$, then for all $n, t > 0$, and all $x \in \mathbb{Z}^d$,*

$$|P_n(x+y) - P_n(x)| \le c\,|y|\, n^{-(d+1)/2}.$$

In particular, if $d \ge 2$,

$$\sum_{n=0}^\infty |P_n(x+y) - P_n(x)| < \infty. \tag{31}$$

By the triangle inequality it suffices to prove the result for $y = e$ where $|e| = 1$. If $p \in L_0^\infty$ and aperiodic, the result follows immediately from (30). More generally, write $p = \alpha p_1 + (1-\alpha) p_2$ where $p_1 \in L_0^\infty$ and aperiodic. Using the notation of (5), we have

$$\begin{aligned} |P_n(x+e) - P_n(x)| &\le \sum_{j=0}^n \mathbb{P}\{M_n = j\} \sum_{z\in\mathbb{Z}^d} P_{n-j,2}(z-x)\, |P_{j,1}(x+e) - P_{j,1}(x)| \\ &\le \sum_{j=0}^n \mathbb{P}\{M_n = j\}\, \frac{c}{(j+1)^{-(d+1)/2}} = O(n^{-(d+1)/2}). \end{aligned}$$

□

Proposition 6.15 *If p is an increment distribution for a transient random walk, then*

$$\lim_{|x|\to\infty} [G(x) + G(-x)] = 2\,\hat{k},$$

where $\hat{k}$ is as defined in (18).

Proof. By the inversion formula, we know that

$$G(x) + G(-x) = \frac{2}{(2\pi)^d} \lim_{\xi\to 1-} \int_{\mathbb{T}^d} \cos(x\cdot\theta)\, \mathrm{Re}\left[\frac{1}{1-\xi\phi(\theta)}\right] d\theta.$$

Recall that $\mathrm{Re}[1/(1-\phi)]$ is nonnegative, Lebesgue integrable, and for every $r > 0$,

$$\begin{aligned} \hat{k} &:= \left[\lim_{\xi\to 1-} \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\xi\,\phi(\theta)}\right] d\theta \right] - \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \mathrm{Re}\left[\frac{1}{1-\phi(\theta)}\right] d\theta \\ &= \left[\lim_{\xi\to 1-} \frac{1}{(2\pi)^d} \int_{|\theta|\le r} \mathrm{Re}\left[\frac{1}{1-\xi\,\phi(\theta)}\right] d\theta \right] - \frac{1}{(2\pi)^d} \int_{|\theta|\le r} \mathrm{Re}\left[\frac{1}{1-\phi(\theta)}\right] d\theta. \end{aligned}$$

For fixed x and $\epsilon > 0$, choose r_ϵ sufficiently small so that $|1 - \cos(\theta\cdot x)| \le \epsilon$ for $|\theta| \le r_\epsilon$. By splitting at this r_ϵ and letting $\xi \to 1-$, we see that

$$\frac{G(x) + G(-x)}{2} = [1 + O(\epsilon)]\, \hat{k} + \frac{1}{(2\pi)^d} \int_{\theta\in\mathbb{T}^d, |\theta|\ge r_\epsilon} \cos(\theta\cdot x)\, \mathrm{Re}\left[\frac{1}{1-\phi(\theta)}\right] d\theta.$$

Letting $\epsilon \to 0+$ and recalling that $\mathrm{Re}[1/1-\phi(\theta)]$ is Lebesgue integrable, we get

$$G(x) + G(-x) = 2\hat{k} + \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \cos(\theta\cdot x)\, \mathrm{Re}\left[\frac{1}{1-\phi(\theta)}\right] d\theta.$$

Now the Riemann-Lebesgue lemma tells us that

$$\lim_{|x|\to\infty} \int_{\mathbb{T}^d} \cos(\theta\cdot x)\, \mathrm{Re}\left[\frac{1}{1-\phi(\theta)}\right] d\theta = 0.$$

Proposition 6.16 *If $d \geq 2$ and p is an increment distribution for a transient walk, then $\hat{k} = 0$, i.e.,*

$$\lim_{|x|\to\infty} G(x) = 0.$$

Remark. This result is not true if $d = 1$. For example if $p(1) = 2/3, p(-1) = 1/3$, then

$$G(x) = 3 \left(\frac{1}{2}\right)^{x \vee 0}.$$

Proof. If $d \geq 3$, this is almost immediate, since

$$G(x) = \sum_{j=0}^{n} P_j(x) + \sum_{j=n+1}^{\infty} P_j(x) \leq c\, n^{-1/2} + \sum_{j=0}^{n} P_j(x).$$

For given n, there are only a finite number of x such that $\sum_{j=0}^{n} P_j(x) \geq c\, n^{-1/2}$; for all other x, $G(x) \leq 2cn^{-1/2}$. We will give a different proof that will also work for $d = 2$.

We first note that $G(0,x)\, G(x,0) \to 0$ as $|x| \to \infty$. To see this, note that the strong Markov property implies that the probability that a random walk starting at 0 visits x and then returns to 0 after that time is given by $G(0,x)G(x,0)/G(0)^2$. But it is easy to see that for every $\epsilon > 0$ we can find a finite set A such that the probability a point in $\mathbb{Z}^d \setminus A$ is visited before the last return to the origin is at most ϵ.

Assume $J = 2\hat{k} > 0$ and let $\epsilon = J/4$. By Proposition 6.14, we can find an n such that for all $x \in \mathbb{Z}^d, |e| = 1$,

$$\sum_{j=n+1}^{\infty} |P_j(0,x) - P_j(0,x+e)| \leq \epsilon.$$

By choosing r sufficiently large, we can guarantee that

$$\sum_{j=0}^{n} |P_j(x,0)| \leq \epsilon$$

for all $|x| \geq r$.. By choosing r larger if necessary, we can also guarantee that $G(x) + G(-x) > J - \epsilon$ and $\min\{G(x), G(-x)\} < \epsilon$ for $|x| \geq r$. In particular, $|G(x) - G(x+e)| \leq 2\epsilon$, and if $G(x) \geq \epsilon$, then $G(x) > J - 2\epsilon$. Now consider $\{y : G(y) < \epsilon\}$. It is easy to see that transience implies that this set is infinite, and $J > 0$ implies that its complement is infinite. Hence, if $d \geq 2$, there exist x, e such that $|x|, |x+e| \geq r$, $|e| = 1$, $G(x) \leq \epsilon$, $G(x+e) > \epsilon$. But this contradicts the two relations: $|G(x) - G(x+e)| \leq \epsilon$, $G(x+e) > J - 2\epsilon$. This gives a contradiction, and hence $\hat{k} = 0$. $\square$

6.2 Another version of the LCLT

Proposition 6.17 *Suppose $p \in L_0^2$ is aperiodic. Then there is a sequence $\epsilon_n \to 0$ such that for all n, x,*

$$|P_n(x) - \bar{P}_n(x)| \leq \frac{\epsilon_n}{|x|^2\, n^{(d-2)/2}}.$$

Remark. This proposition improves Theorem 6.2 for random walks that are just barely in L^2 and for x that are signicantly larger than $\sqrt{n}$.

Proof. We know that

$$P_n(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \phi(\theta)^n\, e^{-ix\cdot\theta}\, d\theta.$$

Note that $\Delta[e^{-ix\cdot\theta}] = -|x|^2\, e^{-ix\cdot\theta}$. Here and for the entire proof, the Laplacian is with repect to $\theta = (\theta^1, \ldots, \theta^d)$. Hence by Green's Theorem,

$$\int_{\mathbb{T}^d} \phi(\theta)^n\, e^{-ix\cdot\theta}\, d\theta = -\frac{1}{|x|^2} \int_{\mathbb{T}^d} \phi(\theta)^n\, \Delta[e^{-ix\cdot\theta}]\, d\theta = -\frac{1}{|x|^2} \int_{\mathbb{T}^d} \Delta[\phi(\theta)^n]\, e^{-ix\cdot\theta}\, d\theta.$$

The boundary terms disappear by periodicity of ϕ. Note that

$$\Delta[\phi(\theta)^n] = n(n-1)\phi(\theta)^{n-2} |\nabla\phi(\theta)|^2 + n\phi(\theta)^{n-1} \Delta\phi(\theta).$$

Since $\nabla\phi$ and $\Delta\phi$ are bounded functions, we can see, as in the proof of Theorem 6.2 that

$$\int_{\mathbb{T}^d} \Delta[\phi(\theta)^n]\, e^{-ix\cdot\theta} = O(e^{-bn^{1/4}}) + \int_{|\theta|\leq n^{-3/8}} \Delta[\phi(\theta)^n]\, e^{-ix\cdot\theta}\, d\theta.$$

By change of variables the integral on the right equals

$$\frac{1}{n^{d/2}} \int_{|\theta|\leq n^{1/8}} n\, \Delta\left[\phi\left(\frac{\theta}{\sqrt{n}}\right)^n\right]\, e^{-ix\cdot\theta/n}\, d\theta.$$

The factor of n appears because we are taking the Laplacian with respect to the variable θ rather than the variable $\theta/\sqrt{n}$. We now use

$$\Delta\left[\phi\left(\frac{\theta}{\sqrt{n}}\right)^n\right] = \Delta[e^{-\theta\cdot\Gamma\theta}]\, [1 + o(1)],$$

where the $o(1)$ is uniform over $\theta \leq n^{1/4}$. The proof then proceeds as in the previous case. $\square$

6.3 Coupling

An equivalent formulation of Proposition 6.13 is

$$\sum_{z\in\mathbb{Z}^d} |P_n(x,z)-P_n(y,z)| \leq c|x-y|\, n^{-1/2}.$$

This can be used to give a "coupling" result for random walks started at different points.

Proposition 6.18 *Suppose p is an aperiodic increment distribution in $\mathbb{Z}^d$. There is a constant c such that for every $x,y\in\mathbb{Z}^d$ and every positive integer N, we can define (W_n,Y_n) on the same probability space so that*

- *W_n is a random walk with increment p starting at x;*
- *Y_n is a random walk with increment p starting at y;*
- *$\mathbb{P}\{W_m = Y_m$ for all $m\geq N\} \geq 1 - c\,|x-y|\,N^{-1/2}$.*

Proof. For ease, assume that $y=0$. It is standard (and can be derived very easily) that we can define (W_N,Y_N) on the same probability space so that W_N has distribution $P_N(x,\cdot)$, W_N has distribution $P_N(0,\cdot)$ and

$$\mathbb{P}\{W_N\neq Y_N\} \leq \frac{1}{2}\sum_{z\in\mathbb{Z}^d} |P_n(x,z)-P_n(0,z)| \leq c\,|x|\,N^{-1/2}.$$

Start with (W_N,Y_N) coupled appropriately. We define $W_n, W_n, n\leq N$, indepedently, using the appropriate conditional distributions. We also define $W_n,\hat{W}_n, n\geq N$, independently, using the random walk distribution conditioned on the value W_N, W_n. For $n\geq N$, set

$$Y_n = \begin{cases} W_n & \text{if } W_N = Y_N \\ \hat{Y}_n & \text{if } W_N\neq Y_n. \end{cases}$$

It is easy to see that this coupling satisfies the conditions. $\square$

In fact, by doing things a little more carefully we can prove a stronger result. Although we state the result for random walks W_n,Y_n, the same result holds for Markov chains.

Proposition 6.19 *If p is an increment distribution of a random walk in $\mathbb{Z}^d$ and $x,y\in\mathbb{Z}^d$, we can construct a probability space $(\Omega,\mathcal{F},\mathbb{P})$ on which are defined random variables (J_n,W_n,Y_n) taking values in $\{0,1\}\times\mathbb{Z}^d\times\mathbb{Z}^d$ such that:*

- *$W_0,W_1,\ldots$ has the distribution of a random walk with increment distribution p with $W_0=x$,*
- *$Y_0,Y_1,\ldots$ has the distribution of a random walk with increment distribution p with $Y_0=y$,*

- *If $m\geq n$, $\mathbb{P}\{J_m = 1 \mid J_n = 1\} = 1$.*
- *$\mathbb{P}\{J_n=1, W_n\neq Y_n\} = 0$.*
- *For each n, the conditional distributions of W_n and Y_n given $\{J_n=0\}$ are mutually singular, i.e., for each n and each $z\in\mathbb{Z}^d$,*

$$\mathbb{P}\{J_n=0, W_n=z\}\,\mathbb{P}\{J_n=0,Y_n=z\} = 0.$$

Proof. We will sketch how to construct this space. Let $\Omega = [\{0,1\}\times\mathbb{Z}^d\times\mathbb{Z}^d]^{\{0,1,2,\ldots\}}$ with the usual product σ-algebra. We will write $\mathbf{v}_n = (\mathbf{j}_n,\mathbf{w}_n,\mathbf{y}_n) = ((j_0,w_0,y_0),\ldots,(j_n,w_n,y_n))$ for elements of $\Omega_n := [\{0,1\}\times\mathbb{Z}^d\times\mathbb{Z}^d]^{\{0,1,\ldots,n\}}$. We will define the probability measure $\mathbb{P}$ as the limit of measures $\mathbb{P}_n$. Each $\mathbb{P}_n$ will be defined by giving a probability measure μ_n on Ω_n with the property

$$\mu_n\{\mathbf{v}_n : j_n = 1, w_n\neq y_n\} = 0.$$

The measure $\mathbb{P}_n$ is defined by saying that its marginal on Ω_n is μ_n and for times $m>n$, the measure is defined by stating that $j_m = j_n$ for $m>n$; x_m moves like a random walk; $y_m = x_m$ if $j_n=1$; and y_m moves like an independent random walk if $j_n=0$. according to the Markov chain where $x_m=y_m$ if $j_n=1$ and x_m,y_m move independently if $j_n=0$. If $x=y$, the measure μ_0 is the point mass on $(1,x,x)$; if $x\neq y$, μ_0 is the point mass on $(0,x,y)$. We now describe how to get μ_{n+1} from μ_n.

- If $j_n=1$ and $w_n=y_n$, then

$$\mu_{n+1}((\mathbf{v}_n,(1,z,z))) = \mu_n(\mathbf{v}_n)\,p(z-w_n).$$

- For every $z\in\mathbb{Z}^d$ let

$$q_{n+1,W}(z) = \sum_{\mathbf{w}\in(\mathbb{Z}^d)^{n+1}}\sum_{\mathbf{y}\in(\mathbb{Z}^d)^{n+1}} \mu_n((\mathbf{0},\mathbf{w}_n,\mathbf{y}_n))\,p(z-w_n),$$

$$q_{n+1,Y}(z) = \sum_{\mathbf{w}\in(\mathbb{Z}^d)^{n+1}}\sum_{\mathbf{y}\in(\mathbb{Z}^d)^{n+1}} \mu_n((\mathbf{0},\mathbf{w}_n,\mathbf{y}_n))\,p(z-y_n),$$

and let $q_{n+1}(z) = \min\{q_{n+1,W}(z),q_{n+1,Y}(z)\}, Q_{n+1}(z) = \max\{q_{n+1,W}(z),q_{n+1,Y}(z)\}$. Then we set

$$\mu_{n+1}(\mathbf{0},\mathbf{w}_n,\mathbf{y}_n),(1,z,z)) =$$

$$\frac{\sum_{\mathbf{y}}\mu_n((\mathbf{0},\mathbf{w}_n,\mathbf{y})\,p(z-w_n)}{q_{n+1,W}(z)}\;\frac{\sum_{\mathbf{w}}\mu_n((\mathbf{0},\mathbf{w}_n,\mathbf{y}_n))\,p(z-y_n)}{q_{n+1,Y}(z)}\;q_{n+1}(z).$$

At this point we have defined μ_{n+1} on $\{j_{n+1}=1\}$ and we will put the remainder of the mass on $\{j_{n+1}=0\}$. By the construction, we have

- If $\mathbf{w} \in (\mathbb{Z}^d)^{n+2}, \mathbf{y} \in (\mathbb{Z}^d)^{n+2}$,

$$\mu_{n+1}\{\mathbf{v} : j_{n+1} = 1, \mathbf{w}_{n+1} = \mathbf{w}\} \le \mathbb{P}\{W_0 = w_0, \ldots, W_{n+1} = w_{n+1}\},$$

$$\mu_{n+1}\{\mathbf{v} : j_{n+1} = 1, \mathbf{y}_{n+1} = \mathbf{y} \le \mathbb{P}\{Y_0 = y_0, \ldots, Y_{n+1} = y_{n+1}\},$$

- For every z, either

$$\mu_{n+1}\{\mathbf{v} : j_{n+1} = 1, w_{n+1} = z\} = \mathbb{P}\{W_{n+1} = z\}$$

or

$$\mu_{n+1}\{\mathbf{v} : j_{n+1} = 1, y_{n+1} = z\} = \mathbb{P}\{Y_n = z\}.$$

.

- In particular,

$$\mu_{n+1}\{j_n = 0\} \le \frac{1}{2} \sum_{z \in \mathbb{Z}^d} |\mathbb{P}\{W_{n+1} = z\} - \mathbb{P}\{Y_{n+1} = z\}|.$$

In fact, these two quantities are equal.

- If $j < n+1$ and $\mathbf{v} \in (\{0,1\} \times \mathbb{Z}^d \times \mathbb{Z}^d)^{j+1}$ with $v_j = 1$,

$$\mu_j(\mathbf{v}) \le \mu_{n+1}\{\mathbf{v}' : (v_0, \ldots, v_j) = \mathbf{v}\}.$$

We finish the construction by:

- Put the rest of the mass of μ_{n+1} on $\{j_{n+1} = 0\}$ in any manner such that for every $n \ge 0$, $\mathbf{w} \in (\mathbb{Z}^d)^{n+2}, \mathbf{y} \in (\mathbb{Z}^d)^{n+2}$,

$$\mu_{n+1}\{\mathbf{v} : \mathbf{w}_{n+1} = \mathbf{w}\} = \mathbb{P}\{W_0 = w_0, \ldots, W_{n+1} = w_{n+1}\},$$

$$\mu_{n+1}\{\mathbf{v} : \mathbf{y}_{n+1} = \mathbf{y}\} = \mathbb{P}\{Y_0 = y_0, \ldots, Y_{n+1} = y_{n+1}\},$$

If $j < n$, let $\tilde{\mu}_{n,j}$ denote the marginal on $(\{0,1\}\tilde{\mathbb{Z}}^d \times \mathbb{Z}^d)^{j+1}$ induced by μ_n. Then if $n \le m$,

$$\frac{1}{2} \sum_{\mathbf{v}} |\tilde{\mu}_{n,j}(\mathbf{v}) - \mu_{m,j}(\mathbf{v})| \le \mu_n\{j_n = 0\} = \frac{1}{2} \sum_{z \in \mathbb{Z}^d} |\mathbb{P}\{W_n = z\} - \mathbb{P}\{Y_n = z\}|.$$

Since the term on the right tends to zero as n goes to infinity, the measures $\tilde{\mu}_{n,j}$ converge to a limiting distribution which we call $\mathbb{P}_j$. These measures are consistent and give the measure $\mathbb{P}$. □

The coupling in the above proposition is sometimes called the *maximal coupling*. Note that

$$\mathbb{P}\{J_n = 0\} = \frac{1}{2} \sum_z |\mathbb{P}\{W_n = z\} - \mathbb{P}\{Y_n = z\}|.$$

In particular, for a random walk increment p, the maximal coupling has the property that

$$\mathbb{P}\{J_n = 0\} \le \frac{c|x-y|}{\sqrt{n}}.$$

There is another important property that this coupling have. Suppose that $|x|, |y| < n$ and

$$\xi_m = \min\{j : |W_j| \ge m\}, \qquad \xi'_m = \min\{j : |Y_j| \ge m\}.$$

For any k, z let $V(k,z,m) = \{\xi_m = k, W_k = z\}, V'(k,z,m) = \{\xi'_m = k, Y_k = z\}$. Then,

$$\mathbb{P}\{J_k = 0, \xi_m = k, W_k = z\}\, \mathbb{P}\{J_k = 0, \xi'_m = k, Y_k = z\} = 0.$$

In particular,

$$\mathbb{P}\{J_{\xi_m} = 0\} = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{|z| \ge m} |\mathbb{P}^x\{\xi_m = k, W_k = z\} - \mathbb{P}^y\{\xi'_m = k, Y_k = z\}|. \tag{32}$$

Suppose that $p = \alpha p_1 + (1-\alpha) p_2$ as in (4) with $p_1 \in L_0^\infty$. If $x, y \in \mathbb{Z}^d$, let $W_n^{(1)}, Y_n^{(1)}$ denote random walks with increment distribution p_1 starting at x, y that are coupled so that

$$\mathbb{P}\{W_n^{(1)} \ne Y_n^{(1)} \text{ for some } n \ge N\} \le c\,|x-y|\, N^{-1/2}.$$

Let $S^{(2)}$ denote a random walk starting at the origin with increment distribution p_2 independent of $(W_n^{(1)}, Y_n^{(1)})$, and let M_n be an independent binomial process with parameter α, i.e., $M_n = Z_1 + \cdots + Z_n$ where the Z_j are independent with $\mathbb{P}\{Z_j = 1\} = 1 - \mathbb{P}\{Z_j = 0\} = \alpha$. Let

$$W_n = X_{M_n}^{(1)} + S_{n-M_n}^{(2)}, \quad Y_n = Y_{M_n}^{(1)} + S_{n-M_n}^{(2)}.$$

Then (X_n, Y_n) are coupled and it is easy to check that for some c_1,

$$\mathbb{P}\{W_n \ne Y_n \text{ for some } n \ge N\} \le c_1\,|x-y|\, N^{-1/2}.$$

Note in this coupling, the two processes take identical "large jumps" and the coupling is done only on "small jumps". We will call this a *finite range coupling* for the random walks. Such a coupling depends on α, p_1, p_2. If $J_n^{(1)}$ denotes the coupling indicator for $(W_n^{(1)}, Y_n^{(1)})$ (i.e., the walks are coupled when $J_n^{(1)} = 1$), then the coupling indicator for (W_n, Y_n) is $J_n := J_{M_n}^{(1)}$. Note that

$$\mathbb{P}\{J_n = 0\} = \mathbb{P}\{J_{M_n}^{(1)} = 0\} \le \mathbb{P}\{M_n \le \frac{\alpha}{2n}\} + \mathbb{P}\{J_{\alpha n/2} = 0\} \le c\, n^{-1} \tag{33}$$

Similarly, suppose that ρ_n denotes the first time m such that $m - M_m \ge n$. Then

$$\mathbb{P}\{J_{\rho_n} = 0\} = \mathbb{P}\{J_{\rho_n}^{(1)} = 0\} \le \mathbb{P}\{M_{\rho_n} \le \frac{\alpha}{2n}\} + \mathbb{P}\{J_{\alpha n/2} = 0\} \le c\, n^{-1} \tag{34}$$

6.4 Another coupling result

Proposition 6.20 *Suppose ∇^q is a difference operator of order k and p is an increment distribution in L_0^2. There is a c such that for all $y \in \mathbb{Z}^d$ and all positive integers n,*

$$| \nabla^q \, \mathbb{P}\{S_n = y\} \, | \leq c \, n^{-(d+k)/2}.$$

Proof. By the inversion formla (16),

$$(2\pi)^d \, \nabla^q \, \mathbb{P}\{S_n = y\} = \int_{\mathbb{T}^d} [\nabla^q \, e^{-iy\cdot\theta}] \, \phi(\theta)^n \, d\theta.$$

As in (23), we get

$$\int_{\mathbb{T}^d} [\nabla^q \, e^{-iy\cdot\theta}] \, \phi(\theta)^n \, d\theta = O(e^{-bn^{1/4}}) + \int_{|\theta|\leq n^{-3/8}} [\nabla^q \, e^{-iy\cdot\theta}] \, \phi(\theta)^n \, d\theta.$$

By change of variables, the integral on the right equals

$$n^{-d/2} \int_{|\theta|\leq n^{1/8}} [\nabla^q \, e^{-iy\cdot\theta/\sqrt{n}}] \, \phi(\theta/\sqrt{n})^n \, d\theta.$$

For $|\theta| \leq n$,

$$|\nabla^q \, e^{-iy\cdot\theta/\sqrt{n}}| \leq c \, |\theta| / n^{k/2}.$$

Hence,

$$| \nabla^q \, \mathbb{P}\{S_n = y\} \, | \leq O(e^{-bn^{1/4}}) + c \, n^{-(d+k)/2} \int_{|\theta|\leq n^{1/8}} |\theta|^k \, \phi(\theta/\sqrt{n})^n \, d\theta.$$

We now estimate the integral as in Theorem 6.2. □

In a similar way, we can prove a difference estimate version of the alternative local central limit theorem.

Proposition 6.21 *Suppose ∇^q is a difference operator of order k and p is an increment distribution in L_0^2. There is a c such that for all $y \in \mathbb{Z}^d$ and all positive integers n,*

$$| \nabla^q \, \mathbb{P}\{S_n = y\} \, | \leq c \, n^{-(d-1)/2} \, |x|^{-2}.$$

For the next proposition, we will use the notation

$$\mathcal{B}_n = \{x \in \mathbb{Z}^d : |x| < n\},$$

$$\xi_n = \min\{j \geq 1 : S_j \notin \mathcal{B}\}.$$

If $x, y \in \mathcal{B}_n$, $z \in \mathbb{Z}^d \setminus \mathcal{B}_n$, let

$$\tilde{h}_n(x,z) = \mathbb{P}^x\{S_{\xi_n} = z\},$$

$$P_j(x,y;\mathcal{B}_n) = \mathbb{P}^x\{S_j = y; \xi_n > j\}.$$

If $x \in \mathcal{B}_n$ and $z \notin \mathcal{B}_n$, by reversing paths we can see that

$$\mathbb{P}^x\{S_{\xi_n} = z; \xi_n = k\} = \mathbb{P}^z\{S_k^* = x; \xi_n^* > k\},$$

where S^* denotes the reversed random walk and ξ_n^* denotes the corresponding quantity for the reversed walk. If $x \in \mathcal{B}_{n/2}$, then by the strong Markov property,

$$\mathbb{P}^z\{S_k = x; \xi_n > k\} = \sum_{j=1}^{k} \sum_{y\in\mathcal{B}_{n/2}} \mathbb{P}^z\{\xi_n \wedge \tau_{\mathcal{B}_{n/2}} = j; S_j = y\} \, P_{k-j}(y,x;\mathcal{B}_n).$$

Recall that we use ∇_x to denote the first order difference operator, $\nabla_x f(y) = f(y+x) - f(y)$.

Proposition 6.22 *Suppose p is an increment distribution in L_0^∞. There is a c such that if n is a positive integer, $|x|, |y| \leq n/4$,*

$$\sum_{k=1}^{\infty} \sum_{z\in\partial\mathcal{B}_n} | \, \nabla_x \, \mathbb{P}^y\{\xi_n = k; S_k = z\} \, | \leq c \, |x| \, n^{-1}.$$

Proof. Note that if $A \subset A'$ and $y, x+y \in A$,

$$\sum_{k=1}^{\infty} \sum_{z\in A'} | \, \nabla_x \, \mathbb{P}^y\{\tau_{\mathbb{Z}^d\setminus A'} = k; S_k = z\} \, | \leq \sum_{k=1}^{\infty} \sum_{z\in A} | \, \nabla_x \, \mathbb{P}^y\{\tau_{\mathbb{Z}^d\setminus A} = k; S_k = z\} \, |,$$

Using this and the triangle inequality we can see that if suffices to prove the proposition for $y = 0$ and $x = e$ where $|e| = 1$. Also, by considering the reversed random walk, we see that it suffices to show that if $y = 0$ then

$$\sum_{k=1}^{\infty} \sum_{z\in\partial\mathcal{B}_n} | \, \nabla_e \, \mathbb{P}^z\{\xi_n > k; S_k = y\} \, | \leq c \, n^{-1}.$$

(The operator ∇_e is applied to the variable y.) Let us choose n sufficiently large so that $p(z_1, z_2) = 0$ for $z_1 \in \mathbb{Z}^d \setminus \mathcal{B}_{n/2}$, $z_2 \in \mathcal{B}_{3n/8}$.

$$\nabla_e \, \mathbb{P}^z\{\xi_n > k; S_k = y\} =$$

$$\sum_{j=1}^{k} \sum_{w\in\mathcal{B}_{n/2}\setminus\mathcal{B}_{3n/8}} \mathbb{P}^z\{S_j = w; \xi_n \wedge \tau_{\mathcal{B}_{n/2}} = j\} \, \nabla_e \, \mathbb{P}^w\{S_{k-j} = y; \xi_m > k - j\}.$$

Therefore,

$$| \sum_{k=1}^{\infty} \nabla_e \, \mathbb{P}^z\{\xi_n > k; S_k = y\} \, | \leq$$

$$\mathbb{P}^z\{\tau_{\mathcal{B}_{n/2}} < \xi_n\} \, \sup \left\{ | \, \nabla_e \, \mathbb{P}^w\{S_l = y; \xi_m > l\} \, | : w \in \mathcal{B}_{n/2} \setminus \mathcal{B}_{3n/8}, y \in \mathcal{B}_{n/4}, l = 0, 1, \ldots \right\}$$

The first term on the right is $O(n^{-1})$ by the gambler's ruin estimate. By the estimates above (BE MORE SPECIFIC), the second term is $O(n^{-(d+1)})$. We now use the fact that $\#(\partial\mathcal{B}_n) = O(n^{-(d+1)})$ to conclude the result, □

Corollary 6.23 *Suppose $p \in L_0^2$ is an increment distribution and $p = \alpha p_1 + (1-\alpha)p_2$ with $p_1 \in L_0^\infty$, There is a $c = c(p,\alpha,p_1) < \infty$ such that if (W_j, Y_j) are coupled using the finite range coupling with $W_0 = x, Y_0 = y$ and*

$$\xi_n = \min\{j : W_j \notin \mathcal{B}_n\}, \quad \xi_n' = \min\{j : Y_j \notin \mathcal{B}_n\},$$

then for $|x|, |y| < n/4$,

$$\mathbb{P}\{J_{\xi_n} = 0\} \le c\,|x-y|\,n^{-1}.$$

Proof. First note that $p_2 \in L_0^2$. Let

$$T_1 = T_{1,n} = \min\{j : |W_{M_j}^{(1)}| \ge \frac{n}{2}\}, \quad T_2 = T_{2,n} = \min\{j : |W_{M_j}^{(2)}| \ge \frac{n}{2}\}.$$

By the estimate for finite range random walk, we can see that

$$\mathbb{P}\{J_{T_1 \wedge n^2} = 0\} \le \mathbb{P}\{M_n \le \frac{n^2\alpha}{2}\} + \mathbb{P}\{J^{(1)}_{(\alpha n^2/2)\wedge \xi_{n/2}} = 0\} \le \frac{c}{n}.$$

Since $p_2 \in L_0^2$, we can use the reflection principle and Chebyshev's inequality to see that

$$\mathbb{P}\{T_2 \le 2^{-k}\,n^2\} \le c\,\mathbb{P}\{|S_{\lfloor 2^{-k}n^2\rfloor}| \ge n/2\} \le c\,n^{-2}\,\mathbb{E}[|S_{\lfloor 2^{-k}n^2\rfloor}|^2] \le c\;2^{-k}.$$

We can write

$$\mathbb{P}\{J_{T_2} = 0\} = \mathbb{P}\{J_{T_2} = 0; T_2 > n^2\} + \sum_{j=0}^{\infty} \mathbb{P}\{J_{T_2} = 0; 2^{-j-1}\,n^2 < T_2 < 2^{-j}\,n^2\}.$$

Note that

$$\mathbb{P}\{J_{T_2} = 0; T_2 > n^2\} \le \mathbb{P}\{M_{n^2} \le \frac{\alpha n^2}{2}\} + \mathbb{P}\{J^{(1)}_{\alpha n^2/2} = 0\} \le \frac{c}{n}.$$

By independence, using the notation of (34),

$$\begin{aligned}\mathbb{P}\{J_{T_2} = 0; 2^{-j-1}\,n^2 < T_2 < 2^{-j}\,n^2\} &\le \mathbb{P}\{J(\rho_{2^{-j-1}n^2}) = 0\}\,\mathbb{P}\{|S^{(2)}_{2^{-j}n^2}| \ge n/2\} \\ &\le c\,(2^{-j-1}n^2)^{-1/2}\,(2^{-j}) \le c\,2^{-j/2}\,n^{-1}.\end{aligned}$$

By summing over j, we get $\mathbb{P}\{J_{T_2} = 0\} \le c/n$. □

7 Harmonic functions

If $A \subset \mathbb{Z}^d$ we write

$$\partial A = \{y \in \mathbb{Z}^d \setminus A : p(y-x) + p(x-y) > 0 \text{ for some } x \in A\}, \quad \overline{A} = A \cup \partial A.$$

Note that our definition of the boundary of A depends on the increment distribution p. A function

$$f : \overline{A} \to \mathbb{Z}^d$$

is *p-harmonic*, $\mathbf{L}$*−harmonic* or just *harmonic in A* if

$$\mathbf{L}f(x) = \mathbb{E}^x[f(S_1)] - f(x) = 0$$

for all $A \in \mathbb{Z}^d$. Similarly, f is *superharmonic* (resp., *subharmonic*) if $\mathbf{L}f(x) \le 0$ (resp., $\mathbf{L}f(x) \ge 0$) for $x \in A$.

Proposition 7.1 *Suppose $f : \overline{A} \to \mathbb{R}$ is harmonic (resp., superharmonic or subharmonic) in A, and $S_n, \tilde{S}_t$ are discrete and continuous time random walks with increment distribution p started at $x \in \overline{A}$, and*

$$M_n = f(S_{n\wedge\bar\tau_{\partial A}}), \quad \tilde{M}_t = f(S_{t\wedge\bar\tau_{\partial A}})$$

If for every $x \in A$ and $n, t \ge 0$, $\mathbb{E}^x[|M_n|] < \infty, \mathbb{E}^x[|\tilde{M}_t|] < \infty$, then $M_n, \tilde{M}_t$ are martingales (resp., supermartingales, submartingales).

Proof. This is immediate from the definition. □

Proposition 7.2 (Dirichlet Problem) *Suppose p is an increment distribution, and $F : \partial A \to \mathbb{R}$ is a bounded function. Suppose that for every $x \in \overline{A}$,*

$$\mathbb{P}^x\{\bar\tau_{\partial A} < \infty\} = 1.$$

Then the unique bounded function $f : \overline{A} \to \mathbb{R}^d$ satisfying

$$\mathbf{L}f(x) = 0, \quad x \in A,$$

$$f(x) = F(x), \quad x \in \partial A,$$

is given by

$$f(x) = \mathbb{E}^x[F(S_{\bar\tau_{\partial A}})]. \tag{35}$$

Proof. Let M_n be as in the previous proposition. Then M_n is a bounded martingale. Therefore, by the optional sampling theorem,

$$f(x) = \mathbb{E}^x[M_0] = \mathbb{E}^x[M_\infty] = \mathbb{E}^x[F(S_{\bar\tau_{\partial A}})].$$

□

Remark. We could have used continuous time random walk as well. In fact, it is easy to see that $\mathbb{E}^x[F(\tilde{S}_{\bar\tau_{\partial A}})] = \mathbb{E}^x[F(S_{\bar\tau_{\partial A}})]$.

Remark. The condition $\mathbb{P}^x\{\bar\tau_{\partial A} < \infty\} = 1$ obviously holds if either A is finite or the random walk is recurrent.

Remark. It is immediate from (35) that under the conditions of the proposition,

$$\sup_{z\in\partial A}|f(z)| = \sup_{z\in\overline{A}}|f(z)|. \tag{36}$$

This is sometimes called the *maximum principle* for harmonic funcitons.

Definition. If $A \subset \mathbb{Z}^d$, the *Poisson kernel on* A or *the hitting measure of* ∂A is the function

$$H_A : \overline{A}\times\partial A \to [0,1],$$

defined by

$$H_A(x,y) = \mathbb{P}^x\{S_{\bar\tau_{\partial A}} = y\}.$$

Note that

$$\sum_{y\in\partial A} H_A(x,y) = \mathbb{P}^x\{\bar\tau_{\partial A} < \infty\}.$$

Proposition 7.3 *For fixed* $y \in \partial A$,

$$\mathbf{L}_x H(x,y) = 0, \quad x\in A; \qquad H(x,y) = \delta(y-x), x \in \partial A.$$

Proof. Immediate from the definition. □

We can restate (35) as

$$f(x) = \sum_{y\in\partial A} H_A(x,y)\,F(y).$$

Proposition 7.4 *Suppose* p *is an increment distribution and* $f : \overline{A} \to \mathbb{R}$ *is a function such that*

$$\sum_{y\in\mathbb{Z}^d} p(y)\,|f(x+y)| < \infty, \quad x \in A.$$

Let S_n *be a random walk with* $S_0 \in \overline{A}$ *and let*

$$M_n = f(S_{n\wedge\bar\tau_A}) - \sum_{j=0}^{(n\wedge\bar\tau_{\partial A})-1} \mathbf{L}f(S_j).$$

Suppose $\mathbb{E}^x[|M_n|] < \infty$ *for every* $x\in A$ *and* $n \geq 0$. *Then* M_n *is a martingale. In particular, if* f *is harmonic in* A, *then* $f(S_{n\wedge\bar\tau_A})$ *is a martingale.*

Proof. If $S_{n-1} \notin A$, then $M_n = M_{n-1}$. If $S_{n-1}\in A$, then $\bar\tau_{A^c} > n-1$ and

$$\begin{aligned}\mathbb{E}[M_n \mid \mathcal{F}_{n-1}] &= f(S_{n-1}) + \mathbf{L}f(S_{n-1}) - \sum_{j=0}^{n-1}\mathbf{L}f(S_j)\\ &= M_{n-1}.\end{aligned}$$

□

Definition. If $A \subset \mathbb{Z}^d$, we define the *escape probability (from* A*)* by

$$\overline{\mathrm{Es}}_A(x) = \mathbb{P}^x\{\bar\tau_A = \infty\} = \mathbb{P}^x\{\tilde\tau_A = \infty\},$$

$$\mathrm{Es}_A(x) = \mathbb{P}^x\{\tau_A = \infty\}.$$

Note that $\overline{\mathrm{Es}}_A(x) = \mathrm{Es}_A(x)$ if $x \notin A$. Also, $\overline{\mathrm{Es}}_A(x)$ is a bounded function on $\mathbb{Z}^d$ that is harmonic on $\mathbb{Z}^d\setminus A$ with boundary condition $\overline{\mathrm{Es}}_A(x) = 0, x\in A$.

Proposition 7.5 *If* p *is the increment distribution of a random walk in* $\mathbb{Z}^d$, *then the only bounded p-harmonic functions on* $\mathbb{Z}^d$ *are constants.*

Proof. We may assume that p is aperiodic. Suppose $f : \mathbb{Z}^d \to \mathbb{R}$ is harmonic with $\|f\|_\infty < \infty$. Then for each x,

$$f(x) = \mathbb{E}^x[f(S_n)] = \sum_{z\in\mathbb{Z}^d} P_n(x,z)\,f(z).$$

Hence,

$$|f(x) - f(y)| \leq \|f\|_\infty \sum_{z\in\mathbb{Z}^d}|P_n(x,z) - P_n(y,z)|.$$

Proposition 6.13 shows that the right hand side goes to zero as $n\to\infty$. □

Remark. The last proposition does not hold for simple random walk on regular trees of vertex degree at least three. We can represent the vertices of such a tree as words $l_1 l_2\cdots l_n$ from the three letter alphabet $\{a,b,c\}$ such that no adjacent letters are the same. We also include the empty word $\emptyset$. Each word has three "nearest neighbors" which are the legal words obtained by adding a letter on the right or removing the last letter. The simple random walk on the tree is the random walk that moves with probability $1/3$ to each of its nearest neighbors. If we let Y_n be the length of the word at time n, we see that Y_n is a Markov chain on $\{0,1,\ldots\}$. By comparison with the random walk on $\mathbb{Z}$ with $p(1) = 2/3, p(-1) = 1/3$, we can see that the random walk is transient. Hence, w.p.1 the length of the word goes to infinity, and hence the "first letter" of the word stays constant from some time afterwords. The function

$$f(x) = \mathbb{P}^x\{\text{final "first letter" is } a\}$$

is a nonconstant harmonic function for this random walk. For any irreducible *recurrent* Markov chain Y_n , the only bounded harmonic functions are constant. The proof of this is similar to the proof of Proposition 7.2.

Definition. A set $A \subset \mathbb{Z}^d$ is called *transient* if there exists an $x\in\mathbb{Z}^d$ such that $\mathrm{Es}_A(x) > 0$. Otherwise A is called *recurrent.*

Proposition 7.6 *If p is an increment distribution in $\mathbb{Z}^d$ and $A \subset \mathbb{Z}^d$, then A is transient if and only if for every $x \in \mathbb{Z}^d$, $\mathbb{P}^x\{V < \infty\} = 1$ where*

$$V = \sum_{n=0}^{\infty} 1\{S_n \in A\}.$$

If $\mathbb{P}^x\{V < \infty\} = 1$ for some $x \in A$, then $\mathbb{P}^x\{V < \infty\} = 1$ for every $x \in A$.

Proof. We may assume that p is aperiodic. Let E be the event that $\{V < \infty\}$. Since $f(x) = \mathbb{P}^x\{V < \infty\}$ is a bounded harmonic function, it is constant, say c. Also, using Proposition 6.13, we can see that the event E is independent of $\{S_0, \ldots, S_n\}$, and hence by the Kolmogorov $0-1$ law must equal 0 or 1. Clearly, if $\mathrm{Es}_A(x) = 0$ for all x, then $\mathbb{P}^x\{V = \infty\} = 1$ for all x. If $\mathrm{Es}_A(x) > 0$ for some x, then $\mathbb{P}^x\{V < \infty\} \geq \mathrm{Es}_A(x) > 0$, and hence for all y, $\mathbb{P}^y\{V < \infty\} = 1$. □

The empty set is transient. If $A \neq \emptyset$ and p is recurrent, then A is recurrent. If A is finite and p is transient, then A is transient. If A is infinite and p is transient, then A might be transient or recurrent. If $\mathbb{E}^x[V] < \infty$ for some (and hence all) $x \in \mathbb{Z}^d$, then A is transient since this implies $\mathbb{P}^x\{V < \infty\} = 1$. There are examples, however, of transient sets A for which $\mathbb{E}^x[V] = \infty$; see the remark after Proposition 14.7. If A is a transient set, then $f(x) = \overline{\mathrm{Es}}_A(x)$ is a nontrivial bounded function that is harmonic on $\mathbb{Z}^d \setminus A$ and equals zero on A. The next proposition, which can be considered as an extension of Proposition 7.2, shows that up to a constant this is the only one.

Proposition 7.7 (Dirichlet problem, transient case) *Suppose p is an increment distribution for a random walk, and $F : \partial A \to \mathbb{R}$ is a bounded function. The only bounded functions $f : \overline{A} \to \mathbb{R}$ satisfying*

$$\mathbf{L}f(x) = 0, \quad x \in A,$$

$$f(x) = F(x), \quad x \in \partial A,$$

are given by

$$f(x) = \mathbb{E}^x[F(S_{\bar\tau_{\partial A}}); \bar\tau_{\partial A} < \infty] + c\,\overline{\mathrm{Es}}_{\partial A}(x), \tag{37}$$

where $c \in \mathbb{R}$.

Proof. We may assume that p is aperiodic and that A is a transient set. Let $\tau_n = \bar\tau_{\partial A} \wedge n$. Then Proposition 6.13 implies that there is a c such that for all $x, y \in \mathbb{Z}^d$,

$$\sum_{z \in \mathbb{Z}^d} |\mathbb{P}^x\{S_{\tau_n} = z\} - \mathbb{P}^y\{S_{\tau_n} = z\}| \leq \mathbb{P}^x\{\bar\tau_{\partial A} < n\} + \mathbb{P}^y\{\bar\tau_{\partial A} < n\} + \frac{c\,|x-y|}{\sqrt{n}}.$$

By considering the martingale $M_n = f(S_{\tau_n})$ and letting $n \to \infty$, we see that

$$|f(x) - f(y)| \leq 2\,\|f\|_\infty \,\left[\mathbb{P}^x\{\bar\tau_{\partial A} < \infty\} + \mathbb{P}^y\{\bar\tau_{\partial A} < \infty\}\right].$$

From this we see that there is a c such that for all x, y,

$$|f(x) - c| \leq 2\,\|f\|_\infty\, \mathbb{P}^x\{\bar\tau_{\partial A} < \infty\}.$$

For each $\epsilon > 0$, let $\sigma_\epsilon = \min\{n : S_n \in A \text{ or } \overline{\mathrm{Es}}_{\partial A}(x) > 1 - \epsilon\}$. By stopping the martingale M_n at time σ_ϵ and letting $\epsilon \to 0$, we see that

$$f(x) = \lim_{\epsilon \to 0+} \mathbb{E}^x[f(S_{\sigma_\epsilon})] = \mathbb{E}^x[F(S_{\bar\tau_{\partial A}}; \bar\tau_{\partial A} < \infty] + c\,\overline{\mathrm{Es}}_{\partial A}(x).$$

□

If $F : \overline{A} \to \mathbb{R}$ is a bounded function, define $f : \{0, 1, 2, \ldots\} \times \overline{A} \to \mathbb{R}$ by

$$f(n, x) = \mathbb{E}^x[F(S_{n \wedge \bar\tau_{\partial A}})].$$

Then f satisfies the discrete "heat equation" with boundary value F on ∂A and initial condition F on A:

$$f(n, x) = F(x), \quad x \in \partial A,$$

$$f(0, x) = F(x), \quad x \in A,$$

$$f(n+1, x) - f(n, x) = \mathbf{L}_x[f(n, x)], \quad x \in A.$$

In fact, f is the unique function satisyfing these three conditions. Uniqueness is easy to prove since $f(n, x)$ is determined for $x \in \partial A$; $f(0, x), x \in A$ is determined; and recursively we can define

$$f(n+1, x) = f(n, x) + \mathbf{L}_x[f(n, x)] = \sum_{y \in \mathbb{Z}^d} p(y)\, f(n, x+y). \tag{38}$$

If $\mathbb{P}^x\{\bar\tau_{\partial A} < \infty\} = 1$ for all $x \in A$, then

$$\lim_{n \to \infty} f(n, x) = \mathbb{E}^x[f(S_{\bar\tau_{\partial A}})],$$

which is the solution of the Dirichlet problem as in Proposition 7.2. In fact, (38) gives a numerical way to find the solution of the Dirichlet problem which is sometimes called the relaxation method. The rate of convergence of this method is closely tied to the number of steps needed by the random walker to reach the boundary.

8 Green's function and potential kernel

If p is the increment distribution for a transient random walk in $\mathbb{Z}^d$, then (1 implies that

$$\mathbf{L}_x G(x, y) \;=\; \lim_{n \to \infty} \mathbf{L}_x \sum_{j=0}^{n} P_j(x, y)$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n} \mathbf{L}_x P_j(x,y)$$
$$= \lim_{n\to\infty} \sum_{j=0}^{n} [P_{j+1}(x,y) - P_j(x,y)]$$
$$= \lim_{n\to\infty} [P_{n+1}(x,y) - P_0(x,y)] = -\delta(y-x). \tag{39}$$

Similarly, $\mathbf{L}_y^* G(x,y) = \mathbf{L}_y^* G^*(y,x) = -\delta(y-x)$. Also, $0 < G(x) \leq G(0) < \infty$ and

$$G(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{e^{-ix\cdot\theta}\, d\theta}{1-\phi(\theta)}. \tag{40}$$

Definition. If p is a recurrent random walk in $\mathbb{Z}^2$, the *potential kernel* is defined by

$$a(x,y) = \sum_{n=0}^{\infty} [P_n(x,x) - P_n(x,y)]. \tag{41}$$

We let $a(x) = a(0,x)$, and note that $a(x,y) = a(y-x)$. If a^* is the potential kernel of the reversed random walk, then $a^*(x) = a(-x)$.

Remark. The potential kernel can also be defined for transient random walks; in fact, $a(x,y) = G(x,x) - G(x,y)$. If $d = 2$, Proposition 6.14 gives $|P_n(0) - P_n(x)| \leq c\,|x|\,n^{-3/2}$, and hence the sum is absolutely convergent. If $d = 1$, one can define the potential kernel by

$$a(x) = \lim_{n\to\infty} \sum_{j=0}^{n} [P_j(0) - P_j(x)].$$

In this case the sum is not absolutely convergent, and a more delicate argument is need to establish the limit. In this book we will only consider the potential kernel for $d = 2$. In this case, we can use the inversion formula for the characteristic function to conclude that

$$a(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{1}{1-\phi(\theta)} [1 - e^{-ix\cdot\theta}]\, d\theta. \tag{42}$$

Since $|1 - e^{-ix\cdot\theta}| \leq |x|\,|\theta|$ and $|1 - \phi\theta| \geq c\,|\theta|^2$, this integral is finite.

Remark. If $d = 2$, and we let $G_n(x) = G_n(0,x) = \sum_{j=0}^{n} P_n(0,x)$, then there is a c such that

$$|G_n(0,0) - G_n(0,x) - a(0,x)| \leq c\, n^{-1/2}. \tag{43}$$

Remark. If $d = 2$, and $|e| = 1$, then

$$\lim_{x\to\infty} [a(x+e) - a(x)] = 0. \tag{44}$$

This can be see by noting that

$$\begin{aligned} a(x+e) - a(x) &= \sum_{j=0}^{\infty} [P_n(x) - P_n(x+e)] \\ &= \sum_{j=0}^{R} [P_n(x) - P_n(x+e)] + \sum_{j=R+1}^{\infty} [P_n(x) - P_n(x+e)]. \end{aligned}$$

The second sum is bounded by $cR^{-1/2}$ and for any fixed R, the first sum tends to zero as $x \to \infty$.

Remark. If $d = 2$ and $p \in L_0^4$, then (26) gives

$$\sum_{j=0}^{n} P_j(0) = \frac{1}{2\pi\sqrt{\det\Gamma}} \log n + C + O\left(\frac{1}{n}\right),$$

and hence,

$$a(x) = \lim_{n\to\infty} \left[\frac{1}{2\pi\sqrt{\det\Gamma}} \log n + C - \sum_{j=0}^{n} P_j(x) \right].$$

Since the sum in (41) is absolutely convergent, (2) gives

$$\mathbf{L}_y^* a(x,y) = \lim_{n\to\infty} \mathbf{L}_y \left[\sum_{j=0}^{n} [P_j(x,x) - P_j(x,y)] \right] = \lim_{n\to\infty} \sum_{j=0}^{n} -\mathbf{L}_y^* P_j(x,y)$$

$$= \lim_{n\to\infty} \sum_{j=0}^{n} [P_j(x,y) - P_{j+1}(x,y)] = \lim_{n\to\infty} [P_0(x,y) - P_{n+1}(x,y)] = \delta(y-x). \tag{45}$$

Also, $\mathbf{L}_x a(x,y) = (\mathbf{L}_x^*)^* a^*(y,x) = \delta(y-x)$. Also note that the potential kernel satsifies $a(0) = 0$ and

$$a(x) \leq 1 + \sum_{1\leq j\leq |x|^2} \frac{c}{j} + \sum_{|x|^2 < j} \frac{c\,|x|}{j^{3/2}} \leq \frac{c}{\log|x|}.$$

Remark. If p is an increment distribution with potential kernel a (resp., Green's function G) and p_α is defined as in (3), then the potential kernel for p_α is a/α (resp., the Green's function for p_α is G/α). One way to see this is to use (40) and (42). For this reason, it suffices to understand the Green's function or the potenial kernel for aperiodic random walks.

The next proposition shows that the Green's function and the potential kernel for continuous time random walk is the same as for the discrete time random walk.

Proposition 8.1 *If $\tilde S$ is a continuous time random walk with transient increment distribution p, then for every $x \in \mathbb{Z}^d$,*

$$\int_0^\infty p_t(x)\,dt = G(x).$$

If $d = 2$, and $\tilde S$ is the continuous time random walk with recurrent increment distribution p, then for every $x \in \mathbb{Z}^2$,

$$\int_0^\infty [p_t(0) - p_t(x)]\,dt = a(x).$$

Proof. Recall that the characteristic function of $\tilde S_t$ is $\tilde\phi(\theta)^t = \exp\{t(1-\phi(\theta)\}$ where $\phi(\theta)$ is the characteristic function of X_1. Hence,

$$\begin{aligned}\int_0^\infty [p_t(0) - p_t(x)]\,dt &= \int_0^\infty \frac{1}{(2\pi)^2}\left[\int_{\mathbb{T}^2}[1-e^{-ix\cdot\theta}]\,\tilde\phi(\theta)^t\,d\theta\right]\,dt\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}[1-e^{-ix\cdot\theta}][\int_0^\infty e^{t\,[1-\phi(\theta)]}\,dt\,]\,d\theta\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}[1-e^{-ix\cdot\theta}]\,\frac{1}{1-\phi(\theta)}\,d\theta = a(x).\end{aligned}$$

A similar argument works for G. □

Definition. If $A \subset \mathbb{Z}^d$, the *Green's function on A* is defined by

$$G_A(x,y) = \mathbb{E}^x\left[\sum_{n=0}^{\tau_{A^c}-1} 1\{S_n = y\}\right] = \sum_{n=0}^\infty \mathbb{P}^x\{S_n = y; n < \tau_{A^c}\}.$$

In other words, $G_A(x,y)$ is the expected number of visits to y before leaving A for a random walk starting at x. Note that $G_A(x,y) = 0$ if $x \notin A$ or $y \notin A$.

Define

$$P_{n,A}(x,y) = \mathbb{P}^x\{S_n = y; n < \tau_{A^c}\}.$$

Then

$$G_A(x,y) = \sum_{n=0}^\infty P_{n,A}(x,y).$$

Note that $P_{n,A}(x,y) = 0$ if $x \notin A$ or $y \notin A$ and

$$\mathbf{L}_x P_{n,A}(x,y) = P_{n+1,A}(x,y) - P_{n,A}(x,y).$$

Unless p is symmetric, it is not true that $P_{n,A}(x,y) = P_{n,A}(y,x)$; in fact, $P_{n,A}(x,y) = P^*_{n,A}(y,x)$ where $P^*_{n,A}$ denotes the corresponding quantity for the reversed random walk.

Proposition 8.2 *Suppose A is a proper subset of $\mathbb{Z}^d$.*

- *If $A \subset A'$, then $G_A(x,y) \le G_{A'}(x,y)$. In particular if p is transient, $G_A(x,y) \le G_{\mathbb{Z}^d}(x,y) = G(x,y)$.*
- *If $x,y \in A$, $G_A(x,y) = \mathbb{P}^x\{\bar\tau_y < \tau_{A^c}\}\,G_A(y,y)$.*
- *If p is recurrent, $z \notin A$, $y \in A$,*

  $$G_A(y,y) \le G_{\mathbb{Z}^d\setminus\{z\}}(y,y) = \frac{1}{\mathbb{P}^y\{\tau_z < \tau_y\}}.$$

  In particular, if A is a proper subset of $\mathbb{Z}^d$, $G_A(x,y) < \infty$ for all x,y.
- *For fixed $y \in A$, the function $G_A(\cdot,y)$ satisfies*

  $$\mathbf{L}_x G_A(x,y) = -\delta(y-x),\quad x\in A;\qquad G_A(x,y) = 0,\quad x \notin A.$$

- *Let $G^*_A(x,y)$ denote the Green's function on A for the reversed random walk. Then $G^*_A(x,y) = G_A(y,x)$. For fixed $x \in A$, the function $G_A(x,\cdot)$ satisfies*

  $$\mathbf{L}^*_y G_A(x,y) = -\delta(y-x),\quad y\in A;\qquad G_A(x,y) = 0,\quad y \notin A.$$

  If p is symmetric, then $G_A(x,y) = G_A(y,x)$.

Proof. Straightforward and left to the reader. □

Proposition 8.3

- *If p is transient, A a proper subset of $\mathbb{Z}^d$, $x,y \in A$,*

  $$G_A(x,y) = G(x,y) - \sum_{z\in\partial A} H_A(x,z)\,G(z,y). \tag{46}$$

- *If p is recurrent in $\mathbb{Z}^2$, A a proper subset of $\mathbb{Z}^2$, and $x \in A, y \in \mathbb{Z}^2$,*

  $$\lim_{N\to\infty} G_N\,\mathbb{P}^x\{\tau_{\partial A} > N\} = G_A(x,y) + a(x,y) - \sum_{z\in\partial A} H_A(x,z)\,a(z,y), \tag{47}$$

  where $G_N = \sum_{n\le N} P_n(0)$.
- *If p is recurrent in $\mathbb{Z}^2$, A a finite subset of $\mathbb{Z}^2$, $x,y \in A$,*

  $$G_A(x,y) = -a(x,y) + \sum_{z\in\partial A} H_A(x,z)\,a(z,y). \tag{48}$$

Proof. If p is transient we can write

$$\sum_{j=0}^{\bar\tau_{A^c}-1} 1\{S_j = y\} = \sum_{j=0}^{\infty} 1\{S_j = y\} - \sum_{j=\bar\tau_{A^c}}^{\infty} 1\{S_j = y\}.$$

Taking expectations gives (48). For the recurrent case, we write a similar formula stopping at time N,

$$\begin{aligned}
\sum_{j=0}^{(\bar\tau_{A^c}\wedge N)-1} 1\{S_j = y\} &= \sum_{j=0}^{N} 1\{S_j = y\} - \sum_{j=\bar\tau_{A^c}}^{N} 1\{S_j = y\} \\
&= \sum_{j=0}^{N} 1\{S_j = y\} - 1\{\bar\tau_{A^c} \le N\} \sum_{j=\bar\tau_{A^c}}^{N+\bar\tau_{A^c}} 1\{S_j = y\} \\
&\qquad +1\{\bar\tau_{A^c} \le N\} \sum_{j=N+1}^{N+\bar\tau_{A^c}} 1\{S_j = y\} \\
&= [-G_N + \sum_{j=0}^{N} 1\{S_j = y\}] - 1\{\bar\tau_{A^c} \le N\}\,[-G_N + \sum_{j=\bar\tau_{A^c}}^{N+\bar\tau_{A^c}} 1\{S_j = y\}] \\
&\qquad +1\{\tau_{A^c} > N\}\, G_N + 1\{\bar\tau_{A^c} \le N\} \sum_{j=N+1}^{N+\bar\tau_{A^c}} 1\{S_j = y\},
\end{aligned}$$

where $G_N = \sum_{j\le N} P_j(0)$. We claim that

$$\lim_{N\to\infty} \mathbb{E}^x \left[1\{\bar\tau_{A^c} \le N\} \sum_{j=N+1}^{N+\bar\tau_{A^c}} 1\{S_j = y\} \right] = 0. \tag{49}$$

Once we establish this, we can let $N \to \infty$ and finish the proof. But,

$$\mathbb{E}^x \left[1\{\bar\tau_{A^c} \le N\} \sum_{n=N+1}^{N+\bar\tau_{A^c}} 1\{S_n = y\} \right] = \sum_{z\in\partial A} \sum_{j=1}^{N} \mathbb{P}^x\{\bar\tau_{\partial A} = j; S_j = z\} \sum_{n=N-j+1}^{N} P_n(y-z).$$

We split the sum over j into three parts, $j \le \sqrt{N}$, $\sqrt{N} < j \le N/2$, and $N/2 < j \le N$.

$$\sum_{z\in\partial A} \sum_{j\le\sqrt{N}} \mathbb{P}^x\{\bar\tau_{\partial A} = j; S_j = z\} \sum_{n=N-j+1}^{N} P_n(y-z) \le \sum_{N-\sqrt{N}\le n\le N} \frac{c}{n} = O(N^{-1/2}).$$

$$\begin{aligned}
\sum_{z\in\partial A} \sum_{\sqrt{N}<j\le N/2} \mathbb{P}^x\{\bar\tau_{\partial A} = j; S_j = z\} \sum_{n=N-j+1} P_n(y-z) &\le \mathbb{P}\{\tau_{\partial A} \ge N^{-1/2}\} \sum_{N/2\le n\le N} \frac{c}{n} \\
&\le c\,\mathbb{P}\{\tau_{\partial A} \ge N^{-1/2}\}.
\end{aligned}$$

For the third sum note that

$$1\{N/2 \le \bar\tau_{A^c} \le N\} \sum_{n=N+1}^{N+\bar\tau_{A^c}} 1\{S_n = y\} \le 1\{\bar\tau_{A^c} \ge N/2\} \sum_{n=N+1}^{2N} 1\{S_n = y\}.$$

However, since $P_j(x) \le c/j$ for all x, j, and the event $\{\bar\tau_{A^c} \ge N/2\}$ depends only on the random walk up through time $N/2$,

$$\mathbb{E}^x \left[\sum_{n=N+1}^{2N} 1\{S_n = y\} \,\Big|\, \bar\tau_{A^c} \ge N/2 \right] \le \sup_{z\in\mathbb{Z}^2} \sum_{N/2\le j\le 2N} P_j(z,y) \le c,$$

and hence

$$\sum_{z\in\partial A} \sum_{N/2\le j\le N} \mathbb{P}^x\{\bar\tau_{\partial A} = j; S_j = z\} \sum_{n=N-j+1}^{N} P_n(y-z) \le c\,\mathbb{P}^x\{\bar\tau_{\partial A} \ge N/2\}.$$

This gives (49).

If A is finite, it is not difficult to see that $\mathbb{P}^x\{\tau_{A^c} > N\}$ decays exponentially to zero. Indeed, there exists an $\epsilon > 0$ and positive integer k such that for all $y \in A$, $\mathbb{P}^y\{\tau_{A^c} \le k\} \ge \epsilon$, and hence $\mathbb{P}^x\{\tau_{A^c} > kN\} \le (1-\epsilon)^N$. Since $G_N \le c\log N$, $G_N\,\mathbb{P}^x\{\tau_{A^c} > N\} \to 0$. $\square$

Remark. The assumption that A is finite is more than is necessary for (48). It suffices that $\mathbb{P}^x\{\tau_{A^c} > N\} = o(1/\log N)$.

Corollary 8.4 *Suppose $p \in L_0^2$ is an increment distribution in $\mathbb{Z}^2$, and $A \subset \mathbb{Z}^2$ is a finite subset containing the origin. Then if $x \in \mathbb{Z}^2 \setminus A$,*

$$\lim_{n\to\infty} \frac{\log n}{2\pi\sqrt{\det\Gamma}}\, \mathbb{P}^x\{\tau_A > n\} = a^*(x) - \sum_{z\in A} \mathbb{P}^x\{S_\tau = z\}\, a^*(z).$$

In particular, if $x \ne 0$,

$$\lim_{n\to\infty} \frac{\log n}{2\pi\sqrt{\det\Gamma}}\, \mathbb{P}^x\{\tau_0 > n\} = a^*(x).$$

Proof. This is (47) with $y = 0$. Note that the local central limit theorem implies that $G_n \sim \log n/[2\pi\sqrt{\det\Gamma}]$. $\square$

Proposition 8.5 (Dirichlet Problem, Inhomogeneous Case) *Suppose p is an increment distribution for a random walk in $\mathbb{Z}^d$ and $A \subset \mathbb{Z}^d$ such that A^c is a recurrent set. Suppose $F : A \to \mathbb{R}$ is a function such that*

$$\sup_{y\in A} \sum_{z\in A} G_A(y,z)\,|F(z)| = K < \infty. \tag{50}$$

Then the unique bounded function $f : \overline{A} \to \mathbb{R}$ satisfying

$$Lf(x) = -F(x), \quad x \in A,$$

$$f(x) = 0, \quad x \in \partial A,$$

is

$$f(x) = \mathbb{E}^x\left[\sum_{j=0}^{\bar\tau_{\partial A}-1} F(S_j)\right] = \sum_{y\in A} G_A(x,y)F(y). \tag{51}$$

Proof. Since $L_x G_A(x,y) = -\delta(y-x)$ for $x \in A$, it is easy to check that (51) gives a solution. Suppose $\tilde f$ satsifies $L\tilde f(x) = -F(x), x \in A$ and $\tilde f \equiv 0$ on ∂A. Let $\tau_n = \tau_{\partial A} \wedge n$. Consider the martingale M_n as in Proposition 7.4,

$$M_n = \tilde f(S_{\tau_n}) - \sum_{j=0}^{\tau_n-1} L\tilde f(S_j) = \tilde f(S_{\tau_n}) + \sum_{j=0}^{\tau_n-1} F(S_j).$$

We claim that $\mathbb{E}^x[M_n^2]$ is uniformly bounded in x, n. Since $\tilde f$ is bounded we need only consider the second term. But

$$\mathbb{E}^x\left[(\sum_{j=0}^{\tau_n-1} F(S_j))^2\right] \le 2\,\mathbb{E}^x\left[\sum_{j=0}^{\infty}\sum_{k=j}^{\infty} 1\{\bar\tau_{\partial A} > j\}\, 1\{\bar\tau_{\partial A} > k\}\, |F(S_j)|\,|F(S_k)|\right] \tag{52}$$

Note that the strong Markov property and (50) imply for every y,

$$\mathbb{E}^x\left[\sum_{k=j}^{\infty} 1\{\bar\tau_{\partial A} > k\}\ |F(S_k)| \mid \bar\tau_{\partial A} > j, S_j = y\right] \le K.$$

Plugging this in shows that the right hand side of (52) is bounded by $2K^2$. In particular, M_n is a uniformly integrable martingale, and we can use the optional sampling theorem to conclude

$$\tilde f(0) = \mathbb{E}^x[M_0] = \mathbb{E}^x[M_{\tau_{\partial A}}] = \mathbb{E}^x\left[\sum_{j=0}^{\tau_{\partial A}-1} F(S_j)\right].$$

□

Remark. By the maximum principle, the supremum in (50) is the same as the supremum over all y with $F(y) \neq 0$. In particular, if $\{y : F(y) \neq 0\}$ is a finite set, (50) holds.

Remark. We call A *connected (with respect to p)* if for every $x, y \in A$, there exist n, m with $P_{n,A}(x,y) > 0, P_{m,A}(y,x) > 0$. If A is connected and $\mathbb{E}^x[\tau_{\partial A}] = \infty$ for some $x \in A$, then $\mathbb{E}^y[\tau_{\partial A}] \ge P_{m,A}(y,x)\,\mathbb{E}^x[\tau_{\partial A}] = \infty$ for all $y \in A$. Note that

$$\mathbb{E}^x[\tau_{\partial A}] = \mathbb{E}\left[\sum_{j=0}^{\tau_{\partial A}-1} 1\right] = \sum_{y\in A} G_A(x,y).$$

If A is connected and $\mathbb{E}^x[\tau_{\partial A}] < \infty$, the function $f(x) = \mathbb{E}^x[\bar\tau_{\partial A}]$ satisfies

$$Lf(x) = -1, \quad x \in A; \qquad f(x) = 0, \quad x \in \partial A.$$

This is true even if $\mathbb{E}^x[\tau_{\partial A}]$ is not uniformly bounded in x. In this case, we find can find an increasing sequence of finite sets A_k with $\cup A_k = A$ and use the monotone convergence theorem to conclude that

$$\mathbb{E}^x[\bar\tau_{\partial A}] = \lim_{k\to\infty} \mathbb{E}^x[\bar\tau_{\partial A_k}].$$

Proposition 8.6 *If p is an increment distribution and A is a proper subset of $\mathbb{Z}^d$, $x \in A$, $z \in \partial A$, then*

$$H_A(x,z) = \sum_{y\in A} G_A(x,y)\, p(z-y).$$

Proof. We use a "last-exit" decomposition. Note that

$$\begin{aligned} H_A(x,z) &= \sum_{j=1}^{\infty} \mathbb{P}^x\{\tau_{A^c} = j; S_j = z\} \\ &= \sum_{j=1}^{\infty}\sum_{y\in A} \mathbb{P}^x\{S_{j-1} = y; \tau_{A^c} > j-1; S_j - S_{j-1} = z\} \\ &= \sum_{y\in A} p(z-y) \sum_{j=0}^{\infty} P_{j,A}(x,y) \\ &= \sum_{y\in A} G_A(x,y)\, p(z-y). \end{aligned}$$

□

Proposition 8.7 *Suppose p is an increment distribution for a random walk in $\mathbb{Z}^d$.*

- *If p is transient, $x \in \mathbb{Z}^d$, and $M_n = G^*(S_{n\wedge\bar\tau_x} - x) = G(x - (S_{n\wedge\bar\tau_x})$, then M_n is a bounded martingale with $M_n \le G(0)$.*
- *If $p \in L_0^2$ is an increment distribution in $\mathbb{Z}^2$, $x \in \mathbb{Z}^2$, and $M_n = a^*(S_{n\wedge\bar\tau_x} - x) = a(x, S_{n\wedge\bar\tau_x})$, then M_n is a martingale.*
- *If A is a proper subset of $\mathbb{Z}^d$, $x \in A$, and $\tau = \bar\tau_x \wedge \bar\tau_{A^c}$,*

  $$M_n = G_A^*(x, S_{n\wedge\tau}) = G(S_{n\wedge\tau}, x),$$

  then M_n is a bounded martingale with $M_n \le G_A^*(x,x) = G_A(x,x)$.

Proof. See Proposition 7.4, (39), (45), and Proposition 8.2. For the second assertion we need to establish that $\mathbb{E}^y[M_n] < \infty$ for all y, but this follows from the existence of the second moment and the estimate $a^*(y-x) \leq c \log|y-x|$.

We have only been discussing discrete time random walks in this section. This is because continuous time walks have the same Green's function and potential kernel as their discrete counterparts. For example, if $\tilde{S}$ is a continuous time random walk with increment distribution p, then

$$G_A(x,y) = \mathbb{E}^x\left[\int_0^{\tilde{\tau}_A} 1\{\tilde{S}_t = y\}\, dt\right].$$

This can be deduced by remembering that the continuous time walk can be considered as a process that waits an exponential amount of time (with mean 1) and then moves according to the discrete chain. Similar arguments show that if p is transient

$$G(x) = \int_0^\infty p_t(x)\, dt,$$

and if p is recurrent in $\mathbb{Z}^2$,

$$a(x) = \int_0^\infty [p_0(x) - p_t(x)]\, dt.$$

An alternative way to derive the last expression is to recall that $\tilde{\phi}(\theta) = \exp\{1-\phi(\theta)\}$ and therefore

$$\begin{aligned}
\int_0^\infty [p_0(x) - p_t(x)]\, dt &= \int_0^\infty \left[\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} [1-e^{-ix\cdot\theta}]\, \tilde{\phi}(\theta)^t\, d(\theta)\right] dt \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} [1-e^{-ix\cdot\theta}] \left[\int_0^\infty e^{t[1-\phi(\theta)]}\, dt\right] d\theta \\
&= \frac{1}{(2\pi)^2}\int_{\mathbb{T}^2} [1-e^{-ix\cdot\theta}]\, \frac{1}{1-\phi(\theta)}\, d\theta = a(x).
\end{aligned}$$

9 Estimating the overshoot

Let

$$\kappa_n = \min\{j \geq 0 : |S_j| \geq n\}.$$

If $\theta \in \mathbb{R}^d$ with $|\theta| = 1$, let

$$\mathcal{H}(\theta,n) = \{x \in \mathbb{Z}^d : 0 < x\cdot\theta < n\}.$$

Let

$$\hat{\kappa}(\theta,n) = \min\{j \geq 0 : S_j \notin \mathcal{H}(\theta,n)\}.$$

Lemma 9.1 *If S_n is a random walk with increment distribtion p, then there is a $\rho > 0$ such for every sufficiently large integer n, every $x \in \mathbb{Z}^d$, and every $|\theta| = 1$, $\mathbb{P}^x\{\hat{\kappa}(\theta,n) \leq n^2\} > \rho$. Hence there exist c, b such that for all x, s, n and $|\theta| = 1$,*

$$\mathbb{P}^x\{\hat{\kappa}(\theta,n) \geq s\,n^2\} \leq c\,e^{-bs},$$

$$\mathbb{P}^x\{\kappa_n \geq s\,n^2\} \leq c\,e^{-bs},$$

and $\mathbb{E}^x[\hat{\kappa}(\theta,n)\,\mathbb{E}^x[\kappa_n] \leq c\,n^2$.

Proof. To prove the first inequality, it suffices to show that there is a ρ such that for all sufficiently large n and all $|\theta| = 1$,

$$\mathbb{P}\{|S_{n^2}\cdot\theta| \geq n\} \geq \rho.$$

If $p \in L_0^2$, this follows from the local central limit theorem; in fact, using that theorem we can see that for every $C < \infty$ there exists a ρ_C such that for all n sufficiently large and all $|\theta| = 1$,

$$\mathbb{P}\{|S_n\cdot\theta| \leq n/C\} \geq \rho_C, \qquad \mathbb{P}\{|S_n|\cdot\theta| \geq C\,n\} \geq \rho_C.$$

For general p, write $p = \alpha p_1 + (1-\alpha)p_2$ with $p_1 \in L_0^\infty$. By using the above estimate for p_1, we can see, using the notation of (5),

$$\mathbb{P}\{|S_{n^2}\cdot\theta| \geq n\} \geq -\mathbb{P}\{M_n \leq \alpha n/2\} + \sup_{j\geq\alpha n/2}\inf_{|\theta|=1}\min\left\{\mathbb{P}\{|S_j^{(1)}\cdot\theta| \leq n\}, \mathbb{P}\{|S_j^{(1)}\cdot\theta| \geq 3n\}\right\},$$

where $S^{(1)}$ denotes a random walk with increment distribution p_1. Since $\mathbb{P}\{M_n \leq \alpha n/2\} \to 0$, we get the first inequality for all p.

Note that the first inequality implies for all $|x| \leq n$,

$$\mathbb{P}^x\{\hat{\kappa}(\theta,n) \leq n^2\} \geq \rho.$$

By the strong Markov property, this implies

$$\mathbb{P}^x\{\kappa(\theta,n) \geq (k+1)n^2 \mid \kappa(\theta,n) \geq kn^2\} \leq (1-\rho),$$

which implies

$$\mathbb{P}^x\{\kappa(\theta,n) \geq kn^2\} \leq (1-\rho)^k = e^{-bk}.$$

where $b = -\log(1-\rho)$. This gives the second inequality. The inequality for κ_n follows immediately since the sphere of radius n is contained between two parallel hyperplanes distance $2n$ apart. $\square$

Lemma 9.2 *If $X_1, X_2, \ldots$ are independent random variables in $\mathbb{R}$ with $\mathbb{E}[X_j] = 0, \mathbb{E}[X_j^2] = \sigma^2$, $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ and T is a stopping time with respect to the natural filtration, such that $\mathbb{P}\{T < \infty\} = 1$, then*

$$\mathbb{E}[S_T^2] \leq \sigma^2\,\mathbb{E}[T].$$

If T is bounded, then $\mathbb{E}[S_T^2] = \sigma^2\,\mathbb{E}(T)$.

Proof. It is easy to check that $M_n = S^2_{n\wedge T} - \sigma^2\,(n\wedge T)$ is a martingale amd hence for each n,
$$\mathbb{E}[S_T^2; T\le n] \le \mathbb{E}[S^2_{T\wedge n}] = \sigma^2\,\mathbb{E}[T\wedge n].$$
If $\mathbb{P}\{T\le n\}=1$ the first inequality is an equality. We now use monotone convergence theorem on the left and right hands sides. □

If $A\subset \mathbb{Z}^d$, we define
$$\operatorname{diam}(A) = \sup\{|x-y| : x,y\in A\},$$
$$\operatorname{rad}(A) = \sup\{|x| : x\in A\}.$$
Note that $\operatorname{diam}(A)$ is a translation invariant quantity, but $\operatorname{rad}(A)$ is not.

Lemma 9.3 *Let p be an increment distribution. If $A\subset\mathbb{Z}^d$, $x\in A$ and $a>0$.*
$$\mathbb{P}^x\{|X_{\tau_{\partial A}}|\ge a\} \le \mathbb{E}^x[\tau_{\partial A}]\,\mathbb{P}\{|X_1|\ge a\}.$$
In particular, there is a c such that
$$\mathbb{P}^x\{|X_{\tau_{\partial A}}|\ge a\} \le c\,[\operatorname{diam}(A)]^2\,\mathbb{P}\{|X_1|\ge a\}.$$

Proof. Note that
$$\begin{aligned}\mathbb{P}^x\{|X_{\tau_{\partial A}}|\ge a\} &= \sum_{n=1}^\infty \mathbb{P}^x\{\tau_{\partial A}=n; |X_n|\ge a\}\\ &\le \sum_{n=1}^\infty \mathbb{P}^x\{\tau_{\partial A}\ge n\}\,\mathbb{P}\{|X_1|\ge a\} = \mathbb{P}\{|X_1|\ge a\}\mathbb{E}^x[\tau_{\partial A}].\end{aligned}$$
By Lemma 9.1, $\mathbb{E}^x[\tau_{\partial A}]\le c\operatorname{diam}(A)^2$. □

If S_n is a simple random walk in $\mathbb{Z}$ starting at $x\in\{1,\ldots,N-1\}$ and $T=T_N=\min\{n: S_n=0 \text{ or } S_n=N\}$, then
$$\mathbb{P}^x\{S_T=N\} = \frac{x}{n}.$$
This is sometimes called the *gambler's ruin estimate*. There are a number of ways to establish this. One way is to consider the martingale $M_n=S_{n\wedge T}$ and use
$$x=\mathbb{E}^x[M_0]=\mathbb{E}^x[M_T]=N\,\mathbb{P}^x\{S_T=N\}.$$
The next proposition generalizes this to the first time an L^2_0 random walk in $\mathbb{Z}^d$ crosses one of two hyperplanes.

Proposition 9.4 (Gambler's Ruin Estimate) *If $p\in L^2_0$ is an increment distribution of a random walk in $\mathbb{Z}^d$, there exist $0<c_1<c_2<\infty$ such that if $\theta\in\mathbb{R}^d$ with $|\theta|=1$, then*
$$\frac{c_1}{\sqrt{n}} \le \mathbb{P}\{S_j\cdot\theta>0, j=1,\ldots,n\} \le \frac{c_2}{\sqrt{n}}. \tag{53}$$
Also, if
$$T=T_{\theta,N}=\min\{n>0: S_n\cdot\theta\le 0 \text{ or } S_n\cdot\theta\ge N\},$$
then, for $0\le x\cdot\theta\le c_2$,
$$\frac{c_1}{N}\le \mathbb{P}^x\{S_T\cdot\theta\ge N\}\le\frac{c_2}{N}. \tag{54}$$

Proof. By irreducibility of the random walk, there exist $\epsilon>0$ and a finite set $\{x_1,\ldots,x_k\}$ with $p(x_1),\ldots,p(x_k)\ge\epsilon$ and
$$\min_{|\theta|=1}\max_{j=1,\ldots,k} x_j\cdot\theta\ge\epsilon.$$
Let $C=\max\{|x_1|,\ldots,|x_k|\}$. We will prove (54) for $0\le x\cdot\theta\le C$. We can see that there exists a $q>0$ and positive integer m, independent of θ, such that if $S_0\cdot\theta\ge 0$, then the probability that the random walk reaches $\{y: y\cdot\theta\ge C\}$ within m steps without entering (or returning to) $\{y:y\cdot\theta\le 0\}$ is at least q.

We start with the upper bound in (53) by using an argument similar to that in Proposition 6.10. Let $v(n,x)=v(n,x,\theta)=\mathbb{P}^x\{S_j\cdot\theta>0; j=1,\ldots,n\}$. Let $x_\theta\in\{x_1,\ldots,x_k\}$ be a point with $x_\theta\cdot\theta>0$. Let
$$U=\{y:\mathbb{Z}^d: y\cdot\theta\le 0, (y+x_\theta)\cdot\theta>0\}.$$
Then, $v(x,n)\ge\epsilon\, v(0,n-1)$ for $x\in U$. Moreover, simple geometry shows that $\#\{x\in U: |x|\le\sqrt{n}\}\ge c\,n^{(d-1)/2}$ (where c is independent of θ). Let $\eta=\eta_n=\max\{j\le n: S_j\cdot\theta\le 0\}$. Hence, a last-exit decomposition gives
$$1=\sum_{j=0}^n\mathbb{P}\{\eta=j\}=\sum_{y\cdot\theta\le 0}\sum_{j=0}^n P_j(y)\,v(n-j,y)\ge\epsilon\sum_{y\in U,|y|\le\sqrt{n}}\ \sum_{j=0}^n P_j(y)\,v(n,0)$$
The local central limit theorem tells us for $|x|\le\sqrt{n}$, $\sum_{j=0}^n P_j(x)\ge c\,n^{(2-d)/2}$. Hence,
$$\sum_{y\in U,|y|\le\sqrt{n}}\ \sum_{j=0}^n P_j(y)\ge c\,n^{1/2},$$
and hence $v(n,0)\le c\,n^{-1/2}$. By the comment in the first paragraph, $v(n,0)\ge q\,v(n-m,x)$ for every $x\cdot\theta\le C$ and hence we get the upper bound for all $x\cdot\theta\le C$.

The upper bound for (54) follows almost immediately. In fact, Lemma 6.11 can be used to see that there is a constant c such that for every x with $x\cdot\theta\ge 0$,
$$\begin{aligned}v(cN^2,x) &\ge \mathbb{P}^x\{S_T\cdot\theta\ge N; (S_j-S_T)\cdot\theta\ge -N/2, j=T+1,\ldots,T+cN^2\}\\ &\ge c\,\mathbb{P}^x\{S_T\cdot\theta\ge N\}.\end{aligned}$$

By the argument in the first paragraph, it suffices to prove the lower bound in (54) for $1 \leq x \cdot \theta \leq C$. By the upper bound in (53), we see that $\mathbb{P}^x\{T \geq j\} \leq c/\sqrt{j}$. Hence,

$$\mathbb{E}^x[T] = \sum_{j=1}^{\infty} \mathbb{P}^x\{T \geq j\} \leq \sum_{j=1}^{N^2} \frac{c}{\sqrt{j}} + \mathbb{P}^x\{T \geq N^2\} \sup_y \mathbb{E}^y[T] \leq c\, N.$$

Note that $M_n := S_{n \wedge T} \cdot \theta$ is a martingale. with $|M_n| \leq N + |M_T|$. By Lemma 9.2, $\mathbb{E}^x[M_T^2] \leq \mathbb{E}[(X_1 \cdot \theta)^2]\, \mathbb{E}^x[T] \leq c\, N$, and hence M_n is uniformly integrable. By the optional sampling theorem,

$$x \cdot \theta = \mathbb{E}^x[M_0] = \mathbb{E}^x[M_T] =$$

$$\mathbb{E}^x[S_T \cdot \theta; S_T \cdot \theta > 2N] + \mathbb{E}^x[S_T \cdot \theta; N \leq S_T \cdot \theta \leq 2N] + \mathbb{E}^x[S_T \cdot \theta; S_T \cdot \theta \leq 0].$$

Note that

$$\mathbb{E}^x[S_T \cdot \theta; S_T \cdot \theta > 2N] \leq \sum_{l=2N-1}^{\infty} \mathbb{P}^x\{S_T \cdot \theta > l\}$$

But, Lemma 9.3 implies

$$\mathbb{P}^x\{S_T \cdot \theta > a\} \leq \mathbb{P}^x\{|X_T| \geq a - N\} \leq \mathbb{E}^x[T]\, \mathbb{P}\{|X_1| \geq a - N\}.$$

Hence,

$$\mathbb{E}^x[S_T \cdot \theta; S_T \cdot \theta > 2N] \leq \mathbb{E}^x[T] \sum_{l=N-1}^{\infty} \mathbb{P}\{|X_1| \geq l\} \leq \mathbb{E}^x[T]\, \mathbb{E}[|X_1|; |X_1| \geq N] = o(1).$$

The last inequality uses $\mathbb{E}^x(T) = O(N)$ and $\mathbb{E}[|X_1|^2] < \infty$. Hence,

$$1 \leq x \cdot \theta \leq o(1) + \mathbb{E}^x[S_T \cdot \theta; N \leq \theta \leq 2N] \leq o(1) + 2N\, \mathbb{P}^x\{N \leq S_T \cdot \theta \leq 2N].$$

This gives the lower bound in (54). Given this, another arugment using Proposition 6.11 gives the lower bound in (53). $\square$

Proposition 9.5 *Suppose $p \in L_0^r$ for some $r \geq 2$. There exists a $c < \infty$ such that for all $|\theta| = 1$, if*

$$T = T_\theta = \min\{j \geq 1 : S_j \cdot \theta \leq 0\},$$

then for all $x \in U = U_\theta := \{x : S_j \cdot \theta > 0\}$,

$$\mathbb{E}^x[|S_T \cdot \theta|^{r-2}] \leq c. \tag{55}$$

Moreover, if $0 < x \cdot \theta \leq 10$,

$$\mathbb{E}^x[|S_T \cdot \theta|^{r-1}] \leq c.$$

Proof. Let $U_k = U_{k,\theta} = \{x \in U : k-1 < S_j \cdot \theta \leq k\}$. We claim there is a c_1 such that for all k,

$$J_k := \sup_{x \in U} \sum_{y \in U_k} G_U(x,y) \leq c_1\, k. \tag{56}$$

To see this, we first note that the supremum is the same if we restrict to $x \in U_k$. There is an integer m and $\epsilon > 0$ such that for every $x \in J_k$, $\mathbb{P}^x\{S_m \cdot \theta < k-1\} > \epsilon$. let $\sigma = \sigma_{\theta,k} = \min\{j \geq m : S_j \in U_k\}$. Then if $x, y \in U_k$, the strong Markov property gives

$$G_U(x,y) \leq m + \mathbb{P}^x\{\sigma < T\}\, J_k.$$

But Proposition 9.6 (using $-\theta$) tells us that $\mathbb{P}^x\{\sigma > T\} \geq c'/k$. Hence,

$$J_k \leq m + (1 - \frac{c'}{k})\, J_k,$$

i.e., $J_k \leq (m/c')\, k$. This gives (56). Using (56), we have

$$\begin{aligned} \mathbb{P}^x\{|S_T \cdot \theta| \geq a\} &= \sum_{y \in U} G_A(x,y)\, \mathbb{P}\{X_1 \cdot \theta \leq -(a+y)\} \\ &\leq \sum_{k=1}^{\infty} \sum_{y \in U_k} G_A(x,y)\, \mathbb{P}\{|X_1| \geq a + k - 1\} \\ &\leq c \sum_{k=1}^{\infty} k\, \mathbb{P}\{|X_1| \geq a + k - 1\}. \end{aligned} \tag{57}$$

Therefore,

$$\begin{aligned} \mathbb{E}^x[|S_T \cdot \theta|^{r-2}] &= (r-2) \int_0^\infty a^{r-3}\, \mathbb{P}\{|S_T \cdot \theta| \geq a\}\, da \\ &\leq c \sum_{k=1}^{\infty} k \int_0^\infty a^{r-3}\, \mathbb{P}\{|X_1| \geq a + k - 1\}]\, da \\ &\leq c \int_0^\infty a^{r-3} \mathbb{E}[|X_1|^2; |X_1| \geq a - 1\}]\, da \\ &\leq c[1 + \int_0^\infty \mathbb{E}[|X_1|^{r-1} \geq a - 1\}]\, da \\ &\leq c[1 + \mathbb{E}[|X_1|^r]. \end{aligned}$$

If $1 < x \cdot \theta \leq 10$ then

$$\sup_{y \in U_k} G_U(x,y) \leq \mathbb{P}^x\{T > \bar{\tau}_{U_k}\}\, J_k.$$

But Proposition 9.6 shows that $\mathbb{P}^x\{T > \bar{\tau}_{U_k}\} \leq c/k$. Hence,

$$\sup_k \sum_{y \in U_k} G_U(x,y) < \infty.$$

The rest of the proof goes in the same way. $\square$

Remark. Although (55) does not say anything if $r = 2$, we can see from (57) that if a is a positive integer,

$$\mathbb{P}^x\{|S_T \cdot \theta| \geq a\} \leq c \sum_{k=1}^{\infty} k\, \mathbb{P}\{|X_1| \geq k + a - 1\} \leq c \sum_{m=a}^{\infty} m\, \mathbb{P}\{|X_1| \geq m\}$$

$$\leq c \sum_{m=a}^{\infty} \sum_{n=m}^{\infty} m\ \mathbb{P}\{n \leq |X_1| < n+1\} \leq \sum_{n=a}^{\infty} n^2 \mathbb{P}\{n \leq |X_1| < n+1\}$$

$$\leq \mathbb{E}[|X_1|^2; |X_1| \geq a] = o(1), \qquad a \to \infty. \tag{58}$$

Note that the bound on $\mathbb{P}^x\{|S_T \cdot \theta| \geq a\}$ is uniform over x.

Proposition 9.6 (Gambler's Ruin Estimate) *If $p \in L_0^2$ is an increment distribution of a random walk in $\mathbb{Z}^d$, there exists $0 < c_1 < c_2 < \infty$ such that if $\theta \in \mathbb{R}^d$ with $|\theta| = 1$ and*

$$T = T_{\theta,N} = \min\{n > 0 : S_n \cdot \theta \leq 0 \text{ or } S_n \cdot \theta \geq N\},$$

Then,

$$\frac{c_1\,[(x\cdot\theta)+1]}{N} \leq \mathbb{P}^x\{S_T \cdot \theta \geq N\} \leq \frac{c_2\,[(x\cdot\theta)+1]}{N}, \qquad 0 < x \cdot \theta \leq \frac{N}{2}.$$

Proof. We have already established the bounds for $0 < x \cdot \theta \leq C$. To prove the upper bound, let $\beta_N(k)$ be the supremum of $\mathbb{P}^x\{S_T \cdot \theta \geq N\}$ where the supremum is over all x with $x \cdot \theta \leq k$. Assume $k - 1 < S_0 \cdot \theta \leq k$, and consider the first time that the random walk reaches a point with $S_n \cdot \theta \leq k-1$ or $S_n \cdot \theta \geq N$. Then by the strong Markov property and the upper bound above, we see for $k \leq N/2$, that

$$\beta_N(k) \leq \beta_{N/2} + \beta_N(k-1) \leq \frac{c}{N} + \beta_N(k-1),$$

and hence $\beta_N(k) \leq ck/N$.

For the lower bound, we claim that there is a C_1 and an $\epsilon > 0$ such that if $0 < x \cdot \theta \leq N/2$ and $\eta = \min\{j \geq 0 : S_j \cdot \theta \leq C_1 \text{ or } S_j \cdot \geq N\}$, then $\mathbb{P}^x\{0 < S_\eta \cdot \theta \leq C_1\} > \epsilon$. In fact, we know that $\mathbb{P}^x\{S_\eta \cdot \theta \leq C_1\| \geq 2\epsilon$ for some ϵ, and (58) shows that for C_1 sufficiently large, $\mathbb{P}^x\{S_\eta \cdot \theta \leq 0\} \leq \epsilon$. Given this claim, we can conclude that

$$\beta_N(k + C_1) \geq \beta_N(k) + \frac{c}{N},$$

since any "good" path (i.e., a path reaching $\{z \cdot \theta \geq N\}$ before reaching $\{z \cdot \theta \leq 0\}$) starting at x with $x \cdot \theta \leq k$ is still good if it is translated so that it starts at an x' with $x' \cdot \theta \geq x + C_1$, but there is a collection of paths of probability of order c/N that are good if they start at x' but are not good if they are translated to start at x. $\square$

Proposition 9.7 *Suppose $p \in L_0^2$. Then there exists a constant R such that for every positive integer k and every $s > 0$*

$$\mathbb{P}\left\{\max_{0\leq j\leq n^2} |S_j| \geq k(R+s)n\right\} \leq (1/2)^k + n^2\, \mathbb{P}\{|X_1| > sn\}. \tag{59}$$

In particular, if $p \in L_0^r$ for some $r \geq 2$, then there is a c_1 such that

$$\mathbb{P}\left\{\max_{0\leq j\leq n^2} |S_j| \geq c_1\, n\, \log n\right\} = o\left(\frac{1}{n^{r-2}\log^r n}\right). \tag{60}$$

Proof. Let $V = V_{n,s}$ be the event that $|X_j| \leq sn$ for $j = 1, \ldots, n^2$. Then $\mathbb{P}(V^c) \leq n^2\, \mathbb{P}\{|X_1| > sn\}$. Hence it suffices to show that

$$\mathbb{P}\left\{\max_{0\leq j\leq n^2} |S_j| \geq k(R+s)n; V\right\} \leq c\, e^{-ab}.$$

By (28), we know that there is an $R < \infty$ such that

$$\mathbb{P}\{\max_{0\leq j\leq n^2} |S_j| \geq Rn\} \leq 1/2.$$

However, by the strong Markov property,

$$\begin{aligned}
&\mathbb{P}\{\max_{0\leq j\leq n^2} |S_j| \geq k(R+s)n; V\}\\
&\quad \leq \mathbb{P}\{\max_{0\leq j\leq n^2} |S_j| \geq R\,n\}\, \mathbb{P}\{\max_{0\leq j\leq n^2} |S_j| \geq k(R+s)n - (R+s)n; V\}\\
&\quad \leq (1/2)\ \mathbb{P}\{\max_{0\leq j\leq n^2} |S_j| \geq (k-1)(R+s)\,n; V\}.
\end{aligned}$$

By iterating this we get (59). For (60), set $s = 1$ amd $k = \lfloor c_1 \log n \rfloor$ with c_1 sufficiently large. Note that

$$\mathbb{P}\{|X_1| \geq c_1\, n\, \log n\} \leq (n \log n)^{-r}\ \mathbb{E}[|X_1|^r; |X_1| \geq c_1 n \log n\} = o(n^{-r}\ \log^{-r} n).$$

$\square$

Remark. The estimate (60) is essentially optimal, since we can bound

$$\begin{aligned}
\mathbb{P}\left\{\max_{0\leq j\leq n^2} |S_j| \geq c\,n\, \log n\right\} &\geq \mathbb{P}\left\{\max_{1\leq j\leq n^2} |X_j| \geq n + c\,n\, \log n\right\}\\
&\sim n^2\, \mathbb{P}\{|X_1| \geq n + c\,n\, \log n\}.
\end{aligned}$$

If $r \geq 2$ and $\epsilon_n = o(n^{2-r}\ \log^{-r} n)$, then we can find a $p \in L_0^r$ such that

$$\limsup_{n\to\infty} \epsilon_n^{-1}\, n^2 \mathbb{P}\{|X_1| \geq n + c\,n\, \log n\} > 0.$$

Proposition 9.8 *Suppose $p \in L_0^2$. Then there exist sequences $v_m, \delta_m \to 0$ such that all positive integers m, n,*

$$\mathbb{P}\left\{\max_{0\le j\le n^2} |S_j| \ge v_m\, m\, n\right\} \le m^{-2}\,\delta_m.$$

Proof. Since $\mathbb{P}\{|X_1| \ge n\} = o(n^{-2})$, we can find a $g(n)$ with $g(n)/n \to 0$ such that $\mathbb{P}\{|X_1| \ge g(n)\} = o(n^{-2})$. Let $\alpha_m = \sup_{n\ge 1} g(mn)/mn$ and note that $\alpha_m \to 0$ as $m \to \infty$. Let $v_m = \alpha_m + m^{-1/4}$.

Consider the events

$$V_1 = V_{1,n,m} = \{|X_j| > g(mn) \text{ for at least one value of } j = 1,\ldots,n^2\},$$

$$V_2 = V_{2,n,m} = \{|X_j| > \sqrt{m}\,n \text{ for at least two values of } j = 1,\ldots,n^2\} \setminus V_1,$$

$$V_3 = V_{3,n,m} = \{|X_j| > \sqrt{m}\,n \text{ for exactly one value of } j = 1,\ldots,n^2\} \setminus V_1,$$

$$V_4 = V_{4,n,m} = \{|X_j| \le \sqrt{m}\,n \text{ for all } j = 1,\ldots,n^2\}.$$

Note that $\mathbb{P}(V_1) \le n^2\,\mathbb{P}\{|X_1| \ge g(mn)\} = o(m^{-2})$. Similarly, $\mathbb{P}(V_3) = o(m^{-1})$ and $\mathbb{P}(V_2) = o(m^{-2})$. Also, note that

$$\mathbb{P}\left\{\max_{1\le j\le n^2} |S_j| \ge v_m\, m\, n \,\middle|\, V_3\right\} \le \mathbb{P}\left\{\max_{1\le j\le n^2} |S_j| \ge m^{3/4}\, n \,\middle|\, V_4\right\},$$

since on the event V_3 there is exactly one increment of absolute value greater than $\sqrt{m}\,n$ and its absolute value is bouned by $g(mn) \le \alpha_m mn$. Hence, it suffices to find $\delta_m \to 0$ such that

$$\mathbb{P}\left\{\max_{1\le j\le n^2} |S_j| \ge m^{3/4} n; V_4\right\} \le \delta_m\, m^{-2}.$$

This can be done as in Proposition 9.7; in fact, we see that the left hand side is bounded by $O(\exp\{-bn^{1/4}\})$ for some b. $\square$

10 Green's function asymptotics

If $d \ge 3$ and $f : \mathbb{R}^d \setminus \{0\} \to (0,\infty)$ is harmonic, radially symmetric, and satisfies $f(x) \to 0$ as $|x| \to \infty$, then $f(x) = c\,|x|^{2-d}$. For this reason $|x|^{2-d}$ is called the Green's function for the Laplacian in $\mathbb{R}^d, d \ge 3$. In analogy, one would expect that for a random walk $p \in L_0^2$ in $\mathbb{Z}^d, d \ge 3$, with covariance matrix $\sigma^2 I$, that $G(x) \sim c\,|x|^{2-d}$ as $|x| \to \infty$. In this section, we will see that this is the case at least if we make sufficient moment conditions on the random walk.

In this section we assume that p is aperiodic and in L_0^2 with $d \ge 3$. We will show how to determine the asymptotic behavior of $G(x)$ as $x \to \infty$. It will be convenient to consider the continuous time random walk, so that

$$G(x) = \int_0^\infty p_t(x)\, dt.$$

The local central limit theorem says roughly that $p_t(x)$ is approximately $c_d\, t^{-d/2}\, e^{-\mathcal{J}(x)^2/(2t)}$ where $c_d = [\sqrt{\det\Gamma}\,(2\pi)^{d/2}]^{-1}$. Note that

$$\int_0^\infty t^{-d/2}\, e^{-\mathcal{J}(x)^2/2t}\, dt = \frac{2^{(d-2)/2}}{(x\cdot\Gamma^{-1}x)^{(d-2)/2}} \int_0^\infty u^{(d-4)/2}\, e^{-u}\, du = \frac{C_d}{\mathcal{J}(x)^{d-2}},$$

where

$$C_d = \frac{\mathbf{\Gamma}(\frac{d-2}{2})}{2\pi^{d/2}\sqrt{\det\Gamma}},$$

and $\mathbf{\Gamma}$ denotes the usual Gamma function. The goal of this section is to estimate the "error terms" in order to say when this approximation is almost correct.

Recall from Theorem 6.3 that if $p \in L_0^{k+\epsilon}$ where $k \ge 2$ is an integer and $0 \le \epsilon < 1$, then

$$\begin{aligned} p_t(x) &= \tilde{Q}_t(x;k) + t^{(2-k-d)/2}\,\epsilon(t,x) \\ &= c_d\, t^{-d/2}\, e^{-\mathcal{J}(x)^2/(2t)} \left[1 + \frac{\tilde{u}_3(x/\sqrt{t})}{t^{1/2}} + \cdots + \frac{\tilde{u}_k(x/\sqrt{t})}{t^{(k-2)/2}}\right] + t^{(2-k-d)/2}\,\epsilon_k(t,x), \end{aligned}$$

where $c_d = [(2\pi)^d \det\Gamma]^{-1/2}$ and $|\epsilon_k(t,x)| \le \rho(t)$ for some fixed decreasing function $\rho(t)$ with $\rho(t) = o(t^{-\epsilon})$ as $t \to \infty$. Also, $|\tilde{Q}_t(x;k)| \le c\,t^{-d/2}\, e^{-b|x|^2/n}\,(|x/\sqrt{t}|^k + 1)$ for some b, c. Let $\kappa(x)$ be another function with $\kappa(x) \to 0$ as $x \to |\infty|$. Then we can write

$$\begin{aligned} G(x) &= \int_0^{|x|^2\kappa(x)} p_t(x)\, dt + \int_{|x|^2\kappa(x)}^\infty p_t(x)\, dt \\ &= \int_0^{|x|^2\kappa(x)} p_t(x)\, dt + \int_{|x|^2\kappa(x)}^\infty \tilde{Q}_t(x;k)\, dt + \int_{|x|^2\kappa(x)}^\infty t^{(2-k-d)/2}\epsilon_k(t,x)\, dt \end{aligned}$$

Note that

$$\begin{aligned} \int_0^{|x|^2\kappa(x)} |\tilde{Q}_t(x;k)|\, dt &\le c \int_0^{|x|^2\kappa(x)} t^{-d/2}(|x/\sqrt{n}|^k + 1)\, e^{-b|x|^2/n}\, dt \\ &= c \int_{1/\kappa(x)}^\infty u^{(d-4)/2}|x|^{2-d}\,(u^{k/2} + 1)\, e^{-bu}\, du \\ &= |x|^{2-d}\, o(e^{-b/\kappa(x)}). \end{aligned}$$

We, therefore, have

$$G(x) = \overline{G}(x;k) + \int_0^{|x|^2\kappa(x)} p_t(x)\, dt + O\left(\int_{|x|^2\kappa(x)}^\infty t^{(2-k-d)/2}\,\rho(t)\, dt\right) + |x|^{2-d}O(e^{-b/\kappa(x)}),$$

where

$$\overline{G}(x;k) = \int_0^\infty \tilde{Q}_t(x;k)\, dt.$$

Note that if $\Gamma = \Lambda^T \Lambda$, the substitution $s = \mathcal{J}(x)^2/t$ gives

$$\int_0^\infty \frac{\tilde{u}_j(x/\sqrt{t})}{t^{(d+j-2)/2}}\, e^{-\mathcal{J}(x)^2/(2t)}\, dt = \frac{1}{\mathcal{J}(x)^{d+j-4}} \int_0^\infty s^{(d+j-6)/2}\, \tilde{u}_j\left(\frac{\sqrt{s}}{\Lambda^{-1}\tilde{x}}\right) e^{-s/2}\, ds,$$

where $\tilde{x} = x/|x|$. Note that

$$\int_0^\infty s^{(d-4)/2}\, e^{-s/2}\, ds = 2^{(d-2)/2}\, \mathbf{\Gamma}\left(\frac{d-2}{2}\right),$$

where we use the bold face $\mathbf{\Gamma}$ to denote the usual Gamma function. Recall that $\mathbf{\Gamma}(1/2) = \sqrt{\pi}, \mathbf{\Gamma}(1) = 1$ and $\mathbf{\Gamma}(m+1) = m\mathbf{\Gamma}(m)$. Define

$$\Theta_j(z) = 2^{(2-d)/2}\, \mathbf{\Gamma}\left(\frac{d-2}{2}\right)^{-1} \int_0^\infty s^{(d+j-6)/2}\, \tilde{u}_j\left(\frac{\sqrt{s}}{\Lambda^{-1} z}\right) e^{-s/2}\, ds,$$

which is defined for $|z| = 1$. Then,

$$\overline{G}(x;k) = \frac{C_d}{\mathcal{J}(x)^{d-2}} \left[1 + \frac{\Theta_3(\tilde{x})}{\mathcal{J}(x)} + \cdots + \frac{\Theta_k(\tilde{x})}{\mathcal{J}(x)^{k-2}}\right], \tag{61}$$

where $\tilde{x} = x/\sqrt{x}$ and

$$C_d = \frac{2^{(d-2)/2}\, \mathbf{\Gamma}(\frac{d-2}{2})}{(2\pi)^{d/2}\, \sqrt{\det \Gamma}} = \frac{\mathbf{\Gamma}(\frac{d-2}{2})}{2\, \pi^{d/2}\, \sqrt{\det \Gamma}}.$$

Note that if p is symmetric, then $\Theta_j \equiv 0$ for odd j. Summarizing, we have

$$G(x) = \frac{C_d}{\mathcal{J}(x)^{d-2}} \left[1 + \frac{\Theta_3(\tilde{x})}{\mathcal{J}(x)} + \cdots + \frac{\Theta_k(\tilde{x})}{\mathcal{J}(x)^{k-2}}\right]$$

$$+ \int_0^{|x|^2 \kappa(x)} p_t(x)\, dt + O\left(\int_{|x|^2 \kappa(x)}^\infty t^{(2-k-d)/2}\, \rho(t)\, dt\right) + |x|^{2-d} O(e^{-b/\kappa(x)}). \tag{62}$$

Theorem 10.1 *Suppose $p \in L_0^2$ is the increment distribution for a random walk in $\mathbb{Z}^d$, $d \geq 3$, with covariance matrix Γ. Let*

$$C_d = \frac{\mathbf{\Gamma}(\frac{d-2}{2})}{2\, \pi^{d/2}\, \sqrt{\det \Gamma}}. \tag{63}$$

If $d = 3$,

$$\lim_{|x| \to \infty} \mathcal{J}(x)^{-1}\, G(x) = C_3.$$

If $d \geq 4$,

$$\liminf_{|x| \to \infty} \mathcal{J}(x)^{2-d}\, G(x) = C_d.$$

If $d = 4$,

$$\limsup_{|x| \to \infty} \frac{|x|^2}{\log |x|}\, G(x) < \infty.$$

If $d \geq 5$,

$$\limsup_{|x| \to \infty} |x|^2\, G(x) < \infty.$$

Proof. For the liminf results, note that for every $a > 0$,

$$G(x) \geq \int_{a|x|^2}^\infty p_t(x)\, dt = \int_{a|x|^2}^\infty [Q_t(x;2) + o(t^{-d/2})]\, dt$$

$$= \int_{a|x|^2}^\infty Q_t(x;2)\, dt + o\left(\frac{1}{(a|x|^2)^{(d/2)-1}}\right).$$

If we first let $|x| \to \infty$ and then let $a \to 0$, we get the lower bound. For $d = 3$, we can find a sequence $\kappa(x) = o(1)$ as $|x| \to \infty$ such that

$$\int_{|x|^2 \kappa(x)}^\infty \rho(t)\, \frac{1}{t^{3/2}}\, dt = o(|x|^{-1}).$$

Since $\kappa(x) \to 0$, $e^{-b/\kappa(x)} \to 0$ as well. Proposition 6.17 tells us that $p_t(x) \leq c\, t^{-1/2}\, |x|^{-2}$ and hence

$$\int_0^{|x|^2 \kappa(t)} p_t(x)\, dt \leq c\kappa(t)^{1/2} |x|^{-1}.$$

Therefore all of the "error terms" in (62) are $o(|x|^{-1})$; this establishes the limit. For the $d \geq 4$ upper bound we can put $\kappa(x) = 1$ in (62). Note that the last two error terms are $O(|x|^{2-d})$ and the first error term is bounded by

$$\int_0^{|x|^2} p_t(x)\, dt \leq \int_0^1 p_t(x)\, dt + \int_1^{|x|^2} \frac{dt}{t^{(d-2)/2}\, |x|^2} \leq c\, r(t)\, |x|^{-2},$$

where $r(t) = \log t$ if $d = 4$ and $r(t) = 1$ if $d \geq 5$. □

Remark. One cannot give much better upper bounds for $d \geq 4$ without making further assumptions about the random walk. To see this, let $\epsilon(n)$ be any sequence tending to zero. Then we can find a sequence $n_j \uparrow \infty$ and an L_0^2 increment distribution p such that

$$p((n_j, 0, \cdots, 0)) \geq \epsilon(n_j)\, n_j^2.$$

Note that $G((n_j, 0, \ldots, 0)) \geq p(n_j, 0, \ldots, 0) \geq \epsilon(n_j)\, n_j^2$. If $d = 4$, a slightly stronger argument can be given to show that $G(n_j, 0, \ldots, 0) \geq c\, \epsilon(n_j)\, n_j^{-2} \log n_j$.

Definition. If $r > 0$, let $\mathcal{U}^r$ denote the set of increment distributions p satisfying

$$p(x) = o(|x|^{-r-d}), \qquad |x| \to \infty. \tag{64}$$

Note that $L^{r+(d)} \subset \mathcal{U}^r \subset L^{r-}$. If $r \geq 1$, we write $\mathcal{U}_0^r = \mathcal{U}^r \cap L_0^1$.

Lemma 10.2 *If $r > 2$ and $p \in \mathcal{U}_0^r$, there is a c such that*

$$\int_0^{c\,|x|^2/\log|x|} p_t(x)\,dt = o(|x|^{4-d-r}\ \log^{-2}|x|).$$

Proof. Choose integer k such that $k(r-2) > r+d-4$ and let

$$\zeta = \zeta_{k,x} = \min\{j : |X_j| \geq |x|/(2k)\}.$$

Let $\beta = \beta(k,x) = c|x|^2/\log|x|$ where the constant c is chosen sufficiently small so tht

$$\mathbb{P}\{\max_{1\leq j\leq\beta} |S_j| \geq |x|/(2k)\} = o(|x|^{2-r}).$$

Proposition 9.7 tells us this can be done. Note that if $R \geq |x|/k$, the strong Markov property implies that

$$\mathbb{P}\{\max_{1\leq j\leq\beta} |S_j| \geq R; \zeta > \beta\}$$

$$\leq \mathbb{P}\{\max_{1\leq j\leq\beta} |S_j| \geq |x|/(2k)\}\ \mathbb{P}\{\max_{1\leq j\leq\beta} |S_j| \geq R - (|x|/k); \zeta \geq \beta\}.$$

By iterating, we can see that

$$\mathbb{P}\{\max_{1\leq j\leq\beta} |S_j| \geq |x|; \zeta > \beta\} \leq [\mathbb{P}\{\max_{1\leq j\leq\beta} |S_j| \geq |x|/(2k)\}]^k = o(|x|^{(2-r)k}).$$

Hence it suffices to consider the expected number of visits to x after the time ζ.

A last-exit decomposition and (64) show that if $p \in \mathcal{U}^r$ and $t \leq \beta$,

$$\mathbb{P}\{\zeta \leq \beta; S_\zeta = y\} \leq \sum_{j=1}^{\beta} \sum_{|z-y|\geq|x|/(2k)} \mathbb{P}\{S_{j-1} = z\} p(y-z) \leq \beta\, o(|x|^{-r-d}).$$

Hence,

$$\begin{aligned}
\mathbb{E}[\int_{\zeta\wedge\beta}^{\beta} 1\{\tilde S_t = x\}\,dt\,] &\leq \sum_{y\in\mathbb{Z}^d} \mathbb{P}\{\zeta \leq \beta; S_\rho = y\} \int_0^\beta p_t(x-y)\,dt \\
&\leq \beta \sum_{y\in\mathbb{Z}^d} o(|x|^{-r-d}) \int_0^\beta p_t(x-y)\,dt \\
&\leq \beta\, o(|x|^{-r-d}) \sum_{z\in\mathbb{Z}^d} \int_0^\beta p_t(-z)\,dt \\
&= \beta^2\, o(|x|^{-r-d}) = o(|x|^{4-r-d}\ \log^{-2}|x|).
\end{aligned}$$

□

Theorem 10.3 *Suppose $k \geq 2$ is a positive integer, $\epsilon > 0$, and $p \in \mathcal{U}_0^{k+\epsilon}$ is an increment distribution in $\mathbb{Z}^d, d \geq 3$. Then there exists an $\alpha = \alpha(k)$ such that*

$$G(x) = \frac{C_d}{\mathcal{J}(x)^{d-2}} \left[1 + \frac{\Theta_3(\tilde x)}{\mathcal{J}(x)} + \cdots + \frac{\Theta_k(\tilde x)}{\mathcal{J}(x)^{k-2}}\right] + o\left(\frac{\log^{(d+k+\epsilon-4)/2}|x|}{|x|^{d+k+\epsilon-4}}\right),$$

where $\tilde x = x/|x|$ and $C_d, \Theta_1, \ldots, \Theta_k$ are as in (62).

Proof. Consider (62) where $\kappa(x) = c/\log|x|$ for c sufficiently small. The previous lemma shows that the first error term is $o(|x|^{4-d-k-\epsilon})$. By choosing c small enough, we can ensure that the third error term is also $o(|x|^{4-d-k-\epsilon})$. The second term is

$$o\left(\int_{c|x|^2/\log|x|}^{\infty} \frac{dt}{t^{(d+k+\epsilon-2)/2}}\right) = o\left(\frac{\log^{(d+k+\epsilon-4)/2}|x|}{|x|^{d+k+\epsilon-4}}\right).$$

□

Corollary 10.4 *If $p \in \mathcal{U}_0^{3+\epsilon}$ for some $\epsilon > 0$, then*

$$G(x) = \frac{C_d}{\mathcal{J}(x)^{d-2}} + O\left(\frac{1}{|x|^{d-1}}\right).$$

If $p \in \mathcal{U}_0^{4+\epsilon}$ for some $\epsilon > 0$ and p is symmetric, then

$$G(x) = \frac{C_d}{\mathcal{J}(x)^{d-2}} + O\left(\frac{1}{|x|^{d}}\right).$$

Proof. This follows immediately from the previous theorem and the fact that $\Theta_3 \equiv 0$ if p is symmetric. □

Corollary 10.5 *If p is the increment distribution for simple random walk in $\mathbb{Z}^d, d \geq 3$. Then,*

$$G(x) = \frac{c_d}{|x|^{d-2}} + O\left(\frac{1}{|x|^d}\right),$$

where

$$c_d = \frac{C_d}{d^{(d-2)/2}} = \frac{d\,\Gamma(\frac{d-2}{2})}{2\,\pi^{d/2}} == \frac{2}{(d-2)\,\omega_d},$$

and ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Proof. Note that $\Gamma = d^{-1} I$ and $|x \cdot \Lambda^{-1} x| = d^{1/2}\,|x|$. □

11 Potential kernel asymptotics

In this section we assume that $p \in L_0^2(\mathbb{Z}^2)$. Recall from Proposition 6.9 that $G_r \sim C_2 \log r$ where

$$G_r = \sum_{n \leq r} P_n(0).$$

Theorem 11.1 *If $p \in L_0^2(\mathbb{Z}^2)$, there is a c such that for all $x \in \mathbb{Z}^2$,*

$$|a(x) - G_{|x|^2}| \leq c.$$

In particular, as $|x| \to \infty$,

$$a(x) \sim 2\, C_2 \, \log |x|,$$

where $C_2 = [2\,\pi \sqrt{\det \Gamma}]^{-1}$. If $p \in L_0^r(\mathbb{Z}^2)$ for some $2 < r \leq 3$, there exists a constant C such that as $|x| \to \infty$,

$$a(x) = 2\, C_2 \log \mathcal{J}(x) + C + o\left(\frac{\log^{(r/2)-1}|x|}{|x|^{r-2}}\right).$$

Proof. . By definition,

$$a(x) = \left[\sum_{n \leq |x|^2} P_n(0,0)\right] - \left[\sum_{n \leq |x|^2} P_n(0,x)\right] + \left[\sum_{n > |x|^2} (P_n(0,0) - P_n(0,x))\right].$$

Thee first term on the right is $G_{|x|^2}$. From Proposition 6.17, we know that $P_n(0,x) \leq c|x|^{-2}$, which shows that the second terms is $O(1)$. Proposition 6.14 gives $|P_n(0,0) - P_n(0,x)| \leq c\,|x|\, n^{-3/2}$, which shows that the final term is $O(1)$.

If $p \in L_0^r$ for $r > 2$, we can write

$$\begin{aligned} a(x) &= \int_0^\infty [p_t(0) - p_t(x)]\, dt \\ &= \int_0^1 p_t(0)\, dt - \int_0^1 p_t(x)\, dt + \int_1^\infty [p_t(0) - \frac{C_2}{t}]\, dt \\ &\quad + \int_1^\infty \frac{C_2}{t}[1 - e^{-\mathcal{J}(x)^2/(2t)}]\, dt + \int_1^\infty [\frac{C_2}{t} e^{-\mathcal{J}(x)^2/(2t)} - p_t(x)]\, dt. \qquad (65) \end{aligned}$$

Since $p_t(0) - (C_2/t) = o(t^{1-r})$,

$$\int_1^\infty [p_t(0) - \frac{C_2}{t}]\, dt < \infty.$$

Using Proposition 9.7, we can see that $\int_0^1 p_t(x)\ dt = o(|x|^{1-r})$. The change of variables $s = \mathcal{J}(x)^2/t$ gives

$$\begin{aligned} \int_1^\infty \frac{1}{t}[1 - e^{-\mathcal{J}(x)^2/(2t)}]\, dt &= \int_0^{\mathcal{J}(x)^2} \frac{1}{s}[1 - e^{-s/2}]\, ds \\ &= 2\log \mathcal{J}(x) + c'' + O\left(e^{-b|x|^2}\right) \qquad (66) \end{aligned}$$

for some constant b where

$$c'' = \int_0^1 \frac{1}{s}[1 - e^{-s/2}]\, ds - \int_1^\infty \frac{e^{-s/2}}{s}\, ds.$$

If we choose $\tilde{c}$ sufficiently small, direct compuation shows that

$$\int_1^{\tilde{c}\,|x|^2/\log|x|} \frac{1}{t}\, e^{-x\cdot\Gamma^{-1}x/(2t)}\, dt. = o(|x|^{-4}).$$

Also Proposition 9.7 can be used to see that for sufficiently small $\tilde{c}$

$$\begin{aligned} \int_1^{\tilde{c}\,|x|^2/\log|x|} p_t(x)\, dt &\leq c\, \log|x|\, \mathbb{P}^x\{\max_{t \leq \tilde{c}|x|^2/\log|x|} |S_t| \geq |x|\} \\ &\leq c\, \log|x|\, o\left((|x|/\log^{1/2}|x|)^{2-r}\log^{-r}|x|\right) \\ &= o(|x|^{2-r}). \end{aligned}$$

Finally, the local central limit theorem gives

$$p_t(x) - \frac{C_2}{t} e^{-x\cdot\Gamma^{-1}x/(2t)} = o(t^{-r/2}),$$

and hence

$$\int_{\tilde{c}|x|^2/\log|x|}^\infty \left[\frac{C_2}{t} e^{-x\cdot\Gamma^{-1}x/(2t)} - p_t(x)\right]\, dt = o(|x|^{1-r}\log^{(r/2)-1}|x|),$$

$\square$

Remark. It follows from the theorem, that for $p \in L_0^2(\mathbb{Z}^2)$, there exists a c_1 such that for all n sufficiently large,

$$\max\{a(x) + 1 : |x| \leq n\} \leq \min\{a(x) : |x| \geq c_1 n\}. \qquad (67)$$

Proposition 11.2

- *If $p \in L_0^r$ is an increment distribution in $\mathbb{Z}^2$ with $r > 3$, then there exists a c such that*

$$a(x) = 2\, C_2 \, \log \mathcal{J}(x) + C + O\left(\frac{1}{|x|}\right),$$

  where $C_2 = [2\pi\sqrt{\det \Gamma}]^{-1}$.

- *If $p \in L_0^r$ is a symmetric increment distribution in $\mathbb{Z}^2$ with $r \geq 4$, then*

$$a(x) = 2\, C_2 \, \log \mathcal{J}(x) + C + O\left(\frac{1}{|x|^2}\right).$$

Proof. We will follow the proof of the last proposition and estimate the size of the errors under the assumptions $p \in L_0^r, r > 3$ and symmetric $p \in L_0^r, r > 4$. We write $a(x)$ as in (65). Note that $\int_0^1 p_t(x)\,dt = o(|x|^{-r})$. By the same argument as above, if $\tilde{c}$ is chosen sufficiently small, then

$$\int_1^{\tilde{c}|x|^2/\log|x|} p_t(x)\,dt = o(|x|^{2-r}).$$

We are left with estimating

$$\int_{\tilde{c}|x|^2/\log|x|}^{\infty} \left[\frac{C_2}{t}\, e^{-x\cdot\Gamma^{-1}x/(2t)} - p_t(x)\right]\,dt.$$

But the local central limit theorem gives

$$p_t(x) = \frac{C_2}{t}\, e^{-x\cdot\Gamma^{-1}x/(2t)}\,[1 + \frac{\tilde{u}_3(x/\sqrt{t})}{t^{1/2}}] + o(t^{-r/2}), \quad r \geq 3,$$

$$p_t(x) = \frac{C_2}{t}\, e^{-x\cdot\Gamma^{-1}x/(2t)}\,[1 + \frac{\tilde{u}_4(x/\sqrt{t})}{t}] + o(t^{-r/2}), \quad r \geq 4, \text{ symmetric.}$$

□

Corollary 11.3 *If a is the potential kernel for simple random walk in $\mathbb{Z}^2$, then there is a constant c such that*

$$a(x) = \frac{2}{\pi}\log|x| + c + O\left(\frac{1}{|x|^2}\right).$$

Proof. Note that $\Gamma = (1/2)\,I$, and hence $C_2 = 1/\pi$, $|x\cdot\Gamma^{-1}x| = 2|x|^2$. Let $c = C + (\log 2/\pi)$. □

Remark. In fact, for simple random walk $c = (2\gamma + \log 8)/\pi$ where γ is Euler's constant. We will not need this fact.

Lemma 11.4 *Suppose $p \in L_0^2$ is an increment distribution in $\mathbb{Z}^2$, $0 \in A \subset \mathbb{Z}^2$ is finite, and $y \in A$. Then the limit*

$$\lim_{|z|\to\infty} H_{A^c}(z,y) = \lim_{|z|\to\infty} \mathbb{P}^z\{S_{\tau_A} = y\}$$

exists.

Proof. We fix A and allow constants in this proof to depend on A. Let $f(x) = H_{A^c}(x,y)$ which is a bounded function that is harmonic on A^c. We claim that there is a c such that for $\beta \geq 2$,

$$\mathbb{P}^x\{\tau_A \leq \beta\,|x|^2\} \leq c\,\frac{\log\beta}{\log|x|}. \tag{68}$$

To see this assume that $\beta < |x|$ and consider the expected number of returns to the origin up to time $(\beta+1)|x|^2$,

$$\alpha = G_{(\beta+1)|x|^2}(x) = \sum_{j=0}^{(\beta+1)|x|^2} P_j(-x) = \sum_{j=0}^{|x|^2} P_j(-x) + \sum_{j=|x|^2+1}^{(\beta+1)|x|^2} P_j(-x).$$

Using the estimate $P_j(-x) \leq c(|x|^2 \vee j)^{-1}$, we see that $\alpha = O(\log\beta)$. But the strong Markov property shows that

$$\alpha \geq \mathbb{P}^x\{\tau_A \leq \beta\,|x|^2\}\, G_{|x|^2} \geq c\,\mathbb{P}^x\{\tau_A \leq \beta\,|x|^2\}\,\log|x|.$$

This gives (68). Proposition 6.18 tells us that for $N \leq |x|, |x'| \leq N\log^2 N$, we can couple random walks starting at x, x' such that, except on an event of probability $O(1/\log^2 N)$, the paths agree for all times $m \geq N^2\log^8 N$. Using (68) we see that, except perhaps on an event of probability $O(\log\log N/\log N)$, the paths agree at the first time they reach A. Therefore,

$$|f(x) - f(x')| \leq c\,\frac{\log\log N}{\log N}, \qquad N \leq |x|, |x'| \leq N\,\log^2 N.$$

Now assume $z \in V = V_N := \{z' : |z'| > N\,\log^2 N\}$, and let $T = T_N$ be the first time n with $|S_n| \leq N\,\log^2 N$. Then for all N sufficiently large,

$$\begin{aligned}
\mathbb{P}^z\{|S_T| \leq N\} &\leq \sum_{y\in V} G_V(z,y)\,\mathbb{P}\{|y + X_1| \leq N\} \\
&\leq \sum_{y\in V}\sum_{|y+y'|\leq N} G_{\mathbb{Z}^2\setminus\{0\}}(y,y)\,p(y') \\
&\leq c\sum_{y\in V}\sum_{|y+y'|\leq N} p(y')\,\log|y| \\
&\leq c\,N^2 \sum_{|y'|\geq N\log^2 N/2} p(y')\,\log|y'| \\
&\leq c\,\log^{-1} N \sum_{|y'|\geq N\log^2 N/2} p(y')\,|y'|^2 \\
&= o(1/\log N).
\end{aligned}$$

The last step uses the fact that $p \in L^2$. Therefore, if $z \in V$,

$$[1 - o(1/\log N)] \inf_{N\leq|x|\leq N\log^2 N} f(x) \leq f(z) \leq o(1/\log N) + \sup_{N\leq|x|\leq N\log^2 N} f(x),$$

and

$$|f(x) - f(x')| \leq c\,\frac{\log\log N}{\log N}, \qquad N \leq |x|, |x'|.$$

□

Definition. If A is a finite subset of $\mathbb{Z}^2$ and $p \in L_0^2$ we define the *harmonic measure* or *hitting measure (from infinity)* by

$$\mathrm{hm}_A(z) = \lim_{|x|\to\infty} \mathbb{P}^x\{S_{\tau_A} = z\}.$$

We write hm_A^* for the corresponding quantity for the reversed random walk.

Proposition 11.5 *Suppose $0 \in A$ is a finite subset of $\mathbb{Z}^2$ and $p \in L_0^{2+\epsilon}$. Define*

$$F_A(x) = \lim_{n\to\infty} (\log n)\, \mathbb{P}^x\{\bar\tau_A > n\}.$$

Then the limit exists. Moreover, F_A is harmonic on A^c; $F_A \equiv 0$ on A; and there exists a constant K_A such that as $|x| \to \infty$,

$$F_A(x) = \log |x \cdot \Gamma^{-1} x| + K_A + o(1).$$

Proof. The existence of the limit was proved in Corollary 8.4; in fact, if we only assume $p \in L_0^2$, we get

$$F_A(x) = 2\pi\sqrt{\det \Gamma}\, \left[a^*(x) - \sum_{z\in A} \mathbb{P}^x\{S_{\bar\tau_A} = z\}\, a^*(z) \right]. \tag{69}$$

This shows that F_A is harmonic in $\mathbb{Z}^2 \setminus A$ and $F \equiv 0$ on A. The previous proposition shows that as $|x| \to \infty$,

$$\sum_{z\in A} \mathbb{P}^x\{S_{\bar\tau_A} = z\}\, a^*(z) = o(1) + \sum_{z\in A} \mathrm{hm}_A(z)\, a^*(-z).$$

If $p \in L_0^{2+\epsilon}$, Theorem 11.1 gives an expansion for $a^*(x)$ from which we conclude this proposition where

$$K_A = 2\pi\sqrt{\det \Gamma}\, \left[C - \sum_{z\in A} \mathrm{hm}_A(z)\, a^*(-z) \right]$$

and C is the constant from Theorem 11.1. □

Remark. Suppose $p \in L_0^2$ is an increment distribution in $\mathbb{Z}^2$, $0 \in A$ is a finite set, and $f : \mathbb{Z}^2 \to \mathbb{R}$ is a bounded function that is harmonic on $\mathbb{Z}^2 \setminus A$. Then

$$\lim_{x\to\infty} f(x) = \sum_{y\in A} \mathrm{hm}_A(y)\, f(y).$$

If $F_A(x)$ is as defined in (69), then only assuming $p \in L_0^2$ we see that $f_A(x) := a^*(x) - F_A(x)$ is a bounded function that is harmonic in $\mathbb{Z}^2 \setminus 0$. Hence, by Proposition 7.2,

$$f_A(x) = \sum_{y\in A} H_{A^c}(x,y)\, f_A(y) = \sum_{y\in A} H_{A^c}(x,y)\, a^*(y).$$

12 Invariance principle

Let $C^d[0,1]$ denote the space of continuous functions $f : [0,1] \to \mathbb{R}^d$ which we consider as a Banach space under the norm $\|f\|_\infty = \max\{|f(t)| : 0 \le t \le 1\}$. A *(driftless) Brownian motion (from time 0 to time 1 starting at the origin) with covariance matrix Γ* is a $C^d[0,1]$-valued random variable, denoted $B_t, 0 \le t \le 1$, such that for each $0 \le t_1 < t_2 < \cdots < t_n \le 1$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent and $B_{t_j} - B_{t_{j-1}}$ is normal, mean zero, covariance matrix $(t_j - t_{j-1})\,\Gamma$. There is only one distribution on $C^d[0,1]$ satisfying these conditions; the distribution is often called *Wiener measure (with drift zero and covariance matrix Γ)*.

If $p \in L_0^2$, let S_n denote the corresponding random walk, and let us define $S_t, 0 \le t \le \infty$ by linear interpolation, i.e., if n is a positive integer and $t \in [0,1]$,

$$S_{n+t} = S_n + t\, X_{n+1}.$$

Let $W_{t,n} = S_t/\sqrt{n}$. Then for each n, $W_{t,n}, 0 \le t \le 1$, is a $C^d[0,1]$-valued random variable.

Theorem 12.1 (Invariance Principle) *If $p \in L_0^2$ with covariance matrix Γ and B_t is a Brownian motion with covariance matrix Γ, then as $n \to \infty$, $W_{\cdot,n} \Longrightarrow B_\cdot$, where $\Longrightarrow$ denotes convergence in distribution as random variables in $C^d[0,1]$.*

We will discuss the proof of this theorem in this section. The basic outline is as follows.

- Show that the sequence of random variables $W_{\cdot,n}$ is "tight" in $C^d[0,1]$.
- Conclude that every subsequence $W_{\cdot,n_j}$ has a further subsequence that converges to a random variable in $C^d[0,1]$.
- Show that the "finite dimensional distributions" of $W_{\cdot,n}$ converge to those of $B_\cdot$.
- Use the fact that $B_\cdot$ is the unique distribution in $C^d[0,1]$ with those finite dimensional distributions to conclude that the only subsequential limit is $B_\cdot$ and hence the sequence converges to Brownian motion.

The third of these steps is the usual central limit theorem and the second and fourth are standard arguments in probability theory, We will not discuss these steps here, but instead will focus on the first step. If $f \in C^d[0,1]$, define the oscillation by

$$\mathrm{osc}(f; r) = \sup\{|f(s) - f(t)| : 0 \le s,t \le 1, |s-t| \le r\}.$$

If $\delta(r)$ is a function with $\delta(0+) = 0$ and $M < \infty$, let

$$K(\delta; M) = \{f \in C^d[0,1] : \|f\|_\infty \le M \text{ and } \mathrm{osc}(f;r) \le \delta(r) \text{ for all } r \in [0,1]\}.$$

Then the Arzela-Ascoli theorem shows that $K(\delta; M)$ is compact. Define

$$\mathrm{osc}(S, r; n) = \max\{|W_{t,n} - W_{s,n}| : 0 \le s,t \le 1,\ |t-s| \le r\}.$$

In order to prove tightness, we need to show that for every $\epsilon > 0$ we can find a $\delta(r)$ with $\delta(0+) = 0$ and an M such that

$$\mathbb{P}\{\|W_{\cdot,n}\|_\infty \leq M; \operatorname{osc}(S,r,n) \leq \delta(r) \text{ for every } r,n\} \geq 1 - 2\epsilon.$$

Since $\lim_{M\to\infty} \sup_n \mathbb{P}\{\|W_{\cdot,n}\|_\infty \geq M\} = 0$, it suffices to find a function $\delta(r)$ with $\delta(0+) = 0$ and

$$\mathbb{P}\{\operatorname{osc}(S,r,n) \leq \delta(r) \text{ for every } r,n\} \geq 1 - \epsilon.$$

We claim that it suffices to find sequences of positive numbers δ_j, u_j tending to zero such that for each j, n.

$$\mathbb{P}\{\operatorname{osc}(S, 2^{-j}; n) \geq \delta_j\} \leq u_j. \tag{70}$$

Indeed by taking a subsequence we see that (70) implies that there exist $\delta_j, \rho_j \to 0$ such that for each j, n,

$$\mathbb{P}\{\operatorname{osc}(S, \rho_j; n) \geq \delta_j\} \leq 2^{-j-1}.$$

By summing over j, we get for all n,

$$\mathbb{P}\{\operatorname{osc}(S, \rho_j; n) \geq \delta_j \text{ for some } j \geq k\} \leq 2^{-k}.$$

Given $\epsilon > 0$, find choose k so that $2^{-k} < \epsilon/2$, choose M so that $\mathbb{P}\{\|W_{\cdot,n}\|_\infty \geq M/2\} \leq \epsilon/2$, and let $\delta(r) = M$ for $r \geq \rho_k$ and

$$\delta(r) = \delta_l, \quad \rho_{l+1} \leq r < \rho_l, \quad l = k, k+1, \ldots.$$

Then for every n,

$$\mathbb{P}\{\operatorname{osc}(S, r; n) \leq \delta(r) \text{ for all } r \in (0,1]\} \geq 1 - \epsilon.$$

This is the required tightness estimate.

We now focus on the required estimate (70). Suppose Let us call (δ_k, u_k) an *oscillation pair* for the random walk if

$$\lim_{k\to\infty} \delta_k = 0, \qquad \lim_{k\to\infty} u_k = 0, \tag{71}$$

and there exist constants c_1, c_2 such that for all k, n,

$$\mathbb{P}\{\operatorname{osc}(S, 2^{-k}; n) \geq c_1\, \delta_k\} \leq c_2\, u_k.$$

Then (70) holds if and only if there exists an oscillation pair. Note that if $S_n = (S_n^{(1)}, \ldots, S_n^{(d)})$ and (δ_k, u_k) are oscillation pairs for each of the one-dimensional random walks $S_n^{(1)}, \ldots, S_n^{(d)}$, then (δ_k, u_k) is also an oscillation pair for S_n. We claim that (δ_k, u_k) is an oscillation pair if and only if (71) holds and there exist c_1, c_2 such that for all k, n,

$$2^k\, \mathbb{P}\{|S_{n/2^k}| \geq c_1\, \delta_k \sqrt{n}\} \leq c_2\, u_k. \tag{72}$$

(If $n/2^k$ is not an integer we use linear interpolation.) To see this, let $r = r_{k,n} = \lfloor n/2^k \rfloor + 1$. The necessity of (72) can be seen by considering the independent events

$$\{|S_r| \geq c_1\ \delta_k \sqrt{n}\}, \{|S_{2r} - S_r| \geq c_1\, \delta_k \sqrt{n}\}, \ldots, \{|S_{r(2^k-1)} - S_{r(2^k-2)}| \geq \delta_k \sqrt{n}\}.$$

The sufficiency can be seen in noting that the event $\{\operatorname{osc}(S, 2^{-k}; n) \geq c\, \delta_k\}$ is contained in the event

$$\left\{ \max_{l=1,\ldots,2^k} \max_{0\leq j\leq r} |S_{j+kr} - S_{kr}| \geq (c/3)\, \delta_k \sqrt{n} \right\}.$$

The probability of this event is bounded above by

$$2^k\, \mathbb{P}\{\max_{0\leq j\leq r} |S_j| \geq (c/3)\, \delta_k \sqrt{n}\} \leq c'\, 2^k\, \mathbb{P}\{|S_r| \geq (c/3)\, \delta_k \sqrt{n}\}.$$

The inequality follows from Lemma 6.11.

We can rewrite (72) as

$$2^k\, \mathbb{P}\{|S_{n/2^k}| \geq c_1\, \delta_k\, 2^{k/2} \sqrt{n/2^k}\} \leq c_2\, u_k.$$

If $p \in L_0^2$, Proposition 9.8 shows that we can find $\delta_k, u_k \to 0$ that satisfies this.

13 A class of random walks

If $p \in L_0^2$ is an increment distribution of a random walk in $\mathbb{Z}^d$, we will write $\mathcal{J}(x) = |\Lambda^{-1} x| = \sqrt{x \cdot \Gamma^{-1} x}$. There exist c_1, c_2 such that

$$c_1\, |x| \leq \mathcal{J}(x) \leq c_2\, |x|.$$

It will be useful to consider the approximate level sets of the Green's function G or the potential kernel a. Under sufficient moment conditions, it will suffices to consider the approximate level sets of $\mathcal{J}$. let

$$\mathcal{C}_n = \{y \in \mathbb{Z}^d : \mathcal{J}(y) < n\}.$$

We write $\hat{G}_n$ for $G_{\mathcal{C}_n}$, $\hat{\tau}_n$ for $\tau_{\mathbb{Z}^d \setminus \mathcal{C}_n}$, and $\hat{H}_n(x,y)$ for $H_{\mathcal{C}_n}(x,y)$. The next defintion will make the moment assumptions that we will use. For some of our results, this assumption is stronger than needed, but it will avoid making things too complicated if we restrict to this class.

Definition. If $d \geq 2$, we define $\mathcal{S}_d$ to be the set of increment distributions p in $\mathbb{Z}^d$ that are in $\mathcal{U}_0^{3+\epsilon}$ for some $\epsilon > 0$.

Remark. We will not define $\mathcal{S}_d$ for $d = 1$. If we write $p \in \mathcal{S}_d$, then the implicit assumption will be that $d \geq 2$.

Remark. The results of the previous section show that if $p \in L_0^{2+d+\epsilon}$ then $p \in \mathcal{S}_d$. If $p \in \mathcal{S}_d$, then $p(x) = o(|x|^{-3-d})$. If $d \geq 3$ and $p \in \mathcal{S}_d$, then Corollary 10.4 tells us that

$$G(x) = \frac{C_d}{\mathcal{J}(x)^{2-d}} + O\left(\frac{1}{|x|^{1-d}}\right), \qquad |x| \to \infty,$$

where C_d is as in (63). If $d = 2$ and $p \in \mathcal{S}_d$, then Proposition 11.2 tells us that

$$a(x) = 2\, C_2\, \log \mathcal{J}(x) + C + O\left(\frac{1}{|x|}\right),$$

where $C_2 = [2\pi\sqrt{\det\Gamma}]^{-1}$ and C is a constant (depending on p).

Remark. Since $p^*(x) = p(-x)$, we can see that $p \in \mathcal{S}_d$ implies $p^* \in \mathcal{S}_d$. Moreover, $\mathcal{J}^* = \mathcal{J}, \mathcal{C}_n^* = \mathcal{C}_n$, so we will not need to use the notations $\mathcal{J}^*, \mathcal{C}^*$.

Remark. Note that if $p \in \mathcal{S}_d$, then there exist constants c_1, c_2 such that for all n,

$$\inf\{G(x) : x \in \mathcal{C}_n\} \geq \frac{c_1}{n^{d-1}} + \sup\{G(x) : x \in \mathbb{Z}^d \setminus \mathcal{C}_{n+c_2}\}, \quad d \geq 3, \tag{73}$$

$$\sup\{a(x) : x \in \mathcal{C}_n\} \leq -\frac{c_1}{n} + \inf\{a(x) : a \in \mathbb{Z}^d \setminus \mathcal{C}_{n+c_2}\}. \tag{74}$$

Also, there exist c_3, c_4 such that

$$|G(x) - G(y)| \leq \frac{c_3\,|x-y|}{|x|^{d-1}}, \qquad |a(x) - a(y)| \leq \frac{c_3\,|x-y|}{|x|}, \tag{75}$$

provided that $|x - y| \leq c_4\,|x|$.

Proposition 13.1 *If $p \in \mathcal{S}_d$, there exists c_1 such that if A is a finite subset of $\mathbb{Z}^d$, $x, y \in A$, $|e| = 1$,*

$$|G_A(y,x) - G_A(y, x+e)| \leq c_1 \left[\frac{1}{\operatorname{dist}(x,\partial A)^{d-1}} + \frac{1}{|y-x|^{d-1}}\right],$$

$$|G_A(x,y) - G_A(x+e, y)| \leq c_1 \left[\frac{1}{\operatorname{dist}(x,\partial A)^{d-1}} + \frac{1}{|y-x|^{d-1}}\right],$$

Proof. Recall from (46 that

$$G_A(y,x) = G(y,x) - \sum_{z\in\partial\mathbb{C}_n} \hat{H}_n(y,z)\,G(z,x), \quad d = 3,$$

$$G_A(y,x) = -a(y,x) + \sum_{z\in\partial\mathbb{C}_n} \hat{H}_n(x,y)\,a(z,x), \quad d = 2.$$

We may assume $x = 0$ and let $m = \min\{\operatorname{dist}(0,\partial A), |y|\}$. By (75),

$$|G(z,0) - G(z,e)| \leq c\,m^{1-d}, \qquad |z| \geq m,\ d \geq 3,$$

$$a(z,0) - a(z,e)| \leq c\,m^{-1}, \qquad |z| \geq m,\ d = 2.$$

Plugging in gives the proposition. ☐

Remark. If ∇^q is a difference operator, we can give similar bounds on $|\nabla_x^q G_A(y,x)|$ if we assume enough moments. Note that

$$\nabla_x^q G_A(y,x) = \nabla_x^q G(y,x) - \sum_{z\in\partial\mathbb{C}_n} \hat{H}_n(y,z)\,\nabla_x^q G(z,x), \quad d \geq 3,$$

$$\nabla_x^q G_A(y,x) = -\nabla_x^q a(y,x) + \sum_{z\in\partial\mathbb{C}_n} \hat{H}_n(y,z)\,\nabla_x^q a(z,x), \quad d = 2.$$

Therefore, if $|x - y| \geq m$ and $\operatorname{dist}(x, \partial\mathcal{C}_n) \geq m$,

$$|\nabla_x^q G_A(y,x)| \leq 2 \sup_{|z|\geq m} |\nabla^q G(z)|, \quad d \geq 3,$$

$$|\nabla_x^q G_A(y,x)| \leq 2 \sup_{|z|\geq m} |\nabla^q a(z)|, \quad d = 2.$$

Proposition 13.2 (Gambler's ruin) *If $p \in \mathcal{S}_d$, there exist c_1, c_2 such that for all n sufficiently large, if $x \in \mathcal{C}_n \setminus \mathcal{C}_{3n/4}$ and $T = T_n$ is the first time k such that $S_k \in \mathcal{C}_{n/2}$ or $S_k \notin \mathcal{C}_n$, then*

$$\frac{c_1\,[\operatorname{dist}(x,\partial\mathcal{C}_n) + 1]}{n} \leq \mathbb{P}^x\{S_T \in \mathcal{C}_{n/2}\} \leq \frac{c_2\,[\operatorname{dist}(x,\partial\mathcal{C}_n) + 1]}{n}.$$

Proof. The upper bound follows immediately from the gambler's ruin estimate for hyperplanes, Proposition 9.6. We will do the lower bound for $d \geq 3$; a similar argument works for $d = 2$ using the potential kernel. Consider the martingale $M_n = G_n^*(S_{n\wedge T})$. By the optional sampling theorem,

$$\begin{aligned} G^*(x) &= \mathbb{E}^x[M_0] = \mathbb{E}^x[M_T] \\ &= \mathbb{E}^x[G^*(S_T); S_T \notin \mathcal{C}_n] + \mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_{n/2} \setminus \mathcal{C}_{n/4}] + \mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_{n/4}]. \end{aligned}$$

By (73), there is an l (independent of n) such that if $x \in \mathcal{C}_{n-l} \setminus \mathcal{C}_{3n/4}$,

$$G^*(x) \geq G^*(z) + \frac{c\operatorname{dist}(x,\partial\mathcal{C}_n)}{n^{d-1}}, \qquad z \notin \mathcal{C}_n,$$

and hence for such x,

$$\mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_{n/2} \setminus \mathcal{C}_{n/4}] + \mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_{n/4}] \geq \frac{c\operatorname{dist}(x,\partial\mathcal{C}_n)}{n^{d-1}}.$$

Using a last-exit decomposition as before we can see that

$$\mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_{n/4}] = \sum_{y\in\mathcal{C}_n\setminus\mathcal{C}_{n/2}} \sum_{z\in\mathcal{C}_{n/4}} G_{\mathcal{C}_n\setminus\mathcal{C}_{n/2}}(x,y)\,p(z-y)\,G^*(z)$$

Since $p \in \mathcal{S}_d$, $p(z-y) = o(n^{-3-d})$ for all $z \in \mathcal{C}_{n/4}, y \notin \mathcal{C}_{n/2}$. Hence,

$$\begin{aligned} \mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_{n/4}] &\leq \mathbb{E}^x[T]\,o(n^{-3-d}) \sum_{z\in\mathcal{C}_{n/4}} G^*(z) \\ &\leq o(n^{-1-d}) \sum_{z\in\mathcal{C}_{n/4}} [1 + |z|]^{2-d} = o(n^{1-d}). \end{aligned}$$

Hence for all n sufficiently large and all $x \in \mathcal{C}_{n-l} \setminus \mathcal{C}_{n/2}$,

$$\frac{c\,\mathrm{dist}(x, \partial\mathcal{C})}{n^{d-1}} \leq \mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_{n/2} \setminus \mathcal{C}_{n/4}] \leq c_1 \frac{\mathbb{P}^x\{S_T \in \mathcal{C}_{n/2} \setminus \mathcal{C}_{n/4}\}}{n^{d-2}}.$$

For $x \in \mathcal{C}_n \setminus \mathcal{C}_{n-l}$ we can use an argument as in the first paragraph of Proposition 9.6. $\square$

Remark. The assumption that $p(z) = o(|z|^{-3-d})$ is more than is needed for this argument. It suffices to assume that $p(z) = o(|z|^{-2-d})$. In this case, however, we must take advantage of the stronger estimate

$$\mathbb{E}^x[T] \leq c\, n\, \mathrm{dist}(x, \partial\mathcal{C}_n).$$

Proposition 13.3 *If $p \in \mathcal{S}_d$, then for all n and all $x \in \mathcal{C}_n$,*

$$\mathbb{E}^x[G(S_{\hat\tau_n})] = \frac{C_d}{n^{d-2}} + O\left(\frac{1}{n^{d-1}}\right), \quad d \geq 3,$$

$$\mathbb{E}^x[a(S_{\hat\tau_n})] = 2\,C_2\,\log n + C + O\left(\frac{1}{n}\right), \quad d = 2.$$

$$\hat G_n(0,0) = G(0,0) - \frac{C_d}{n^{d-2}} + O\left(\frac{1}{n^{d-1}}\right), \quad d \geq 3,$$

$$\hat G_n(0,0) = 2\,C_2\,\log n + C + O\left(\frac{1}{n}\right), \quad d = 2.$$

$$\hat G_n(x,0) = \frac{C_d}{\mathcal{J}(x)^{d-2}} - \frac{C_d}{n^{d-2}} + O\left(\frac{1}{|x|^{d-1}}\right), \quad d \geq 3,$$

$$\hat G_n(x,0) = 2\,C_2[\log n - \log \mathcal{J}(x)] + O\left(\frac{1}{|x|}\right), \quad d = 2.$$

Proof. Note that

$$\mathbb{E}^x[G(S_{\hat\tau_n})] = \sum_y \hat H_n(x,y)\,G(y), \quad d \geq 3,$$

$$\mathbb{E}^x[a(S_{\hat\tau_n})] = \sum_y \hat H_n(x,y)\,a(y), \quad d = 2.$$

But (75) tells us that for $y \notin \mathcal{C}_n$,

$$|G(y) - \frac{C_d}{n^{d-2}}| \leq c\,\frac{1 + \mathrm{dist}(y, \mathcal{C}_n)}{n^{d-1}}, \quad d \geq 3,$$

$$|a(y) - 2\,C_2\,\log n - C| \leq c\,\frac{1 + \mathrm{dist}(y, \mathcal{C}_n)}{n}, \quad d = 2.$$

Also, since $p \in L_0^3$, Proposition 9.5 (applied to $\mathcal{C}_n$ rather than a hyperplane but the proof is virtually the same) implies that

$$\sum_y H_n(x,y)\,\mathrm{dist}(y, \mathcal{C}_n) \leq c.$$

This gives the first two equalities. For the last four equalities we use (46), (48), which give us

$$\hat G_n(x,0) = G(x,0) - \sum_{y \notin \mathcal{C}_n} \hat H_n(x,y)\,G(y,0), \quad d \geq 3$$

$$\hat G_n(x,0) = -a(x,0) + \sum_{y \notin \mathcal{C}_n} \hat H_n(x,y)\,a(y,0), \quad d = 2.$$

$\square$

Remark. By applying the last proposition to the reversed random walk we also obtain

$$\hat G_n(0,x) = \frac{C_d}{\mathcal{J}(x)^{d-2}} - \frac{C_d}{n^{d-2}} + O\left(\frac{1}{|x|^{d-1}}\right), \quad d \geq 3, \tag{76}$$

$$\hat G_n(0,x) = 2\,C_2\,[\log n - \log \mathcal{J}(x)] + O\left(\frac{1}{|x|}\right), \quad d = 2. \tag{77}$$

If $d \geq 3$,

$$\mathbb{P}\{\tau_0 > \hat\tau_n\} = \frac{1}{\hat G_n(0,0)} = \frac{1}{G(0,0)} + O\left(\frac{1}{n^{d-2}}\right),$$

$$\mathbb{P}^x\{\tau_0 < \hat\tau_n\} = \frac{\hat G_n(x,0)}{\hat G_n(0,0)} = \frac{C_d}{G(0,0)}\left[\frac{1}{\mathcal{J}(x)^{d-2}} - \frac{1}{n^{d-2}}\right] + O\left(\frac{1}{|x|^{d-1}}\right).$$

If $d = 2$,

$$\mathbb{P}\{\tau_0 > \hat\tau_n\} = \frac{1}{\hat G_n(0,0)} = \frac{1}{2\,C_2\log n + C}\left[1 + O\left(\frac{\log n}{n}\right)\right],$$

$$\mathbb{P}^x\{\tau_0 < \hat\tau_n\} = \frac{\hat G_n(x,0)}{\hat G_n(0,0)} = \frac{\log n - \log \mathcal{J}(x)}{\log n - (C/2C_2)}\left[1 + O\left(\frac{\log |x|}{|x|}\right)\right].$$

By comparison, we also see that

$$\hat G_n(x,x) = G(0,0) - O\left(\frac{1}{|n - \mathcal{J}(x)|^{d-2}}\right), \quad d \geq 3,$$

$$\hat G_n(x,x) = C_2\log n + O\left(\log n - \log \mathcal{J}(x)\right), \quad d = 2.$$

Hence, if $d \geq 3$,

$$\mathbb{P}\{\tau_x < \hat\tau_n\} = \frac{\hat G_n(0,x)}{\hat G_n(x,x)} = \frac{C_d}{G(0,0)}\left[\frac{1}{\mathcal{J}(x)^{d-2}} - \frac{1}{n^{d-2}}\right] + O\left(\frac{1}{|x|^{d-1}}\right),$$

and if $d = 2$,

$$\mathbb{P}\{\tau_x < \hat{\tau}_n\} = \frac{\hat{G}_n(0,x)}{\hat{G}_n(x,x)} = \frac{\log n - \log \mathcal{J}(x)}{\log n} \left[1 + O\left(\frac{1}{\log|x|}\right)\right].$$

Proposition 13.4 *If $p \in \mathcal{S}_d$, there exist c_1, c_2 such that for all n sufficiently large, if $x \in \mathcal{C}_n \setminus \mathcal{C}_{3n/4}$, then*

$$c_1 \frac{\text{dist}(x, \partial\mathcal{C}_n) + 1}{n^{d-1}} \le \hat{G}_n(x,0) \le c_2 \frac{\text{dist}(x, \partial\mathcal{C}_n) + 1}{n^{d-1}}, \tag{78}$$

$$c_1 \frac{\text{dist}(x, \partial\mathcal{C}_n) + 1}{n^{d-1}} \le \hat{G}_n(0,x) \le c_2 \frac{\text{dist}(x, \partial\mathcal{C}_n) + 1}{n^{d-1}}. \tag{79}$$

Proof. Let T be as in Proposition 13.2. Then by the optional sampling theorem applied to the martingale $M_n = G(S_{n\wedge T}, 0)$,

$$G_n(x,0) = \sum_{y \in \mathcal{C}_{n/2}} \mathbb{P}^x\{S_T = y\}\, G_n(y,0).$$

Since $G_n(y,0) \ge cn^{2-d}$, the lower bound in (78) follows from the lower bound in Proposition 13.2. Also the proof of Proposition 13.2 shows that

$$\sum_{y \in \mathcal{C}_{n/4}} \mathbb{P}^y\{S_T = y\} G_n(y,0) = \text{dist}(x, \partial\mathcal{C}_n)\, o(n^{-1-d}).$$

Also, for $y \in \mathcal{C}_{n/2} \setminus \mathcal{C}_{n/4}$, $G_n(y,0) \le c\, n^{2-d}$. Hence the upper bound in (78) follows from the upper bound in Proposition 13.2. The inequality (79) is obtained by considering the reversed random walk. □

Remark. This proof will work if $p(z) = o(|z|^{-2-d})$.

Lemma 13.5 *If $p \in \mathcal{S}_d$, and $r < 1$, there exist c such that for all n sufficiently large and all $x, y \in \mathcal{C}_{rn}$, $\hat{G}_n(x,y) \ge c\, n^{2-d}$. Morevoer, there exist c_1, c_2 such that if $x \in \mathcal{C}_n \setminus \mathcal{C}_{n/2}$ and $y \in \mathcal{C}_{n/4}$,*

$$c_1 \frac{\text{dist}(x, \partial\mathcal{C}_n) + 1}{n^{d-1}} \le \hat{G}_n(x,y) \le c_2 \frac{\text{dist}(x, \partial\mathcal{C}_n) + 1}{n^{d-1}}, \tag{80}$$

Proof. If $x, y \in \mathcal{C}_{n/4}$, this follows from the arguments as above. For other x, we can use the invariance principle[4] to see that there $q_r > 0$ such that for all n sufficiently large, if $x \in \mathcal{C}_{rn}$, with probability at least q_r, the random walk reaches $\mathcal{C}_{n/4}$ before leaving $\mathcal{C}_n$. The proof the second inequality is then essentially the same as the proof of (78). □

[4] We could have used the arguments of this chapter equally well, but the invariance principle only assumes L_0^2.

Proposition 13.6 *If $p \in \mathcal{S}_d$, there is a constant c such that if $x \in \mathcal{C}_n \setminus \mathcal{C}_{n/2}$, $y, y' \in \mathcal{C}_{n/4}$,*

$$|\hat{G}_n(x,y) - \hat{G}_n(x,y')| \le c\, \frac{|y - y'|}{n^{1-d}}.$$

Moreover, there exist c_1, c_2 such that if n is sufficiently large and $x \in \mathcal{C}_{3n/4} \setminus \mathcal{C}_{n/2}$, $y \in \mathcal{C}_{n/4}$,

$$c_1\, n^{2-d} \le \hat{G}_n(x,y) \le c_2\, n^{2-d}$$

Proof. If $d \ge 3$, (46) implies that

$$|\hat{G}_n(x,y) - \hat{G}_n(x,y')| \le |G(x,y) - G(x,y')| + \sum_z \hat{H}_n(x,z)\, |G(z,y) - G(z,y')|.$$

and a similar formula holds for $d = 2$ with a replacing G. The first inequality then follows from (75). The lower bound for the second inequality follows from the previous lemma and the upper bound follows from $\hat{G}_n(x,y) \le G(x,y)$ for $d = 3$ and a similar argument for $d = 2$. □

Proposition 13.7 *If $d \ge 3$, $p \in \mathcal{S}_d$, $n < m$, and*

$$T_{n,m} = \min\{j : S_j \in \mathcal{C}_n \text{ or } S_j \notin \mathcal{C}_m\},$$

then if $x \in \mathcal{C}_m \setminus \mathcal{C}_n$,

$$\mathbb{P}^x\{S_{T_{n,m}} \in \mathcal{C}_n\} = \frac{\mathcal{J}(x)^{2-d} - m^{2-d}}{n^{2-d} - m^{2-d}}\, [1 + O\left(\frac{1}{n}\right)].$$

In particular, if $x \in \mathbb{Z}^d \setminus \mathcal{C}_n$,

$$\mathbb{P}^x\{S_{\tau_{\mathcal{C}_n}} < \infty\} = \lim_{m\to\infty} \mathbb{P}^x\{S_{T_{n,m}} \in \mathcal{C}_n\} = \frac{\mathcal{J}(x)^{2-d}}{n^{2-d}}\, [1 + O\left(\frac{1}{n}\right)]. \tag{81}$$

Proof. Let $T = T_{n,m}$ and let $q = q(n,m,x) = \mathbb{P}^x\{S_{T_{n,m}} \in \mathcal{C}_n\}$. By the optional sampling theorem,

$$G^*(x) = \mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_n] + \mathbb{E}^x[G^*(S_T); S_T \notin \mathcal{C}_m].$$

Doing estimates have we done before, we get

$$G^*(x) = C_d\, \mathcal{J}(x)^{2-d} + O(|x|^{1-d}),$$

$$\mathbb{E}^x[G^*(S_T); S_T \in \mathcal{C}_n] = q\, C_d\, n^{2-d} + O(n^{1-d}),$$

$$\mathbb{E}^x[G^*(S_T); S_T \notin \mathcal{C}_m] = (1-q)\, C_d\, m^{2-d} + O(m^{1-d}).$$

By solving for q we get the proposition. □

Proposition 13.8 *If $p \in \mathcal{S}_2$, $n < m$, and*

$$T_{n,m} = \min\{j : S_j \in \mathcal{C}_n \text{ or } S_j \notin \mathcal{C}_m\},$$

then if $x \in \mathcal{C}_m \setminus \mathcal{C}_n$,

$$\mathbb{P}^x\{S_{T_{n,m}} \in \mathcal{C}_n\} = \frac{\log m - \log \mathcal{J}(x)}{\log m - \log n}\,[1 + O\left(\frac{1}{n}\right)]$$

Proof. This is proved in the same way as the previous proposition. □

Proposition 13.9 (Harnack Principle) *If $p \in L_0^\infty$, then for every open $V \subset \mathbb{R}^d$ and compact, connected $K \subset V$, there exists a $c = c(K,V) < \infty$ such that the following holds. Let $V_n = \{x \in \mathbb{Z}^d : x/n \in V\}, K_n = \{x \in \mathbb{Z}^d : x/n \in K\}$. Suppose $f : \overline{V}_n \to (0,\infty)$ is p-harmonic in V_n. Then for every $x, y \in K_n$,*

$$f(x) \leq c\, f(y).$$

Proof. We first consider the case where $V = \tilde{V} := \{x \in \mathbb{R}^d : |x \cdot \Gamma^{-1} x| < 1|$ and $K = \tilde{K} := \{x \in \mathbb{R}^d : |x \cdot \Gamma^{-1}x| \leq 1/8\}$. Then $\tilde{V}_n = \mathcal{C}_n$ and $\tilde{K}_n \subset \mathcal{C}_{n/4}$. Since

$$f(x) = \sum_{z \in \partial\mathcal{C}_n} \hat{H}_n(x,z)\, f(z),$$

it suffices to show that there is a c' such that for all $x, y \in \mathcal{C}_{n/4}$ and all $z \in \partial\mathcal{C}_n$,

$$\hat{H}_n(x,z) \leq c'\, \hat{H}_n(y,z). \tag{82}$$

Note that (80) shows that there is a c such that $\hat{G}_n(x,w) \leq \hat{G}_n(y,w)$ for $x, y \in \mathcal{C}_{n/4}$ and $w \in \mathcal{C}_n \setminus \mathcal{C}_{n/2}$. Since $p \in L_0^\infty$, there exists $\epsilon > 0, c_3 < \infty$ such that for each $z \in \partial V_n$,

$$\epsilon \leq \sum_{w \in \mathcal{C}_n \setminus \mathcal{C}_{n-c_3}} P(w,z) \leq 1, \qquad \sum_{w \in \mathcal{C}_{n-c_3}} P(w,z) = 0.$$

Hence,

$$\begin{aligned} \hat{H}_n(x,z) &= \sum_{w \in \mathcal{C}_n \setminus \mathcal{C}_{n-c_3}} \hat{G}_n(x,w)\, P(w,z) \\ &\leq c \sum_{w \in \mathcal{C}_n \setminus \mathcal{C}_{n-c_3}} \hat{G}_n(y,w)\, P(w,z) = c\, H_n(y,z) \end{aligned}$$

For future reference we also note that (for $p \in L_0^\infty$)

$$\hat{H}_n(0,z) = \sum_{w \in \mathcal{C}_n \setminus \mathcal{C}_{n-c_3}} \hat{G}_n(0,w)\, P(w,z) \asymp n^{-d+1}. \tag{83}$$

For general V, K, compactness tell us that we can cover K by a finite number, say $N(K)$, open balls $\mathcal{B}(x_j, \delta_j) := \{x \in \mathbb{R}^d : |x - x_j| < \delta_j\}$ with the property that for some $s > 0$,

$$\mathcal{B}(x_j, \delta_j) \subset (s/2)\,\tilde{K} + x_j \subset s\,\tilde{V} + x_j \subset V.$$

For n sufficiently large, By connectedness of K, if $y, y' \in K_n$, we can a sequence $y = y_0 < y_1 < \cdots < y_k = y'$ with $k \leq N(K)$ and such that y_i, y_{i+1} are both contained in $[\mathcal{B}(x_j,\delta_j)]_n$ for some j. By using the bound for $\tilde{V}, \tilde{K}$ with constants c_1, c_2 we get the bounds for V, K with bounds $c_1^{N(K)}, c_2^{N(K)}$. (This proves it for n sufficiently large, but for small n and bounded V there are only a finite number of positive harmonic functions. It suffices to prove the result for bounded V.) □

Remark. If $p \notin L_0^\infty$, we do not necessarily have a Harnack inequality, even if p has all moments. For example, suppose there is a $y \in \mathbb{Z}^d \setminus \mathcal{C}_n$ such that $p(y) > 0$, but $p(y - x) = 0$ for all $x \in \mathcal{C}_n \setminus \{0\}$, Then,

$$\hat{H}_n(x,y) = \hat{G}_n(x,0)\, p(y).$$

But, $\hat{G}_n(x,0)$ and $\hat{G}_n(0.0)$ are not comparable for all $|x| < cn$. The next proposition gives a weaker result that does hold more generally.

Proposition 13.10 *If $p \in \mathcal{S}_d$, there is a c such that for all $x, y \in \mathcal{C}_n$,*

$$\sum_{z \in \partial\mathcal{C}_n} |\hat{H}_n(0,z) - \hat{H}_n(x,z)| \leq c\,\frac{|x|}{n}. \tag{84}$$

Proof. We start by making an easy observation: if $x, y \in A \subset A'$, then

$$H_{A'}(x,z') = \sum_{z \in \partial A} H_A(x,z)\, H_{A'}(z,z'),$$

and similarly for $H_{A'}(y,z)$. Hence,

$$\begin{aligned} \sum_{z' \in \partial A'} |H_{A'}(x,z') - H_{A'}(y,z')| &\leq \sum_{z \in \partial A} \sum_{z' \in \partial A'} |H_{A'}(x,z) - H_{A'}(y,z)|\, H_A(z,z') \\ &\leq \sum_{z \in \partial A} |H_A(x,z) - H_Z(y,z)|. \end{aligned}$$

It suffices to prove (84) for $|x| \leq c_1\, n$, and by the last inequality and the triangle inequality, we see that it suffices to prove it for $x = e$ with $|e| = 1$. Note that

$$\sum_{z \in \partial\mathcal{C}_n} |\hat{H}_n(0,z) - \hat{H}_n(e,z)| \leq \sum_{y \in \mathcal{C}_n} \sum_{z \in \partial\mathcal{C}_n} |\hat{G}_n(0,y) - \hat{G}_n(e,y)|\, p(y,z).$$

Let c_1 be such that $|z - y| \geq c_1 n$ for $z \in \partial\mathcal{C}_n, y \in \mathcal{C}_{n/2}$. Since $p \in L^3$, Lemma 9.3 shows that

$$\mathbb{P}\{|S_j - S_{j-1}| \geq c_1\, n \text{ for some } j \leq \hat{\tau}_n\} \leq o(1/n).$$

Therefore,

$$\sum_{z\in\partial\mathcal{C}_n} |\hat{H}_n(0,z) - \hat{H}_n(x,e)| \leq o\left(\frac{1}{n}\right) + \sum_{y\in\mathcal{C}_n\setminus\mathcal{C}_{n/2}} \sum_{z\in\partial\mathcal{C}_n} |\hat{G}_n(0,y) - \hat{G}_n(e,y)|\, p(y,z).$$

We claim that

$$|\hat{G}_n(x,0) - \hat{G}_n(x,e)| \leq c\,\frac{\mathrm{dist}(|x|,\partial\mathcal{C}_n)+1}{n^d}, \qquad x\in\mathcal{C}_n\setminus\mathcal{C}_{n/2}.$$

To see this, let $T = \tau_{\partial[\mathcal{C}_n\setminus\mathcal{C}_{n/2}]}$. Then

$$\hat{G}_n(x,0) - \hat{G}_n(x,e) = \sum_{y\in\mathcal{C}_{n/2}} \mathbb{P}^x\{S_T = y\}\,[\hat{G}_n(y,0) - \hat{G}_n(y,e)].$$

For $y\in\mathcal{C}_{n/2}\setminus\mathcal{C}_{n/4}$, we have $|\hat{G}_n(y,0) - \hat{G}_n(y,e)| \leq n^{1-d}$. Therefore, using Proposition 13.2

$$\begin{aligned}\sum_{y\in\mathcal{C}_{n/2}\setminus\mathcal{C}_{n/4}} \mathbb{P}^x\{S_T=y\}\,|\hat{G}_n(y,0)-\hat{G}_n(y,e)| &= \mathbb{P}^x\{S_T\in\mathcal{C}_{n/2}\}\,O(n^{1-d})\\ &\leq [1+\mathrm{dist}(x,\partial\mathcal{C}_n)]\,O(n^{-d}).\end{aligned}$$

For $y\in\mathcal{C}_{n/4}$, we use the estimate $|\hat{G}_n(y,0)-\hat{G}_n(y,e)| \leq [1+|y|]^{1-d}$. (This follows from (75) and the corresponding estimate for $G(y,z)-G(y,z+e)$.) Hence,

$$\sum_{y\in\mathcal{C}_{n/4}} \mathbb{P}^x\{S_T=y\}\,|\hat{G}_n(y,0)-\hat{G}_n(y,e)| \leq c\,n^2 \sum_{y\in\mathcal{C}_{n/4}} o(n^{-3-d})\,[1+|y|]^{1-d} = o(n^{-d}).$$

If $y\in\mathcal{C}_n$,

$$\sum_{z\in\partial\mathcal{C}_n} p(z-y) \leq \mathbb{P}\{|X_1|\geq \mathrm{dist}(y,\partial\mathcal{C}_n)\} = o(\mathrm{dist}(y,\partial\mathcal{C}_n)^{-3}).$$

Since $\#(\mathcal{C}_m\setminus\mathcal{C}_{m-1}) = O(m^{d-1})$,

$$\sum_{y\in\mathcal{C}_{n-k}\setminus\mathcal{C}_{n-k-1}} \sum_{z\in\partial\mathcal{C}_n} |\tilde{G}_n(0,y)-\tilde{G}_n(e,y)|\,p(y,z) \leq n^{-1}\,o(k^{-2}).$$

By summing over k we get the result. $\square$

Remark. The previous proposition is very similar to Proposition 7.4 which tells us that

$$\sum_{z\in\mathbb{Z}^d} |P_{n^2}(0,z) - P_{n^2}(y,z)| \leq c\,\frac{|y|}{n}.$$

Corollary 13.11 *If $p\in\mathcal{S}_d$, there is a $c<\infty$, such that if $A\subset\mathbb{Z}^d$, $f:\overline{A}\to\mathbb{R}$ is harmonic in A, and $x,y\in A$ with $|x-y|\leq \mathrm{dist}(x,\partial A)/2$, then*

$$|f(x)-f(y)| \leq c\,\|f\|_\infty\,\frac{|x-y|}{\mathrm{dist}(x,\partial A)}.$$

Proof. It suffices to prove the result for $x=0$, $|y|=1$. There is a c such that $\mathcal{C}_{cn}\subset\{|z|<n\}$, and

$$\begin{aligned}|f(0)-f(y)| &\leq \|f\|_\infty \sum_{z\in\partial A} |H_{\partial A}(0,z) - H_{\partial A}(y,z)|\\ &\leq \|f\|_\infty \sum_{z\in\partial\mathcal{C}_{cn}} |\hat{H}_{cn}(0,z)-\hat{H}_{cn}(y,z)|.\end{aligned}$$

$\square$

Proposition 13.12 (Weak Harnack Principle) *If $p\in\mathcal{S}_d$, then for every open $V\subset\mathbb{R}^d$ and compact, connected $K\subset V$, there exists a $\rho=\rho(K,V)>0$ such that the following holds. Let $V_n=\{x\in\mathbb{Z}^d : x/n\in V\}, K_n=\{x\in\mathbb{Z}^d : x/n\in K\}$. Then if n is sufficiently large and $x,y\in K_n$,*

$$\sum_{z\in\partial V_n} |H_{V_n}(x,z) - H_{V_n}(y,z)| \leq 1-\rho.$$

Proof. Without loss of generality, assume that $0\in V$. By (84) there is a $c=c(V)$ such that the result holds if $K'=\{z\in\mathbb{R}^d : |z|\leq c\}$. with constant ρ'. However, using the invariance principle and compactness, we can see that there is a $u=u(K,V)>0$ such that a random walk starting at $x\in K_n$ has probability at least u of entering K_n' before leaving V_n. Using the strong Markov property we see that the result holds with $\rho=u^2\rho'$.

Proposition 13.13 *If $p\in L_0^2$, there is a $c>0$ such that for all n and all $x\in\mathcal{C}_n$,*

$$\mathbb{P}^x\{\mathrm{dist}(S_{\hat{\tau}_n},\partial\mathcal{C}_n)\leq c\} \geq \frac{1}{2}.$$

Proof. The argument is the same as that to derive (58). $\square$

Proposition 13.14 *Suppose $p\in L_0^2$. Then if $z\in\partial\mathcal{C}_n$, $x\in\mathcal{C}_{n/4}$,*

$$\hat{H}_n(x,z) \geq \frac{c}{n^{d-1}} \sum_{y\in\mathcal{C}_n} p(z-y).$$

Proof. The argument is the same as that to derive (83). $\square$

Proposition 13.15 *Suppose $p\in\mathcal{S}_d$ and let*

$$b(n) = n^2\,\mathbb{P}\{|X_1|\geq n\}.$$

There exist constants c,c_1 such that if $f:\bar{\mathcal{C}}_n\to(0,\infty)$ is harmonic in $\mathcal{C}_n$ and $x,y\in\mathcal{C}_{n/2}$,

$$f(x)\leq c\,[f(y) + b(c_1 n)\,\|f\|_\infty].$$

In particular, if $p\in L_0^r$,

$$f(x)\leq c\,f(y) + o(n^{2-r})\,\|f\|_\infty.$$

Proof. We choose c_1 sufficiently small so that $\mathrm{dist}(\mathcal{C}_{n/2}, \partial\mathcal{C}_{3n/4}) \geq 2\,c_1\,n$, $\mathrm{dist}(\mathcal{C}_{3n/4}, \partial\mathcal{C}_{7n/8}) \geq 2\,c_1\,n$, $\mathrm{dist}(\mathcal{C}_{7n/8}, \partial\mathcal{C}_n) \geq 2\,c_1\,n$. Let $\sigma_n = \min\{j : |X_j| \geq c_1 n\}$ and let $\hat\tau'_n = \hat\tau_n \wedge (\sigma_n - 1)$. Let

$$\hat G'_n(x,y) = \mathbb{E}^x\Big[\sum_{j=0}^{\hat\tau'_n} 1\{S_j = y\}\Big],$$

$$H'_n(x,z) = \mathbb{P}^x\{S_{\hat\tau_n} = z; \hat\tau_n < \sigma_n\}.$$

Since $\mathbb{P}\{\sigma_n < \hat\tau_n\} \leq c\,n^2\,\mathbb{P}\{X_j| \geq c_1\,n\}$,

$$\sum_{z\in\partial\mathcal{C}_n} H'_n(x,z)\,f(z) \leq f(x) \leq c\,b(c_1 n)\,\|f\|_\infty + \sum_{z\in\partial\mathcal{C}_n} H'_n(x,z)\,f(z).$$

Hence, it suffices to show that there is a c such that for $x,y\in\mathcal{C}_{n/2}$ and $z\in\partial\mathcal{C}_n$, $H'_n(x,z) \leq c\,H'_n(y,z)$. By a last-exit decomposition,

$$H'_n(x,z) = \sum_{w\in\mathcal{C}_n, |z-w|\leq c_1 n} G'_n(x,w)\,p(z-w).$$

Hence it suffices to show that there is a c such that for $x,y\in\mathcal{C}_{n/2}$ and $w\in\mathcal{C}_n\setminus\mathcal{C}_{7n/8}$, $G'_n(x,w) \leq c\,G'_n(y,w)$. By considering the reversed random walk, it suffices to show $G'_n(w,x) \leq c\,G'_n(w,y)$. Let $T = T_n = \inf\{j : S_j \in \mathcal{C}_{3n/4}$ or $\partial\mathcal{C}_n\}$. Then the strong Markov property implies

$$G'_n(w,x) = \sum_{w_1\in\mathcal{C}_{3n/4}\setminus\mathcal{C}_{5n/4}} \mathbb{P}^w\{S_{T\wedge\sigma_n} = w_1\}\,G'_n(w_1,y),$$

and similarly for $G'_n(w,y)$. Hence it suffices to show that $G'_n(w_1,x) \leq c\,G_n(w_1,y)$ for $w_1\in\mathcal{C}_{3n/4}\setminus\mathcal{C}_{5n/8}$, $x,y\in\mathcal{C}_{n/2}$. But,

$$G'_n(w_1,x) = G_n(w_1,x) + O(n^2)\,b(c_1 n),$$

and $G'_n(w,x) \asymp n^{2-d}$. Therefore, since $p\in\mathcal{S}_d$, $G'_n(w_1,x) \asymp G_n(w_1,x) \asymp n^{2-d}$. This finishes the first inequality and the second follows from the fact that $\mathbb{P}\{|X_1| \geq n\} = o(n^{-r})$ if $p\in L^r_0$.
□

14 Capacity, Transient Case

For this section A will always be a finite subset of $\mathbb{Z}^d$ and we will write $h_A(x,y)$ for $H_{A^c}(x,y)$. We will also write $h_A(x,\infty) = \overline{\mathrm{Es}}_A(x)$ so that for all x,

$$\sum_{y\in A\cup\{\infty\}} h_A(x,y) = 1$$

For $x,y\in A$, we write $K_A(x,y) = \mathbb{P}^x\{\tau_A = y\}$. If p is a recurrent random walk, then K_A is the transition matrix for a finite Markov chain with state space A. If p is a transient random walk, then K_A is a transition matrix for a subMarkov chain, or equivalently, it gives the transition matrix for a Markov chain with state space $A\cup\{\infty\}$ where ∞ is an absorbing state and

$$K_A(x,\infty) = 1 - \sum_{y\in A} K_A(x,y).$$

If p is symmetric, then K_A is symmetric. Otherwise, $K_A(y,x) = K^*_A(x,y)$, where $K^*_A(x,y)$ denotes the corresponding quantity for the reversed random walk. Note that

$$K_A(x,y) = \sum_{z\in\mathbb{Z}^d} p(z-x)\,h_A(z,y).$$

Lemma 14.1 *Suppose p is a transient increment distribution and $f:\mathbb{Z}^d\to\mathbb{R}$ is a bounded function that is harmonic on $\mathbb{Z}^d\setminus A$ where A is a finite set. If p is transient, let $f(\infty)$ be the constant from Proposition 7.7 so that for $x\notin A$,*

$$f(z) = f(\infty)\,\mathbb{P}^z\{\tau_A = \infty\} + \sum_{y\in A} h_A(z,y)\,f(y).$$

Then

$$\sum_{x\in\mathbb{Z}^d} \mathbf{L}f(x) = \sum_{x\in A}\mathbf{L}f(x) = \sum_{x\in A}\sum_{y\in A\cup\{\infty\}} K_A(x,y)\,[f(y)-f(x)].$$

If f is nonnegative, and $f(\infty)=0$, then

$$\sum_{x\in A}\mathbf{L}f(x) \leq 0. \tag{85}$$

Proof. The first inequality is immediate. However, since f is harmonic on $\mathbb{Z}^d\setminus A$,

$$\sum_{z\in\mathbb{Z}^d} p(z-x)\,f(z) = \sum_{z\in\mathbb{Z}^d}\sum_{y\in A\cup\{\infty\}} p(z-x)\,h_A(z,y)\,f(y) = \sum_{y\in A\cup\{\infty\}} K_A(x,y)\,f(y).$$

Also, if $f(\infty)=0$,

$$\begin{aligned}\sum_{x\in A}\sum_{y\in A\cup\{\infty\}} K_A(x,y)\,f(y) &= \sum_{y\in A} f(y)\sum_{x\in A} K^*_A(y,x)\\ &= \sum_{y\in A} f(y)\,[1-\mathrm{Es}^*_A(y)] \leq \sum_{x\in A} f(x).\end{aligned}$$

□

If p is a transient increment distribution, we define

$$\mathrm{cap}(A) = \sum_{z\in A}\mathrm{Es}_A(z) = \sum_{z\in\mathbb{Z}^d}\mathbf{L}\overline{\mathrm{Es}}_A(x).$$

Suppose $z \in A$ and let $\sigma_A = \max\{j : S_j \in A\}$. Then a last-exit decomposition gives

$$\mathbb{P}^z\{S_{\sigma_A} = w\} = G(z,w)\,\mathrm{Es}_A(w).$$

Therefore, for every $z \in A$,

$$1 = \sum_{w \in A} G(z,w)\,\mathrm{Es}_A(w).$$

More generally, for every $z \in \mathbb{Z}^d$,

$$\overline{\mathrm{Es}}_A(z) = \sum_{w \in A} G(z,w)\,\mathrm{Es}_A(w).$$

If $f : \mathbb{Z}^d \to \mathbb{R}$ is a function, we define the potential

$$Gf(x) = \sum_{w \in \mathbb{Z}^d} G(x,w)\,f(w).$$

This is well defined provided that

$$\sum_{w \in \mathbb{Z}^d} G(x,w)\,|f(w)| < \infty$$

for every $x \in \mathbb{Z}^d$. Note that

$$\mathbf{L}Gf(x) = \sum_{w \in \mathbb{Z}^d} \mathbf{L}_x G(x,w)\,f(w) = -f(x).$$

For this reason the operator $-G$ is often called the *inverse of the Laplacian.* Note that if f is supported on A, then Gf is harmonic on $\mathbb{Z}^d \setminus A$.

Proposition 14.2 *If p is transient,*

$$\mathrm{cap}(A) = \sup \sum_{w \in A} f(w),$$

where the supremum is over all $f \geq 0$ supported on A such that $Gf(x) \leq 1$ for all x.

Proof. We have already shown that $f(w) = \mathrm{Es}_A(w)$ satisfies $Gf \leq 1$. Now suppose f is a nonnegative function supported on A and let $F(x) = Gf(x)$. Then F is the unique nonnegative function on $\mathbb{Z}^d$ that is harmonic on $\mathbb{Z}^d \setminus A$; takes boundary value Gf on A; and $F(\infty) = 0$. Proposition 7.7 establishes the uniqueness of this function. Let $\tilde{F}$ be the corresponding solution of the Dirichlet problem with boundary conditions $1 - F$ on A and 0 at ∞. Then, using (85),

$$\sum_{y \in A} f(y) = -\sum_{y \in A} \mathbf{L}F(x) \leq -\sum_{y \in A} \mathbf{L}F(x) - \sum_{y \in A} \mathbf{L}\tilde{F}(x) = \sum_{y \in A} \mathrm{Es}_A(x).$$

$\square$

Let $\mathcal{B}_n = \{x \in \mathbb{Z}^d : |x| < n\}$ and let $\xi_n = \tau_{\mathbb{Z}^d \setminus \mathcal{B}_n}$. Note that if $A \subset \mathbb{Z}^d$ is finite,

$$\begin{aligned}
\mathrm{cap}(A) &= \sum_{x \in A} \mathrm{Es}_A(x) \\
&= \lim_{n\to\infty} \sum_{x \in A} \mathbb{P}^x\{\xi_n < \tau_A\} \\
&= \lim_{n\to\infty} \sum_{x \in A} \sum_{y \in \mathbb{Z}^d \setminus \mathcal{B}_n} \mathbb{P}^x\{S_{\tau_A \wedge \xi_n} = y\} \\
&= \lim_{n\to\infty} \sum_{y \in \mathbb{Z}^d \setminus \mathcal{B}_n} \sum_{x \in A} \mathbb{P}^y\{S^*_{\tau_A \wedge \xi_n} = x\} \\
&= \lim_{n\to\infty} \sum_{y \in \mathbb{Z}^d \setminus \mathcal{B}_n} \mathbb{P}^y\{S^*_{\tau_A \wedge \xi_n} \in A\}, \qquad (86)
\end{aligned}$$

where we use S_j^* to denote the reversed random walk. We see that escape probabilities of A by S are closely related to hitting probabilities of A by S^*.

Proposition 14.3 *Suppose p is the increment distribution of a transient random walk in $\mathbb{Z}^d$.*

- *If $A \subset B$ are finite sets, then* $\mathrm{cap}(A) \leq \mathrm{cap}(B)$.
- *If A, B are finite subsets, then*

$$\mathrm{cap}(A \cup B) \leq \mathrm{cap}(A) + \mathrm{cap}(B) - \mathrm{cap}(A \cap B).$$

- *If $d \geq 2$ and A_n is an increasing sequence of sets such that $\cup A_n$ is infinite, then*

$$\lim_{n\to\infty} \mathrm{cap}(A_n) = \infty.$$

Proof. The first assertion is immedate using either Proposition 14.2 or (86). The second assertion follows from (86) and the observation that if $A, B \subset \mathcal{B}_n$ and $y \in \mathbb{Z}^d \setminus \mathcal{B}_n$,

$$\begin{aligned}
\mathbb{P}^y\{S^*_{\sigma_{A\cup B}} \in A \cup B\} &= \mathbb{P}^y\{S^*_{\sigma_A} \in A\} + \mathbb{P}^y\{S^*_{\sigma_B} \in B\} - \mathbb{P}^y\{S^*_{\sigma_A} \in A, S^*_{\sigma_B} \in B\} \\
&\leq \mathbb{P}^y\{S^*_{\sigma_A} \in A\} + \mathbb{P}^y\{S^*_{\sigma_B} \in B\} - \mathbb{P}^y\{S^*_{\sigma_{A\cap B}} \in A \cap B\},
\end{aligned}$$

where $\sigma_V = \tau_V \wedge \xi_n$. For the third assertion note that

$$\mathbb{P}^y\{S^*_{\sigma_A} \in A, S^*_{\sigma_x} = x\} \leq \epsilon(A,x)\,[\,\mathbb{P}^y\{S^*_{\sigma_A} \in A\} + \mathbb{P}^y\{S^*_{\sigma_x} = x\}\,],$$

where

$$\epsilon(A,x) = \max\left\{1 - \mathrm{Es}_A(x),\ \max\{1 - \mathrm{Es}_{\{x\}}(z) : z \in A\}\right\}.$$

Letting $n \to \infty$, we see that

$$\mathrm{cap}(A \cup \{x\}) \geq [1 - \epsilon(A,x)]\,[\mathrm{cap}(A) + \mathrm{cap}(\{x\})] = [1 - \epsilon(A,x)]\,[\mathrm{cap}(A) + G(0,0)^{-1}].$$

But if $d \geq 2$, for fixed finite A, Proposition 6.16 implies that $\lim_{|x|\to\infty} \epsilon(A,x) = 0$. Hence, if A is finite and B is infinite, we can find $x \in B$ with

$$\mathrm{cap}(A \cup \{x\}) \geq \mathrm{cap}(A) + (1/2)\, G(0,0)^{-1}.$$

This gives the third assertion. □

Definition. If $d \geq 3$, let $\mathcal{R}_d$ be the set of increment distributions p of random walks in $\mathbb{Z}^d$, that are in L_0^2 and such that there exist c_1, c_2 such that for all

$$\frac{c_1}{(1+|x|)^{d-2}} \leq G(x) \leq \frac{c_2}{(1+|x|)^{d-2}}. \tag{87}$$

We will only define $\mathcal{R}_d$ for $d \geq 3$; in particular, the statement "$p \in \mathcal{R}_d$" implies that $d \geq 3$.

Remark. Recall that if $d = 3$, then $p \in \mathcal{R}_3$ if and only if $p \in L_0^2$. Also, $\mathcal{S}_d \subset \mathcal{R}_d$.

Corollary 14.4 *Suppose p_1, p_2 are two increment distributions for transient walks in $\mathbb{Z}^d$ with Green's functions G_1, G_2 respectively. Suppose there exists a c_1 such that*

$$G_1(x,y) \leq c_1\, G_2(x,y)$$

for all $x, y \in \mathbb{Z}^d$. Then if $\mathrm{cap}_1, \mathrm{cap}_2$ denote the capacities with respect to p_1, p_2, respectively,

$$\mathrm{cap}_2(A) \leq c_1\, \mathrm{cap}_1(A)$$

for every finite $A \in \mathbb{Z}^d$.

Proof. Let $f_2(x) = \mathrm{Es}_{A,2}(x)$ be the escape probability using p_2. Since $G_2 f_2 \leq 1$, $G_1(f_2/c_1) \leq 1$ and hence

$$\mathrm{cap}_2(A) = \sum_{y\in A} f_2(x) = c_1 \sum_{y \in A} c_1^{-1}\, f_2(x) \leq c_1\, \mathrm{cap}_1(A).$$

□

Remark. If $p \in \mathcal{R}_d$, then $\mathrm{cap}(\mathcal{B}_n) \asymp n^{d-2}$. To see this, note that the previous corollary shows that it suffices to show this for $p \in L_0^\infty$; suppose that $p(z) = 0$ for $|z| \geq k$. Then for such p we can use a last-exit decomposition,

$$1 = \sum_{z \in \mathcal{B}_n} G(0,z)\, \mathrm{Es}_{\mathcal{B}_n}(z) = \sum_{z \in \mathcal{B}_n \setminus \mathcal{B}_{n-k}} G(0,z)\, \mathrm{Es}_{\mathcal{B}_n}(z)$$

$$\asymp n^{2-d} \sum_{z \in \mathcal{B}_n \setminus \mathcal{B}_{n-k}} \mathrm{Es}_{\mathcal{B}_n}(z) = n^{2-d}\, \mathrm{cap}(\mathcal{B}_n).$$

For an improved estimate, see Corollary 14.9.

Proposition 14.5 *If $p \in \mathcal{R}_d$ and $r < 1$, there exist c_1, c_2 such that for all n sufficiently large and all $x, y \in \mathcal{B}_{rn}$,*

$$c_1\, |x-y|^{2-d} \leq G_{\mathcal{B}_n}(x,y) \leq c_2\, |x-y|^{2-d}.$$

Proof. This follow easily from (87) and the estimate

$$G(x,y) - \sup_{|z|>n} G(z,y) \leq G_{\mathcal{B}_n}(x,y) \leq G(x,y).$$

□

Proposition 14.6 *If $p \in \mathcal{R}_d$, there exist constants c_1, c_2 such that if $A \subset \mathcal{B}_n$ and $|x| \geq 2n$, then*

$$c_1\, \frac{\mathrm{cap}(A)}{|x|^{d-2}} \leq \mathbb{P}^x\{\tau_A < \infty\} \leq c_2\, \frac{\mathrm{cap}(A)}{|x|^{d-2}}.$$

Also there exists a b such that if $2n \leq |x| \leq 4n$,

$$\mathbb{P}^x\{\tau_A < \xi_n\} \geq (c_1/2)\, \frac{\mathrm{cap}(A)}{|x|^{d-2}}.$$

Proof. By a last-exit decomposition,

$$\mathbb{P}^x\{\tau_A < \infty\} = \sum_{z \in A} G_A(x,z)\, \mathrm{Es}_A(z) \asymp \sum_{z\in A} \frac{\mathrm{Es}_A(z)}{|x-z|^{d-2}} \asymp \frac{\mathrm{cap}(A)}{|x|^{d-2}}.$$

The last expression holds since $|x| \geq 2n$ and $|z| \leq n$. By choosing b sufficiently large, we can see that for $|z| \geq bn$, $\mathbb{P}^z\{\tau_A < \infty\} \leq (c_1/2)\,(4n)^{2-d}$. If $2n \leq |x| \leq 4n$, since $\mathbb{P}^x\{\tau_A < \infty\} \geq c_1\,(4n)^{2-d}$, we get the last inequality. □

Proposition 14.7 (Wiener's Test) *Suppose $p \in \mathcal{R}_d$. Then a set A is transient if and only if*

$$\sum_{n=1}^{\infty} \frac{\mathrm{cap}(A_n)}{2^{n(d-2)}} < \infty,$$

where $A_n = A \cap \{2^n < |x| \leq 2^{n+1}\}$.

Proof. Without loss of generality, we will assume that A is contained in the positive hyperplane, $\mathcal{H} = \{(x^1, \ldots, x^d) \in \mathbb{Z}^d : x^1 \geq 0\}$. If this is not the case, we can consider separately $A \cap \mathcal{H}$ and $A \cap (-\mathcal{H})$. Let U_n be the event $\{\tau_{A_n} < \infty\}$. Then, $\mathbb{P}(U_n) \asymp \mathrm{cap}(A_n)/2^{n(d-2)}$. If

$$\sum_{n=1}^{\infty} \frac{\mathrm{cap}(A_n)}{2^{n(d-2)}} < \infty,$$

then $\mathbb{P}\sum(U_n) < \infty$ and hence by the Borel-Cantelli Lemma $\mathbb{P}\{U_n \text{ i.o.}\} = 0$ and A is transient.

Now suppose that

$$\sum_{n=1}^{\infty} \frac{\mathrm{cap}(A_n)}{2^{n(d-2)}} = \infty. \tag{88}$$

let $\xi^m = \xi_{2^m}$. By the previous proposition, we can find $c > 0$ and positive integer k such that if $|x| \leq 2^{n+2}$, then $\mathbb{P}^x\{\tau_{A_n} < \xi^{n+k}\} \geq c\,\mathrm{cap}(A_n)/2^{n(d-2)}$. Let

$$E_n = E_{n,k} = \{\exists j \text{ with } \xi^n \leq j < \xi^{n+k} \text{ and } S_j \in A_n\}.$$

If (88) holds then there exists a $j \in \{0, 1, \ldots, k-1\}$ such that

$$\sum_{n=1}^{\infty} \frac{\mathrm{cap}(A_{kn+j})}{2^{(kn+j)(d-2)}} = \infty.$$

For notational ease we will assume this holds for $j = 0$, but the argument is the same for other j. Let $\mathcal{G}_n$ denote the σ-algebra generated by $\{S_j : j \leq \xi^{nk}\}$. Note that

$$\mathbb{P}[E_{kn} \mid \mathcal{G}_n] \geq c\,\frac{\mathrm{cap}(A_{kn})}{2^{kn(d-2)}} - 1\{|S_{\xi^{kn}}| \geq 2^{kn+2}\}.$$

We claim that

$$\mathbb{P}\{|S_{\xi^n}| \geq 2^{n+2} \text{ i.o.}\} = 0.$$

In fact,

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{P}\{|S_{\xi^n}| \geq 2^{n+2}\} &\leq c\sum_{n=1}^{\infty} 2^{2n}\,\mathbb{P}\{|X_1| \geq 2^{n+1}\} \\
&= c\sum_{n=1}^{\infty} 2^{2n} \sum_{m=n+1}^{\infty} \mathbb{P}\{2^m \leq |X_1| < 2^{m+1}\} \\
&= c\sum_{m=2}^{\infty} \mathbb{P}\{2^m \leq |X_1| < 2^{m+1}\} \sum_{n=1}^{m-1} 2^{2n} \\
&\leq c\sum_{m=2}^{\infty} 2^{2m}\,\mathbb{P}\{2^m \leq |X_1| < 2^{m+1}\} < \infty.
\end{aligned}$$

The last step uses $p \in L^2$. We therefore have a sequence of nonnegative numbers v_n with $\sum v_n = \infty$ such that

$$\mathbb{P}\left\{\mathbb{P}[E_{kn} \mid \mathcal{G}_n] \leq v_n \text{ i.o.}\right\} = 0.$$

This implies $\mathbb{P}\{E_{kn} \text{ i.o.}\} = 1$, which implies that A is recurrent. □

Remark. Suppose $A \subset \mathbb{Z}^d$ and $p \in \mathcal{R}_d$. Let

$$V_A = \sum_{j=0}^{\infty} 1\{S_j \in A\}$$

be the total number of visits to A. If $\mathbb{E}[V_A] < \infty$, then $V_A < \infty$ w.p.1 and hence A is transient. We will know show that it is possible for A to be transient but for $\mathbb{E}[V_A] = \infty$. Let z_n be a sequence of points with $|z_n| \sim (3/2)\,2^n$ and let $\mathcal{B}_n$ denote the discrete ball of radius $r_n < (1/2)\,2^n$ centered at z_n. Then $\mathrm{cap}(\mathcal{B}_n) \asymp r_n^{d-2}$ and hence A is transient if and only if

$$\sum_{n=1}^{\infty} \left(\frac{r_n}{2^n}\right)^{d-2} < \infty.$$

However, $G(y) \asymp 2^{n(2-d)}$ for $y \in \mathcal{B}_n$ and hence

$$\mathbb{E}[V_A] = \sum_{n=1}^{\infty} \sum_{z \in \mathcal{B}_n} G(z) \asymp \sum_{n=1}^{\infty} r_n^2 \left(\frac{r_n}{2^n}\right)^{d-2} < \infty.$$

If we choose $r_n = c\,2^n\,n^{-2}$ then the first sum converges but the second one does not.

Proposition 14.8 *If $p \in \mathcal{S}_d, d \geq 3$ and $A \subset \mathbb{Z}^d$ is finite, then*

$$\lim_{|x|\to\infty} J(x)^{d-2}\,\mathbb{P}^x\{\tau_A < \infty\} = C_d\,\mathrm{cap}(A).$$

In fact, there is a $c < \infty$ such that for all A and all $|x| > \mathrm{rad}(A)$,

$$\left|\mathbb{P}^x\{\tau_A < \infty\} - C_d\,J(x)^{2-d}\,\mathrm{cap}(A)\right| \leq c\,[\mathrm{rad}(A) + 1]\,|x|^{1-d}.$$

Proof. The last-exit decomposition gives for $x \notin A$,

$$\mathbb{P}^x\{\tau_A < \infty\} = \sum_{y \in A} G(x,y)\,\mathrm{Es}_A(y).$$

But,

$$G(x,y) = G(x,0) + O\left(\frac{|y|}{|x|^{d-1}}\right) = \frac{C_d}{|x|^{d-2}} + O\left(\frac{\mathrm{rad}(A)+1}{|x|^{d-1}}\right).$$

□

Corollary 14.9 *If $p \in \mathcal{S}_d, d \geq 3$, then*

$$\mathrm{cap}(\mathcal{C}_n) = \frac{n^{d-2}}{C_d} + O(n^{d-1}).$$

Proof. By (81), if $x \in \mathbb{Z}^d \setminus \mathcal{C}_n$,

$$\mathbb{P}^x\{\tau_{\mathcal{C}_n} < \infty\} = \frac{\mathcal{J}(x)^{2-d}}{n^{2-d}}\,[1 + O\left(\frac{1}{n}\right)].$$

□

Lemma 14.10 *If $p \in \mathcal{S}_d, d \geq 3$, there is a constant c such that the following holds. Suppose $r > 0$, $A \subset \{|y| < r\}$, and $2r \leq |x| < n$. Let $h_n(x,z;A) = \mathbb{P}^x\{S_{\xi_n} = z \mid \xi_n < \tau_A\}$ and let $h_n(z) = h_n(0,z;\emptyset) = \mathbb{P}\{S_{\xi_n} = z\}$. Then,*

$$\sum_{z \in \partial \mathcal{B}_n} |h_n(x,z;A) - h_n(z)| \leq c\, \frac{|x|}{n}.$$

Proof. We know from (84) that there is a c' such that for all $|y| \leq |x|$,

$$\sum_{z \in \partial \mathcal{B}_n} |h_n(y,z;\emptyset) - h_n(z)| \leq c'\, \frac{|x|}{n}.$$

Hence, by the strong Markov property,

$$\sum_{z \in \partial \mathcal{B}_n} |\mathbb{P}^x\{S_{\xi_n} = z \mid \xi_n > \tau_A\} - h_n(z)| \leq c'\, \frac{|x|}{n}.$$

Using transience, we can see that there is a c'' such that $q = q_{x,n} := \mathbb{P}^x\{\xi_n < \tau_A\} \geq c''$. However, note that

$$|h_n(x,z;\emptyset) - h_n(z)| \geq q\,|h_n(x,z;A) - h_n(z)| - (1-q)\,|\mathbb{P}^x\{S_{\xi_n} = z \mid \xi_n > \tau_A\} - h_n(z)|,$$

and hence

$$\sum_{z \in \partial \mathcal{B}_n} |h_n(x,z;A) - h_n(z)| \leq$$

$$q^{-1} \sum_{z \in \partial \mathcal{B}_n} \Big(|h_n(x,z;\emptyset) - h_n(z)| + |\mathbb{P}^x\{S_{\xi_n} = z \mid \xi_n > \tau_A\} - h_n(z)| \Big).$$

□

Remark. Suppose A is a finite set and consider $G(x,y)$ restricted to A as an $\#(A) \times \#(A)$ matrix. Let G_A^{-1} denote the inverse of this matrix, i.e., if $x, y \in A$,

$$\sum_{z \in A} G_A^{-1}(x,z)\, G(z,y) = \delta(y-x).$$

Since

$$G(x,y) = \delta(y-x) + \sum_{z \in A} \mathbb{P}^x\{S_{\tau_A} = z\}\, G(z,y),$$

we can see that

$$G_A^{-1}(x,y) = \delta(y-x) - K_A(x,y).$$

Note that

$$\sum_{y \in A} G_A^{-1}(x,y) = 1 - \sum_{z \in A} \mathbb{P}^x\{S_{\tau_A} = z\} = \mathrm{Es}_A(x),$$

and hence

$$\sum_{x \in A} \sum_{y \in A} G_A^{-1}(x,y) = \mathrm{cap}(A). \tag{89}$$

The left-hand side is another way to define $\mathrm{cap}(A)$. Note that under this definition it is immediate that $\mathrm{cap}(A)$ is the same for the random walk and the reversed random walk.

15 Logarithmic capacity

In this section we will consider increment distributions $p \in L_0^2$ of random walks in $\mathbb{Z}^2$. As before, we let a denote the potential kernel. Recall that $a(x) \sim 2\,C_2 \log|x|$ and if $p \in L_0^{2+}$, there is a C such that

$$a(x) = C_2 \log|x \cdot \Gamma^{-1} x| + C + o(1), \quad |x| \to \infty.$$

If A is a finite subset of $\mathbb{Z}^d$ containing the origin, let

$$f_A(x) = a(x,0) - \sum_{y \in A} H_{\mathbb{Z}^2 \setminus A}(x,y)\, a(y,0).$$

In Corollary 8.4, it was shown that

$$f_A(x) = \lim_{n \to \infty} C_2\,(\log n)\, \mathbb{P}^x\{\bar\tau_A > n\}. \tag{90}$$

where $C_2 = [2\pi\sqrt{\det\Gamma}]^{-1}$. Also (47) shows that if $x, y \in \mathbb{Z}^d \setminus A$,

$$f_A(x) = G_{\mathbb{Z}^2 \setminus A}(x,y) + a(x,y) - \sum_{z \in A} \mathbb{P}^x\{S_{\tau_A} = z\}\, a(z,y).$$

Here is a similar result.

Proposition 15.1 *If A is a finite subset of $\mathbb{Z}^2$,*

$$f_A(x) = \lim_{n \to \infty} 2\,C_2\,(\log n)\, \mathbb{P}^x\{\bar\tau_A > \xi_n\},$$

where as before $\xi_n = \min\{j : |S_j| \geq n\}$ and $C_2 = [2\pi\sqrt{\det\Gamma}]^{-1}$.

Proof. Let $z \in A$, $T_n = \xi_n \wedge \bar\tau_A$ and consider the martingale $a(S_{k \wedge T_n}, z)$. Then by optional sampling

$$a(x,z) = \mathbb{E}^x[a(S_{T_n}, z)] = \mathbb{E}^x[a(S_{T_n}, z); T_n = \xi_n] + \sum_{y \in A} \mathbb{P}^x\{S_{T_n} = y\}\, a(y,z).$$

Note that

$$\lim_{n \to \infty} \sum_{y \in A} \mathbb{P}^x\{S_{T_n} = y\}\, a(y,z) = \sum_{y \in A} H_{\mathbb{Z}^2 \setminus A}(x,y)\, a(y,z).$$

Since $a(x) \sim 2\,C_2 \log|x|$, we can see that as $n \to \infty$,

$$\begin{aligned}\mathbb{E}^x[a(S_{T_n}, z); T_n = \xi_n] &\sim C_2\,\mathbb{E}^x[\log|S_{T_n}|; T_n = \xi_n] \\ &\sim C_2\,[\,\mathbb{P}^x\{\xi_n < \bar\tau_A\}\,\log n + \mathbb{E}^x[\log\frac{|S_{T_n}|}{n}; T_n = \xi_n]\,].\end{aligned}$$

The result will follow provided that we show that

$$\lim_{n\to\infty} \mathbb{E}^x[\log\frac{|S_{T_n}|}{n}; T_n = \xi_n] = 0.$$

Note that

$$\mathbb{P}\{|S_{T_n}| \geq rn; T_n = \xi_n\} \leq c\,\mathbb{E}^x[T_n]\,\mathbb{P}\{|X_1| \geq rn\} = \mathbb{E}^x[T_n]\,o(r^{-2}n^{-2}).$$

However, by comparison to a singleton set, we can see that $\mathbb{E}^x[T_n] \leq c_x\,n^2/\log n$. Hence,

$$\mathbb{P}\{|S_{T_n}| \geq rn; T_n = \xi_n\} \leq c\,r^{-2}\,(\log n)^{-1}.$$

From this we get the result. $\square$

Note that $f_A \equiv 0$ on A; $\mathbf{L}f_A(x) = 0, x \in \mathbb{Z}^2 \setminus A$; and as $|x| \to \infty$, $f_A(x) \sim 2\,C_2 \log|x|$. In fact, by Lemma 11.4,

$$f_A(x) = a(x,0) - \sum_{y\in A} \mathrm{hm}_A(y)\,a(y,0) + o(1), \quad |x| \to \infty.$$

Since $f_A \equiv 0$ on A and $f_A(x) > 0$ for all x sufficiently large, the maximum principle implies that $f_A(x) \geq 0$ for all x. If $z \in A$, we could also define

$$f_{A,z}(x) = a(x,z) - \sum_{y\in A} H_{\mathbb{Z}^2\setminus A}(x,y)\,a(y,z).$$

Using (44), we can see that $a(x,0) - a(x,z) = o(1)$ as $x \to \infty$, and hence $f_A - f_{A,z}$ is a bounded harmonic function that equals zero on A. Hence, $f_A = f_{A,z}$. Note that if $y \in A$,

$$\mathbf{L}f_A(y) = \lim_{n\to\infty} 2\,C_2\,(\log n)\,\mathbb{P}\{\tau_A > \xi_n\}.$$

Proposition 15.2 *Suppose $p \in L_0^2$ is a random walk in $\mathbb{Z}^2$, A is a finite set, and $h_0 : A \longrightarrow \mathbb{R}$. Suppose h is an extension of h_0 to $\mathbb{Z}^2$ such that h is harmonic on $\mathbb{Z}^2 \setminus A$.*

- *If $h(x) = o(\log|x|)$ as $|x| \to \infty$, then*

$$h(x) = \mathbb{E}^x[h(S_{\bar\tau_A})] = \sum_{y\in A} \mathbb{P}^x\{S_{\bar\tau_A} = y\}\,h(y).$$

- *If $r \in \mathbb{R}$ and $h(x) \sim 2\,C_2\,r \log|x|$ as $x \to \infty$, then*

$$h(x) = r f_A(x) + \mathbb{E}^x[h(S_{\bar\tau_A})] = r f_A(x) + \sum_{y\in A} \mathbb{P}^x\{S_{\bar\tau_A} = y\}\,h(y).$$

Proof. By the proof of Proposition 15.1, we can see that if $h(x) = o(\log|x|))$, then

$$\lim_{n\to 0} \mathbb{E}[h(S_{\xi_n\wedge\bar\tau_A}); \xi_n < \bar\tau_A] = 0.$$

Hence, but the optional sampling theorem,

$$h(x) = \mathbb{E}^x[h(S_{\xi_n\wedge\bar\tau_A})] = \lim_{n\to\infty} \mathbb{E}^x[h(S_{\xi_n\wedge\bar\tau_A}); \xi_n > \bar\tau_A] = \mathbb{E}^x[h(S_{\bar\tau_A})].$$

For the second assertion, note that $h - rf_A$ is harmonic on $\mathbb{Z}^2\setminus A$ and $h(x) - rf_A(x) = o(\log|x|)$ as $x \to \infty$. $\square$

Definition. If $p \in L_0^2$ is an increment distribution of a random walk in $\mathbb{Z}^2$ and A is a finite subset of $\mathbb{Z}^2$ define the *(logarithmic) capacity* of A by

$$\mathrm{cap}(A) = \sum_{y\in A} \mathrm{hm}_A(y)a(y,z),$$

where $z \in A$. (As we have just shown, this definition is independent of the choice of $z \in A$.)

Remark. Note that the capacity is translation invariant. If $p \in L_0^{2+}$, then

$$f_A(x) = C_2 \log|x \cdot \Gamma^{-1}x| + C - \mathrm{cap}(A) + o(1), \quad x \to \infty.$$

Assuming only $p \in L_0^2$ we can write

$$f_A(x) = a(x) - \mathrm{cap}(A) + o(1), \quad x \to \infty. \tag{91}$$

Proposition 15.3 *Suppose $p \in L_0^2$ is an increment distribution of a random walk in $\mathbb{Z}^2$.*

- *If $A \subset B$, then $f_A(x) \geq f_B(x)$ for all x. In particular, $\mathrm{cap}(A) \geq \mathrm{cap}(B)$.*
- *If A, B are finite subsets of $\mathbb{Z}^2$, then*

$$f_{A\cup B}(x) + f_{A\cap B}(x) \geq f_A(x) + f_B(x). \tag{92}$$

In particular,

$$\mathrm{cap}(A\cup B) \leq \mathrm{cap}(A) + \mathrm{cap}(B) - \mathrm{cap}(A\cap B).$$

Proof. The inequality $f_A(x) \geq f_B(x)$ follows immediately from (90). The inequality (92) follows from (90) and the observation

$$\begin{aligned}\mathbb{P}^x\{\bar\tau_{A\cup B} > n\} &= \mathbb{P}^x\{\bar\tau_A > n\} + \mathbb{P}^x\{\bar\tau_B > n\} - \mathbb{P}^x\{\bar\tau_A > n, \bar\tau_B > n\} \\ &\geq \mathbb{P}^x\{\tau_A > n\} + \mathbb{P}^x\{\bar\tau_B > n\} - \mathbb{P}^x\{\bar\tau_{A\cap B} > n\}.\end{aligned}$$

The inequalities for cap then follow from (91). $\square$

Proposition 15.4 *Suppose $p \in L_0^2$ and A is a finite set. Then if $x, y \in A$,*

$$a(x,y) + \delta(y-x) = \mathbf{L} f_A(x) + \sum_{z \in A} \mathbb{P}^x\{S_{\tau_A} = z\}\, a(z,y).$$

Remark. For a transient random walk, it is easy to see that

$$G(x,y) = \delta(y-x) + \mathbb{E}^x[G(S_{\tau_A}, y)] = \delta(y-x) + \sum_{x \in A} \mathbb{P}^x\{S_{\tau_A} = z\}\, a(x,y).$$

This proposition is the analogue of this equation for mean zero, finite variance walks in $\mathbb{Z}^2$.

Proof. For ease, assume $y = 0$. Then

$$\begin{aligned} a(x,0) + \delta(x) &= \lim_{n\to\infty} 2C_2(\log n)\, \mathbb{P}^x\{\tau_0 > \xi_n\} \\ &= \lim_{n\to\infty} 2C_2(\log n)\, [\mathbb{P}^x\{\tau_A > \xi_n\} + \mathbb{P}^x\{\tau_A < \xi_n, \tau_0 > \xi_n\}] \\ &= \mathbf{L} f_A(x) + \lim_{n\to\infty} 2C_2(\log n) \sum_{z \in A z \neq 0} \mathbb{P}^x\{S_{\tau_A \wedge \xi_n} = z\}\, \mathbb{P}^z\{\tau_0 < \xi_n\} \\ &= \mathbf{L} f_A(x) + \sum_{z \in A} \mathbb{P}^x\{S_{\tau_A} = z\}\, a(z,0). \end{aligned}$$

□

Proposition 15.5 *Suppose $p \in L_0^2$ and A is a finite set. If $x \in A$, then $\mathbf{L} f_A(x) = \mathrm{hm}_A^*(x)$.*

Proof. Let $r_n = 2C_2 \log n$ and let $G_n = G_{\mathcal{B}_n}$. If $y \in A$, a last-exit decomposition gives

$$1 = \sum_{x \in A} G_n(y,x)\, \mathbb{P}^x\{\xi_n < \tau\}.$$

For fixed y, x, as $n \to \infty$, $G_n(y,x) \sim r_n$, Hence,

$$1 \sim \sum_{x \in A} r_n\, \mathbb{P}^x\{\xi_n < \tau\} \sim \sum_{x \in A} \mathbf{L} f_A(x).$$

Since the left and right hand sides to not depend on n, they are equal.

Then

$$\begin{aligned} \mathbf{L} f(x) &= \lim_{n\to\infty} r_n\, \mathbb{P}^x\{\xi_n < \tau_A\} \\ &= \lim_{n\to\infty} r_n \sum_{|z| \geq n} \mathbb{P}^x\{S_{\xi_n \wedge \tau_A} = z\} \\ &= \lim_{n\to\infty} r_n \sum_{|z| \geq n} \mathbb{P}^z\{S^*_{\xi_n \wedge \tau_A} = x\} \\ &= \lim_{n\to\infty} r_n \sum_{|z| \geq n} \mathbb{P}^z\{\xi_n > \tau_A\}\, \mathbb{P}^z\{S^*_{\tau_A} = x \mid \xi_n > \tau_A\} \end{aligned}$$

(ADD SOME DETAILS) As $n \to \infty$, this approaches $\mathrm{hm}_A^*(x)$. □

Proposition 15.6 *Suppose $p \in L_0^2$ and A is a finite set. Then*

$$\mathbb{P}^x\{S_{\bar{\tau}_A} = y\} = \mathrm{hm}_A(y) - a(x,y) + \sum_{z \in A} a(x,z)\, \mathbb{P}^z\{S_{\tau_A} = y\}.$$

Proof. Let $g(x) = g_A(x,y)$ denote the right hand side of the equation above. Then g is harmonic on $\mathbb{Z}^2 \setminus A$. Also, using (44), we can see that

$$\lim_{x\to\infty} \left[-a(x,y) + \sum_{z \in A} a(x,z)\, \mathbb{P}^z\{S_{\tau_A} = y\} \right] = 0,$$

and hence $g(x) \to \mathrm{hm}_A(y)$ as $x \to \infty$. We need to show that $g(x) = \delta(y-x)$ for $x \in A$. But, if $x, y \in A$,

$$\begin{aligned} \sum_{z \in A} a(x,z)\, \mathbb{P}^z\{S_{\tau_A} = y\} &= \sum_{z \in A} a^*(z,x)\, \mathbb{P}^y\{S^*_{\tau_A} = x\} \\ &= a^*(y,x) + \delta(y-x) - \mathbf{L} f_A^*(x) \\ &= a(x,y) + \delta(y-x) - \mathrm{hm}_A(x). \end{aligned}$$

□

Proposition 15.7 *Suppose $p \in L_0^2$ and A is a finite set. Then if $x, y \in A$,*

$$\sum_{z \in A} a(x,z)\, \mathbf{L} f_A(z) = \sum_{y \in A} a(y,z)\, \mathbf{L} f_A(z). \tag{93}$$

Proof. By a last-exit decomposition.

$$\sum_{z \in A} G_{\mathcal{B}_n}(x,z)\, \mathbb{P}^z\{\xi_n < \tau_A\} = 1 = \sum_{z \in A} G_{\mathcal{B}_n}(y,z)\, \mathbb{P}^z\{\xi_n < \tau_A\}.$$

Hence,

$$2\, C_2\, (\log n) \sum_{z \in A} [G_{\mathcal{B}_n}(0,0) - G_{\mathcal{B}}(x,z)]\, \mathbb{P}^z\{\xi_n < \tau_A\}$$

$$= 2\, C_2\, (\log n) \sum_{z \in A} [G_{\mathcal{B}_n}(0,0) - G_{\mathcal{B}}(y,z)]\, \mathbb{P}^z\{\xi_n < \tau_A\}.$$

By letting $n \to \infty$, we get (93). □

Note that the proposition does not give the value of the sum in (93). In fact, we have already discussed this value. Recall that $L f_A(x) = \mathrm{hm}_A^*(x)$. Then For the time being let

$$R(A) = \sum_{z \in A} a(x,z)\, \mathbf{L} f_A(z) = \sum_{z \in A} a^*(z,x) \mathbf{L} f_A^*(z) = \mathrm{cap}^*(A).$$

We use the notation cap* in order to indicate that this is the capacity using the reversed random walk; below we will show that $\mathrm{cap}^*(A) = \mathrm{cap}(A)$. For a given A, consider $a(x,y)$ as an $\#(A) \times \#(A)$ matrix and let $a_A^{-1}(x,y)$ denote the entries of the inverse matrix, i.e.,

$$\sum_{y \in A} a_A^{-1}(x,y)\, a(y,z) = \delta(z-x).$$

The next proposition show that a_A^{-1} exists and gives a formula for it.

Proposition 15.8 *Suppose $p \in L_0^2$ and A is a finite set. Then, if $x, y \in A$*

$$a_A^{-1}(x,y) = \mathbb{P}^x\{S_{\tau_A} = y\} - \delta(y-x) + \frac{\mathbf{L} f_A(x)\, \mathbf{L} f_A^*(y)}{\mathrm{cap}(A)}.$$

Proof. Suppose $x, z \in A$. Then,

$$G_{\mathcal{B}_n}(x,z) = \delta(z-x) + \sum_{y \in A} \mathbb{P}^x\{S_{\tau_A \wedge \xi_n} = y\}\, G_{\mathcal{B}_n}(y,z).$$

Hence,

$$G_{\mathcal{B}_n}(0,0) - G_{\mathcal{B}_n}(x,z) = -\delta(z-x)$$
$$+ G_{\mathcal{B}_n}(0,0)\, \mathbb{P}^x\{\xi_n < \tau_A\} + \sum_{y \in A} \mathbb{P}^x\{S_{\tau_A \wedge \xi_n} = y\}\, [G_{\mathcal{B}_n}(0,0) - G_{\mathcal{B}_n}(y,z)].$$

Letting $n \to \infty$, this gives

$$\delta(z-x) = -a(x,z) + \mathbf{L} f_A(x) + \sum_{y \in A} \mathbb{P}^x\{S_{\tau_A} = y\}\, a(y,z).$$

However, the previous proposition applied to the reversed random walk gives

$$\sum_{y \in A} \mathbf{L} f_A^*(y)\, a(y,z) = \mathrm{cap}(A).$$

Hence we can write the previous equation as

$$\delta(z-x) = \sum_{y \in A} \left[\mathbb{P}^x\{S_{\tau_A} = y\} - \delta(y-x) + \frac{\mathbf{L} f_A(x)\, \mathbf{L} f_A^*(y)}{\mathrm{cap}(A)} \right] a(y,z).$$

□

Remark. Note that

$$\sum_{x \in A} a_A^{-1}(x,y) = \frac{\mathbf{L} f_A^*(y)}{\mathrm{cap}(A)} = \frac{\mathrm{hm}_A(y)}{\mathrm{cap}(A)},$$

$$\sum_{y \in A} a_A^{-1}(x,y) = \frac{\mathbf{L} f_A(x)}{\mathrm{cap}(A)} = \frac{\mathrm{hm}_A^*(x)}{\mathrm{cap}(A)},$$

$$\sum_{x \in A} \sum_{y \in A} a_A^{-1}(x,y) = \frac{1}{\mathrm{cap}(A)}.$$

Note that an alternative definition for cap is

$$\mathrm{cap}(A) = \left[\sum_{x \in A} \sum_{y \in A} a_A^{-1}(x,y) \right]^{-1}.$$

From this we can see that cap is the same for the random walk and the reversed random walk. Note the similarily and difference between this and (89).

If $h : \mathbb{Z}^2 \to \mathbb{R}$ is a function of bounded support, let

$$ah(x) = \sum_{y \in \mathbb{Z}^2} a(x,y)\, h(y).$$

Note that if h is supported on the finite set A, then ah is harmonic on $\mathbb{Z}^2 \setminus A$ and

$$ah(x) \sim 2\, C_2 \log|x| \sum_{y \in A} h(y), \qquad x \to \infty.$$

Also

$$\sum_{x \in A} \mathbf{L} ah(x) = \sum_{x \in A} h(x).$$

We have seen that if $h_A(y) = \mathrm{hm}_A^*(y)/\mathrm{cap}(A)$, then $ah_A(x) = 1,\ x \in A$, and

$$\sum_{x \in A} h_A(x) = \frac{1}{\mathrm{cap}(A)}.$$

If $x \in \mathbb{Z}^2 \setminus A$ and x is not "separated from infinity by A", i.e., if $\mathbb{P}^x\{\xi_n < \tau_A\} > 0$ for all n, then $f_A(x) > 0$ and $ah_A(x) > 1$.

Proposition 15.9 *If $p \in L_0^2$ and A is a finite set, then*

$$\mathrm{cap}(A) = \left[\sup \sum_{y \in A} h(y) \right]^{-1},$$

where the supremum is over all nonnegative functions h supported on A with $ah(x) \le 1$ for $x \in A$.

Lemma 15.10 *Suppose A is a finite set and $g : A \times A \to (0,\infty)$ is an $\#(A) \times \#(A)$ matrix with strictly positive entries. Let*

$$M = \sup \sum_{x \in A} h(x),$$

where the supremum is over all $h : A \to [0,\infty)$ satisfying

$$\sum_{y \in A} g(x,y)\, h(y) \leq 1, \quad x \in A.$$

There there is an $h_0 : A \to [0,\infty)$ with $\sum_{x\in A} h_0(x) = M$ such that

$$\sum_{y \in A} g(x,y)\, h_0(y) = 1 \quad \text{if} \quad h_0(x) > 0.$$

Proof. Compactness arguments imply that the supremum is obtained by some (not necessarily unique) h. Let $\hat{h}$ be a function that obtains the supremum such that $\#\{x : \hat{h}(x) > 0\}$ is as small as possible. We will find an h_0 satisfying the lemma supported on $\{x : \hat{h}(x) > 0\}$. Without loss of generality, we may assume that $A = \{x : \hat{h}(x) > 0\}$, i.e., that every function which obtains the supremum is strictly positive on A.

Suppose that h_0 obtains the supremum and let $x \in A$. Simple linear algebra tells us that there is a $v : A \to \mathbb{R}$ such that $\sum_{y\in A} v(y) \geq 0$ and such that

$$\sum_{y \in A} g(x',y)\, v(y) = 0, \quad x' \in A \setminus \{x\}.$$

Let $h_\epsilon = h_0 + \epsilon\, v$. Note that for $\epsilon \geq 0$,

$$\sum_{y \in A} g(x',y)\, h_\epsilon(y) = \sum_{y \in A} g(x',y)\, h_0(y), \quad x' \in A \setminus \{x\}. \tag{94}$$

$$\sum_{y \in A} h_\epsilon(y) \geq \sum_{y \in A} h_0(y).$$

Let ϵ_0 be the supremum of all $\epsilon > 0$ such that h_ϵ is strictly positive on A and

$$\sum_{y \in A} g(z,y)\, h_\epsilon(y) \leq 1, \quad z \in A.$$

Then h_{ϵ_0} is a maximizer satisfying (94). Since h_{ϵ_0} must be strictly positive, we must have

$$\sum_{y \in A} g(x,y)\, h_{\epsilon_0}(y) = 1.$$

By continuing this argument for each point in A we produce a stictly positive maximizer h with $\sum g(x,y)\, h(y) = 1$ for $x \in A$. $\square$

Proof of Proposition 15.9. Let

$$M = \sup \sum_{x \in A} h(x),$$

where the supremum is over all $h : A \to [0,\infty)$ with

$$\sum_{y \in A} a(x,y)\, h(y) \leq 1, \quad x \in A.$$

Since $h_A(x) = \mathrm{hm}^*_A(x)/\mathrm{cap}(A)$ satisfies the constraints, we have

$$M \geq \sum_{x \in A} \frac{\mathrm{hm}^*_A(x)}{\mathrm{cap}(A)} = \frac{1}{\mathrm{cap}(A)}.$$

By the previous lemma we can find an h_0 satisfying the constraints with $\sum_A h_0(x) = M$ and such that

$$\sum_{y \in A} a(x,y)\, h_0(y) = 1, \quad x \in A', \tag{95}$$

where $A' = \{x \in A : h_0(x) > 0\}$. Since a restricted to A' is invertible, there is a unique h_0 on A' satisfying (95),

$$h_0(x) = h_{0,A'}(x) = \frac{\mathrm{hm}^*_{A'}(x)}{\mathrm{cap}(A')}.$$

If $A' = \{x : \mathrm{hm}^*_A(x) > 0\}$, then $h_0 = h_A$ as above. If A' is smaller than this set, i.e., if there exists an $x \in A \setminus A'$ with $\mathrm{hm}^*_A(x) > 0$, then $f_{A'}(x) > 0$. But this implies and, as mentioned before, $a h_0(x) > 1$. Hence, for those sets $h_{0,A'}$ does not satisfy the constraints. Hence, the supremum is obtained by h_A. $\square$

16 Some estimates

In this section we do some estimates similar to those done earlier. The difference here is that we will only assume that the walk $p \in L^2_0$. Consequently, the conclusions will be somewhat weaker than before. As before let $\mathcal{B}_n = \{z \in \mathbb{Z}^d : |z| < n\}$ and $\xi_n = \min\{j \geq 1 : |S_j| \geq n\}$. Note that if $|x|, |y| \leq (1-\epsilon)n$,

$$G_{\mathcal{B}_{\epsilon n}}(0, y-x) \leq G_{\mathcal{B}_n}(x,y) \leq G_{\mathcal{B}_{2(1-\epsilon)n}}(0, y-x).$$

Proposition 16.1 *If $p \in L^2_0$, then for every $r < 1$, there is a c_r such that*

$$G_{\mathcal{B}_n}(x,y) \geq c_r\, n^{2-d}, \quad |x|, |y| \leq rn.$$

Proof. By the invariance principle it suffices to show that there is a c and an $\epsilon > 0$ such that the estimate holds for $|x|, |y| < \epsilon n$, and (by making ϵ a little smaller if necessary) we may assume that $x = 0$. From the local central limit theorem, we can find an ϵ such that for $|x| < \epsilon$,

$$\sum_{j \leq \delta n} \mathbb{P}\{S_j = y\} \geq \frac{1}{2} \sup_{|z| \geq n} \sum_{j \leq \delta n} \mathbb{P}^z\{S_j = y\}.$$

Hence,

$$G_{\mathcal{B}_n}(0, y) \geq \frac{1}{2} \sum_{j \leq \delta n} \mathbb{P}\{S_j = y\} \geq c\, n^{2-d}.$$

□

Proposition 16.2 *Suppose $p \in L_0^2(\mathbb{Z}^2)$. There is a c such that for every positive integer n and every $|x| < n$,*

$$|G_{\mathcal{B}_n}(0, x) - G_{n^2} + G_{|x|^2}| \leq c, \qquad |G_{\mathcal{B}_n}(x, 0) - G_{n^2} + G_{|x|^2}| \leq c.$$

Here G_r is as in Proposition 6.9. In particular,

$$G_{\mathcal{B}_n}(0, 0) \sim 2\, C_2 \log n, \quad n \to \infty.$$

Proof. We will do the first inequality; the second will follow by considering the reversed walk. It suffices to prove the result for $|x| \leq n/2$. We write

$$G_{\mathcal{B}_n}(0, x) = \sum_{j=0}^{n^2} \mathbb{P}\{S_j = x; j < \xi_n\} + \sum_{j=n^2+1}^{\infty} \mathbb{P}\{S_j = x; j < \xi_n\}.$$

The second sum on the right hand side is $O(1)$ by the argument in Proposition 16.2. Also, the expected number of visits to x in n^2 steps by a random walk starting at y with $|y| \geq n$ is $O(1)$ (since $p_j(y, x) \leq c/n^2$). Hence, by the strong Markov property,

$$\begin{aligned}\sum_{j=0}^{n^2} \mathbb{P}\{S_j = x; j < \xi_n\} &= -O(1) + \sum_{j=0}^{n^2} P_j(0, x) \\ &= G_{n^2} - a(x) + O(1) = G_{n^2} - G_{|x|^2} + O(1).\end{aligned}$$

Proposition 16.3 *If $p \in L_0^2(\mathbb{Z}^2)$, there is a c such that for all x, y,*

$$G_{\mathcal{B}_n}(x, y) \leq c \log\left(1 + \frac{n}{|y - x|}\right).$$

Proof. We first show that there is a c such that for any x, y,

$$\sum_{j=n^2}^{\infty} \mathbb{P}^x\{S_j = y; j < \xi_n\} \leq c. \tag{96}$$

Indeed, for every positive integer k,

$$\begin{aligned}\sum_{j=kn^2}^{(k+1)n^2} \mathbb{P}^x\{S_j = y; j < \xi_n\} &\leq \mathbb{P}^x\{\xi_n > (k - \frac{1}{2})n^2\} \sum_{j=kn^2}^{(k+1)n^2} \mathbb{P}^x\{S_j = y \mid \xi_n > (k - \frac{1}{2})n^2\} \\ &\leq c\, e^{-bk}\, [n^2(1/n^2)] \leq c\, e^{-bk},\end{aligned}$$

for some $b > 0$. Summing over k gives (96). Therefore, it suffices to show that

$$\sum_{j=0}^{n^2} \mathbb{P}^x\{S_j = y\} \leq c \log\left(1 + \frac{n}{|y - x|}\right).$$

However the local central limit theorem (both forms) gives

$$\mathbb{P}\{S_j = y\} \leq c \min\{\frac{1}{j}, \frac{1}{y^2}\}.$$

Therefore,

$$\sum_{j=0}^{n^2} \mathbb{P}^x\{S_j = y\} \leq c \left[\sum_{j=0}^{|y|^2} \frac{1}{|y|} + \sum_{j=|y|^2+1}^{n^2} \frac{1}{j}\right] \leq c \log\left(1 + \frac{n}{|y - x|}\right).$$

□

Proposition 16.4 *If $p \in L_0^2(\mathbb{Z}^3)$, there is a c such that for all x, y,*

$$G_{\mathcal{B}_n}(x, y) \leq \frac{c}{|y - x|}.$$

Proof. This follows from the upper bound for $G(x, y)$. □

Proposition 16.5 *If $p \in L_0^2(\mathbb{Z}^d)$, $d = 2$ or $d = 3$, and $r < 1$, there is a $c = c(r)$ such that if A is a subset of $\mathcal{B}_n$ satisfying*

$$\#\{z \in A : j \leq |z| < j + 1\} \geq 1, \quad j = 0, 1, \ldots, n - 1,$$

and $|x| < 2rn$, then for all sufficiently large n,

$$\mathbb{P}^x\{\tau_A < \xi_{2n}\} \geq c \quad d = 2,$$

$$\mathbb{P}^x\{\tau_A < \xi_{2n}\} \geq \frac{c}{\log n}, \quad d = 3.$$

Remark. One example of a set A satisfying the conditions is

$$A_{n,+} := \{je_1 : j = 0, \ldots, n-1\}.$$

Proof. By the invariance principle, we can see that it suffices to prove the result for $r = 1/2$. Without loss of generality we may assume that $A = \{y_0, y_1, \ldots, y_{n-1}\}$ where $j \leq |y_j| < j+1$. Let $V = V_{n,A}$ denote the number of visits to A before leaving $\mathcal{B}_{2n}$,

$$V = \sum_{j=0}^{\infty} 1\{S_j \in A; j < \xi_{2n}\}.$$

If $|x| \leq rn$, then the previous propositions imply

$$\mathbb{E}^x[V] \geq c_1\, n^{3-d}.$$

However, for any z,

$$\#\{y \in A : |z - y| \leq r\} \leq 2\,(r+1).$$

Also,

$$G_{\mathcal{B}_{2n}}(y,z) \leq c\, \log\left(1 + \frac{n}{|y-x|}\right), \quad d = 2,$$

and $G_{\mathcal{B}_{2n}}(y,z) \leq c/|y-z|$ if $d = 3$. Therefore, for every z,

$$\mathbb{E}^z[V] \leq \begin{cases} c\,n, & d=2 \\ c\,\log n, & d=3. \end{cases}$$

Therefore,

$$c\,n^{3-d} \leq \mathbb{E}^x[V] = \mathbb{P}^x\{V \geq 1\}\,\mathbb{E}^x[V \mid V \geq 1] \leq c\,\mathbb{P}^x\{V \geq 1\}\, s_n,$$

where $s_n = n$ if $d = 2$ and $s_n = \log n$ if $d = 3$. $\square$

Proposition 16.6 *Suppose $p \in L_0^2(\mathbb{Z}^2)$. There is a c such that for every sufficiently large positive integer n, every $2n \leq |x| \leq 4n$, every positive integer $m \geq 4n$, and every $A \subset \mathcal{B}_n$ satisfying*

$$\#\{z \in A : j \leq |z| < j+1\} \geq 1, \quad j = 0, 1, \ldots, n-1,$$

we have

$$\mathbb{P}^x\{\xi_m < \tau_{A_{n,+}}\} \leq \frac{c}{\log(m/n)}.$$

Proof. Without loss of generality we may assume that $m \geq 8n$ and $A = \{y_0, y_1, \ldots, y_{n-1}\}$ with $j \leq |y_j| < j+1$. Let $V = V_{n,A}$ denote the number of visits to A before leaving $\mathcal{B}_m$,

$$V = \sum_{j=0}^{\infty} 1\{S_j \in A, j \leq \xi_m\}.$$

Note that

$$\mathbb{E}^z[V] = \sum_{w \in \mathcal{B}_m} G_{\mathcal{B}_m}(z,w).$$

If $|z|, |w| \leq 4n$, Proposition 16.2 implies

$$\begin{aligned} G_{\mathcal{B}_m}(z,w) &= G_{\mathcal{B}_m}(0, w-z) + O(1) \\ &= O(1) + \sum_{|w-z|^2 < j \leq m^2} P_j(0) \\ &= 2\,C_2\,[\log m - \log(1 + |z-w|)] + O(1). \end{aligned}$$

Using this, if $2n \leq |x| \leq 4n$,

$$\mathbb{E}^x[V] = n\,[O(1) + 2\,C_2\,\log(m/n)].$$

However, we get a bound for all z,

$$\mathbb{E}^z[V] \leq O(n) + n \sum_{n^2 < j \leq m^2} P_j(0) + 2\sum_{l=1}^{n} \sum_{l^2 < j < n^2} P_j(0).$$

But,

$$\sum_{l=1}^{n} \sum_{l^2 < j < n^2} P_j(0) \leq c \sum_{l=1}^{n} \sum_{l^2 < j < n^2} \frac{1}{j} \leq c \sum_{j=1}^{n^2} \frac{1}{j} \sum_{l \leq j^2} 1 = O(n).$$

Hence, for all $|z| \leq 4n$,

$$\mathbb{E}^z[V] = n\,[O(1) + 2\,C_2\,\log(m/n)].$$

Hence,

$$\mathbb{P}^x[V \geq 1] = \frac{\mathbb{E}^x[V]}{\mathbb{E}^x[V \mid V \geq 1]} \geq 1 - O\left(\frac{1}{\log(m/n)}\right).$$

$\square$

The preceding proof can be used to prove the following generalization.

Proposition 16.7 *Suppose $p \in L_0^2(\mathbb{Z}^2)$ and $\kappa > 0$. There is a $c = c_\kappa$ such that for every sufficiently large positive integer n, every $2n \leq |x| \leq 4n$, every positive integer $m \geq 4n$, and every $A \subset \mathcal{B}_n$ satisfying $\#(A) \geq \kappa n$ and for every $y \in A$ and*

$$\#\{z \in A : |z - y| \leq k\} \leq \kappa^{-1}\,k, \quad k = 1, 2, \ldots, n,$$

we have

$$\mathbb{P}^x\{\xi_m < \tau_{A_{n,+}}\} \leq \frac{c}{\log(m/n)}.$$

Proof. As in the previous proof, if V is the number of visits to A and $|z| \leq 4n$,

$$\mathbb{E}^z[V] = \#(A)\,[O(1) + 2\,C_2\,\log(m/n)].$$

In particular,

$$\mathbb{E}^z[V \mid V \geq 1] = \#(A)\,[O(1) + 2\,C_2\,\log(m/n)].$$

$\square$

Proposition 16.8 *Suppose $p \in L_0^2$ is an increment distribution in $\mathbb{Z}^2$. There is a c such that for every sufficiently large positive integer n, every $2n \leq |x|$, every positive integer $m \geq 4n$, we have*

$$\mathbb{P}^x\{\xi_m > \xi_n\} \geq \frac{c}{\log(m/n)}.$$

Proof. Using the invariance principle, we can see that it suffices to find a c_2 such that the estimate holds for all n sufficiently large and all $|x| \geq c_2 n$. By (67), we can find a c_2 such that for all $|x| \geq c_2 n$ and all $|y| \leq n$, then (for n sufficiently large),

$$a(y) \leq a(x) - 1.$$

Hence if we consider the martingale $M_j = a(S_{j\wedge\xi_n\wedge\xi_m}, 0) - a(y)$, we get

$$\mathbb{E}^x[a(S_{\xi_m}, 0) - a(y); \xi_m < \xi_n] \geq 1.$$

A fairly easy estimate can be used to see that

$$\mathbb{E}^x[a(S_{\xi_m}, 0); S_{\xi_m} \geq 2m] = o(1).$$

Also,

$$\begin{aligned}\mathbb{E}^x[a(S_{\xi_m}, 0) - a(y); \xi_m < \xi_n] &= [G_{m^2} - G_{|y|^2} + O(1)]\,\mathbb{P}^x\{\xi_m < \xi_n\}\\ &\leq c\,\log(m/n)\,\mathbb{P}^x\{\xi_m < \xi_n\}.\end{aligned}$$

$\square$

17 One-dimensional killed walks

A *defective increment distribution (on $\mathbb{Z}$)* is a set of nonnegative numbers $\{p_k : k \in \mathbb{Z}\}$ with $\sum p_k < 1$. Given a defective increment distribution, we have the corresponding random walk with killing S_j. This is a Markov chain with state space $\mathbb{Z} \cup \{\infty\}$, where ∞ is an absorbing state, and

$$\mathbb{P}\{S_{j+1} = k + l \mid S_j = k\} = p_l, \qquad \mathbb{P}\{S_{j+1} = \infty \mid S_j = k\} = p_\infty.$$

Here $p_\infty = 1 - \sum p_k$. We also have the corresponding reversed random walk $S_j^* = -S_j$ which is the random walk with defective increment distribution p_k^*,

$$p_k^* = p_{-k}, k \in \mathbb{Z}, \qquad p_\infty^* = p_\infty.$$

We let

$$T = \min\{j : S_j = \infty\}$$

denote the *killing time* for the random walk. Note that $\mathbb{P}\{T = j\} = p_\infty\,(1 - p_\infty)^{j-1}, j \in \{1, 2, \ldots\}$.

Examples.

- Suppose $p(j)$ is the increment distribution of a one-dimensional random walk and $s \in [0, 1)$. Then $p_j = s\,p(j)$ is a defective increment distribution corresponding to the random walk killed with geometric rate $1 - s$. Conversely, if p_j is a defective increment distribution, and $p(j) = p_j/(1 - p_\infty)$, then $p(j)$ is an increment distribution of a one-dimensional random walk (not necessarily aperiodic or irreducible). If we then kill this walk at rate $1 - p_\infty$ we get back p_j.

- Suppose $\bar{S}_j$ is a random walk in $\mathbb{Z}^d, d \geq 2$ which we write $\bar{S}_j = (Y_j, Z_j)$ where Y_j is a random walk in $\mathbb{Z}$ and Z_j is a random walk in $\mathbb{Z}^{d-1}$. Suppose the random walk is killed at rate $1 - s$ and let $\hat{T}$ denote the killing time. Let

$$\tau = \min\{j \geq 1 : Z_j = 0\}. \tag{97}$$

  Then set

$$p_k = \mathbb{P}\{Y_\tau = k; \tau < T\}.$$

  Note that

$$p_k = \sum_{j=1}^{\infty} \mathbb{P}\{\tau = j; Y_j = k; j < \hat{T}\} = \sum_{j=1}^{\infty} s^j\,\mathbb{P}\{\tau = j; Y_j = k\} = \mathbb{E}[s^{\hat{T}}; Y_{\hat{T}} = k; \hat{T} < \infty].$$

  If Z is transient, then we can let $s = 1$.

- Suppose $\bar{S}_j = (Y_j, Z_j)$ as in the previous example and suppose $A \subset \mathbb{Z}^{d-1} \setminus \{0\}$. Let

$$\sigma_A = \min\{j : Z_j \in A\},$$

$$p_k = \mathbb{P}\{Y_\tau = k; \tau < \sigma_A\}.$$

  If $\mathbb{P}\{Z_j \in A \text{ for some } j\} > 0$, then $\{p_k\}$ is a defective increment distribution.

Given the defective increment distribution $\{p_k\}$ define the events

$$V_+ = \{S_j > 0 : j = 1, \ldots, T - 1\}, \quad \bar{V}_+ = \{S_j \geq 0 : j = 1, \ldots, T - 1\},$$

$$V_- = \{S_j < 0 : j = 1, \ldots, T-1\}, \quad \bar{V}_- = \{S_j \le 0 : j = 1, \ldots, T-1\}.$$

Note that

$$\mathbb{P}(V_+ \cap \bar{V}_-) = \mathbb{P}(\bar{V}_+ \cap V_-) = \mathbb{P}\{T = 1\} = p_\infty. \tag{98}$$

Also, $\mathbb{P}(V_+) = \mathbb{P}^*(V_-), \mathbb{P}(V_-) = \mathbb{P}^*(V_+)$, where we write $\mathbb{P}^*$ to denote probabilities using the reversed random walk, e.g.,

$$\mathbb{P}^*(V_+) = \mathbb{P}\{S_j^* > 0 : j = 1, \ldots, T-1\}.$$

Define the defective increment distribution $p_{k,-}$, which is supported on $k = 0, -1, -2, \ldots$, by saying that $p_{k,-}$ is the probability that the first visit to $\{\cdots, -2, -1, 0\}$ after time 0 occurs at position k and this occurs before the killing time T, i.e.,

$$p_{k,-} = \sum_{j=1}^{\infty} \mathbb{P}\{S_j = k \; ; \; S_l > 0, l = 1, \ldots, k-1 \; ; \; j < T\}.$$

Define $p_{k,+}$ similarly. Note that

$$\mathbb{P}(\bar{V}_+) = \mathbb{P}(V_+) + p_{0,-}\, \mathbb{P}(\bar{V}_+),$$

and similarly for $\bar{V}_-$, i.e.,

$$\mathbb{P}(V_+) = (1 - p_{0,-})\, \mathbb{P}(\bar{V}_+), \qquad \mathbb{P}(V_-) = (1 - p_{0,+})\, \mathbb{P}(\bar{V}_-). \tag{99}$$

In the next proposition we prove a nonintuitive fact: the events $\bar{V}_+$ and V_- are independent! Also, it shows that $p_{0,+} = p_{0,-}$.

Proposition 17.1

$$\mathbb{P}(V_-)\, \mathbb{P}(V_+) = \frac{p_\infty}{1 - p_{0,+}} = \frac{p_\infty}{1 - p_{0,-}}.$$

In particular, if the random walk is symmetric

$$\mathbb{P}(V_+) = \sqrt{\frac{p_\infty}{1 - p_{0,+}}}.$$

Remark. It follows from the proposition that $p_{0,+} = p_{0,-}$. This can be shown directly using the fact that

$$\mathbb{P}\{[S_0, S_1, \ldots, S_{j-1}, S_j] = [0, k_1, \ldots, k_{j-1}, 0]\} =$$
$$\mathbb{P}\{[S_0, S_1, \ldots, S_{j-1}, S_j] = [0, -k_{j-1}, \ldots, -k_1, 0]\}.$$

Proof. Given (98) and (99), we can see it suffices to show that $\mathbb{P}(V_- \cap \bar{V}_+) = \mathbb{P}(V_-)\, \mathbb{P}(\bar{V}_+)$. We will prove the equivalent statement $\mathbb{P}(V_- \cap \bar{V}_+^c) = \mathbb{P}(V_-)\, \mathbb{P}(\bar{V}_+^c)$.

Let

$$\rho = \max\{k \in \mathbb{Z} : S_j = k \text{ for some } j = 1, \ldots, T-1\},$$
$$\eta = \min\{j \ge 1 : S_j > 0\}, \qquad \xi_k = \max\{j \ge 0 : S_j = k; j < T\}.$$

Then,

$$\mathbb{P}(V_- \cap \bar{V}_+^c) = \sum_{k=1}^{\infty} \mathbb{P}\{\rho = -k\} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \mathbb{P}\{\rho = -k; \xi_{-k} = j\}.$$

Note that the event $\{\rho = -k; \xi_{-k} = j\}$ is the same as the event

$$\{S_j = -k \; ; \; j < T \; ; \; S_l \le -k, l = 1, \ldots, j-1 \; ; \; S_l < -k, l = j+1, \ldots, T-1\}.$$

Hence,

$$\mathbb{P}\{\rho = -k; \xi_{-k} = j\} = \mathbb{P}(V_-)\, \mathbb{P}\{S_j = -k \; ; \; j < T \; ; \; S_l \le -k, l = 1, \ldots, j-1\}.$$

However, by path reversal we see that

$$\mathbb{P}\{S_j = -k \; ; \; j < T \; ; \; S_l \le -k, l = 1, \ldots, j-1\} = \mathbb{P}^*\{\eta = j; j < T; S_j = k\},$$

where we write $\mathbb{P}^*$ to denote probabilities with respect to the reversed random walk. But,

$$\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \mathbb{P}^*\{\eta = j; j < T; S_j = k\} = \mathbb{P}^*\{\eta < T\} = \mathbb{P}^*(\bar{V}_-^c) = \mathbb{P}(\bar{V}_+^c).$$

□

Let ϕ denote the characteristic function for the proper increment distribution

$$p(0,k) := \frac{p_k}{1 - p_\infty},$$

and let $\hat{\phi}(\theta) = (1 - p_\infty)\, \phi(\theta) = \sum e^{i\theta k} p_k$. Then

$$\mathbb{E}[e^{i\theta S_j}; T > j] = \hat{\phi}(\theta)^j = (1 - p_\infty)^j\, \phi(\theta)^j.$$

We now introduce functions

$$\Theta_+(z) = \exp\left\{ -\sum_{j=1}^{\infty} \frac{1}{j}\, \mathbb{E}[z^{S_j}; S_j > 0; j < T] \right\}$$
$$= \exp\left\{ -\sum_{k=1}^{\infty} z^k \sum_{j=1}^{\infty} \frac{1}{j}\, \mathbb{P}\{S_j = k; j < T\} \right\},$$

$$\Theta_-(z) = \exp\left\{ -\sum_{j=1}^{\infty} \frac{1}{j}\, \mathbb{E}[z^{S_j}; S_j < 0; j < T] \right\}$$

$$= \exp\left\{-\sum_{k=1}^{\infty} z^{-k} \sum_{j=1}^{\infty} \frac{1}{j}\,\mathbb{P}\{S_j = -k; j < T\}\right\},$$

$$\Theta_0(z) = \Theta_0 = \exp\left\{-\sum_{j=1}^{\infty} \frac{1}{j}\,\mathbb{P}\{S_j = 0; j < T\}\right\}.$$

Since $\mathbb{P}\{j < T\} = (1-p_\infty)^j$, we can see Θ_+ is analytic on $\{|z| < 1\}$ and continuous on $\{|z| \le 1\}$ while Θ_- is analytic on $\{|z| > 1\}$ and bounded and continuous on $\{|z| \ge 1\}$. Also,

$$\begin{aligned}
\Theta_+(e^{i\theta})\,\Theta_-(e^{i\theta})\,\Theta_0(e^{i\theta}) &= \exp\left\{-\sum_{j=1}^{\infty} \frac{1}{j}\,\mathbb{E}[e^{i\theta S_j}; j < T]\right\} \\
&= \exp\left\{-\sum_{j=1}^{\infty} \frac{1}{j}\,(1-p_\infty)^j\,\phi(\theta)^j\right\} \\
&= 1 - \hat{\phi}(\theta). \qquad (100)
\end{aligned}$$

Proposition 17.2 *Let $\tau = \inf\{j \ge 1 : S_j > 0\}$. Then*

$$\sum_{j=0}^{\infty} \mathbb{E}[e^{i\theta S_j}; j < \tau \wedge T] = \frac{1 - \mathbb{E}[e^{i\theta S_\tau}; \tau < T]}{1 - \hat{\phi}(\theta)}. \qquad (101)$$

$$\Theta_+(e^{i\theta}) = 1 - \mathbb{E}[e^{i\theta S_\tau}; \tau < T]. \qquad (102)$$

In particular,

$$\mathbb{P}\{\tau > T\} = \Theta_+(1) = \exp\left\{-\sum_{j=1}^{\infty} \frac{1}{j}\,\mathbb{P}\{S_j > 0; j < T\}\right\}$$

Proof. Let

$$K_\theta = \sum_{j=0}^{T-1} e^{i\theta S_j}.$$

Note that

$$\mathbb{E}[K_\theta] = \sum_{j=0}^{\infty} \mathbb{E}[e^{i\theta S_j}; j < T] = \sum_{j=0}^{\infty} \hat{\phi}(\theta)^j = \frac{1}{1-\hat{\phi}(\theta)}.$$

Also $\mathbb{E}^k[K_\theta] = e^{i\theta k}\mathbb{E}[K_\theta]$. Hence by the strong Markov property,

$$\mathbb{E}[\sum_{j=\tau}^{(T\wedge\tau)-1} e^{i\theta S_j}] = \mathbb{E}[e^{i\theta S_\tau}; \tau < T]\,\mathbb{E}[K_\theta] = \mathbb{E}[e^{i\theta S_\tau}; \tau < T]\,\frac{1}{1-\hat{\phi}(\theta)}.$$

Since the left-hand side of (101) is

$$\mathbb{E}[\sum_{j=0}^{\tau-1} e^{i\theta S_j}],$$

this gives (101). Note that

$$\Psi(z) := 1 - \mathbb{E}[z^{S_\tau}; \tau < T] = 1 - \sum_{k=1}^{\infty} z^k\,\mathbb{P}\{\tau < T; S_\tau = k\}$$

is analytic on $\{|z| < 1\}$ and continuous on $\{|z| \le 1\}$, while

$$\Phi(z) := \sum_{j=0}^{\infty} \mathbb{E}[z^{S_j}; j < \tau \wedge T] = \sum_{k=0}^{\infty} z^{-k} \sum_{j=0}^{\infty} \mathbb{P}\{S_j = -k; j < \tau \wedge T\}$$

is analytic on $\{|z| > 1\}$ and bounded, continuous on $\{|z| \ge 1\}$. Combining (100) and (101), we see that

$$\Phi(z)\,\Theta_0\,\Theta_-(z) = \frac{\Psi(z)}{\Theta_+(z)}, \qquad |z| = 1.$$

The left-hand side is analytic for $|z| > 1$ and the right-hand side is analytic for $|z| < 1$. Let

$$g(z) = \begin{cases} \Phi(z)\,\Theta_0\,\Theta_-(z), & |z| \ge 1 \\ \Psi(z)/\Theta_+(z), & |z| \le 1. \end{cases}$$

Then g is bounded, continuous on $\mathbb{C}$ and analytic for $|z| \ne 1$. By Morera's Theorem we can see that g is, in fact, analytic for all z, and hence by Liouville's Theorem g must be a constant. In particular, there is a c such that

$$\Psi(z) = c\,\Theta_+(z), \qquad |z| \le 1.$$

But $\Psi(0) = \Theta_+(0) = 1$. Hence $c = 1$. This establishes (102).

Corollary 17.3

$$\mathbb{P}\{\tau > T\} = \sqrt{\frac{1-p_\infty}{\Theta_0}}\,\exp\left\{-\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j}\,[\,\mathbb{P}\{S_j > 0; j < T\} - \mathbb{P}\{S_j < 0; j < T\}\,]\right\}.$$

Proof. We use $\Theta_+(1)\,\Theta_-(1)\,\Theta_0 = 1 - p_\infty$, to give

$$\Theta_+(1)^2 = \frac{1-p_\infty}{\Theta_0}\,\frac{\Theta_+(1)}{\Theta_-(1)}.$$

But,

$$\frac{\Theta_+(1)}{\Theta_-(1)} = \exp\left\{-\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j}\,[\,\mathbb{P}\{S_j = k; j < T\} - \mathbb{P}\{S_j = -k; j < T\}\,]\right\},$$

$$\Theta_+(1)\,\Theta_-(1)\,\Theta_0 = 1 - p_\infty.$$

Definition If S is a one-dimensional random walk, we will say that S is *weakly symmetric* if
$$\lim_{j\to\infty}\sum_{j=1}^{N}\frac{1}{j}\left[\mathbb{P}\{S_j>0\}-\mathbb{P}\{S_j<0\}\right]=\kappa\in(-\infty,\infty)$$
exists.

Note that if S is symmetric, then S is weakly symmetric with $\kappa=0$. Fukai has shown that if $S_n=(Y_n,Z_n)$ is a two-dimensional random walk with increment distribution $p\in L_0^{3+}(\mathbb{Z}^2)$ and τ is defined as in (97), then the distribution of Y_τ is weakly symmetric.

18 Avoiding a half-line

In this section $S_n=(Y_n,Z_n)$ will denote a two-dimensional random walk with increment distribution $p\in L_0^2$ (we will make some stronger assumptions later in the section). Here Y_n,Z_n are one-dimensional random walks. We let $\sigma_0=0$,
$$\sigma=\sigma_1=\min\{j\ge 1:Z_j=0\},$$
and for $m>1$,
$$\sigma_m=\min\{j>\sigma_{m-1}:Z_j=0\}.$$
Let $W_n=Y_{\sigma_n}$. Then W_n is a one-dimensional random walk. However, the increment distribution of W_n is not in L_0^2; in fact, it is not even in L^1. We can see this heuristically by noting that Proposition 6.10 applied to the one-dimensional walk Z implies $\mathbb{P}\{\sigma>n^2\}\asymp n^{-1}$, and hence we would expect that $\mathbb{P}\{Y_\sigma\ge n\}\asymp n^{-1}$.

Suppose $s\in(0,1)$ and we let $T'=T'_s$ be a geometric random variable, independent of S, such that $\mathbb{P}\{T'>n\}=s^n$. Let $p_k=p_{k,s}$ denote the defective increment distribution,
$$p_k=\mathbb{P}\{Y_\sigma=k;\sigma<T'\}.$$
Let $T=T_s=\max\{j:\sigma_j<T'\}$. Let $V_+,\bar V_+,V_-,\bar V_-$ be as in the previous section, using W,T, e.g.,
$$V_+=\{W_j>0:j=1,\ldots,T-1\}=\{S_j\notin\bar A_-,j=1\ldots,T'-1\},$$
where $\bar A_-=\{-ke_1,k=0,1,\ldots\}$. We similarly write $A_-=\bar A_-\setminus\{0\},\bar A_+=-\bar A_-,A_+=-A_-$. We also let
$$v_n=\mathbb{P}\{S_j\notin\bar A_-,j=1,\ldots,n\},\quad \bar v_n=\mathbb{P}\{S_j\notin A_-,j=1,\ldots,n\},$$
and we write $v_n^*,\bar v_n^*$ for the corresponding probabilities using the reversed random walk S^* (which is the same as the probabilities using S but replacing $\bar A_-,A_-$ with $\bar A_+,A_+$). Note that
$$\mathbb{P}(V_+)=(1-s)\sum_{n=0}^{\infty}s^n\,v_n,\quad \mathbb{P}(\bar V_+)=(1-s)\sum_{n=0}^{\infty}s^n\,\bar v_n.$$

Note that
$$\mathbb{P}(V_+\cap\bar V_-)=\mathbb{P}(\bar V_+\cap V_-)=\mathbb{P}\{Z_j\ne 0:j=1,\ldots,T-1\}\sim\sqrt{\Gamma}\,(1-s)^{1/2}.$$
Here $\Gamma=\mathbb{E}[Z_1^2]$ is the variance of the second component and the asymptotics follow from the proof of Proposition 6.10.

Corollary 18.1 *If $p\in L_0^2(\mathbb{Z}^2)$, s $s\to 1-$,*
$$\mathbb{P}(V_+)\,\mathbb{P}(V_-)\sim(1-q)\,\sqrt{\Gamma}\,(1-s)^{1/2},$$
where $\Gamma=\mathbb{E}[Z_1^2]$ and q is the probability that the first visit to $\{0,e_1,2e_1,\ldots\}$ by S after time 0 is to the origin. If Y is symmetric, then
$$\mathbb{P}(V_+)\sim\sqrt{(1-q)\,\sqrt{\Gamma}}\,(1-s)^{1/4}.$$

Corollary 18.2 *If Y_{σ_j} is a weakly symmetric random walk, then there is a c such that*
$$\mathbb{P}(V_+)\sim c\,(1-s)^{1/4},\quad s\to 1-.$$

We will need a slight generalization of these results. Suppose l is a positive integer. Let
$$A_{+,l}=l\,A_+=\{le_1,2le_1,3le_1,\ldots\},$$
and define $\bar A_{+,l},A_{-,l},\bar A_{-,l}$, similarly. Let
$$V_{+,l}=\{S_j\notin A_{-,l},j=1,\ldots,n\},$$
and define $\bar V_{+,l},V_{-,l},\bar V_{-,l}$, respectively. A similar argument to that above shows that $\mathbb{P}(V_{+,l}\cap\bar V_{-,l})\asymp(1-s)^{1/2}$, where the constant in the asymptotics depends on l as well as on p.

Corollary 18.3 *If Y_{σ_j} is a weakly symmetric random walk and l is a positive integer, then there is a c_l such that*
$$\mathbb{P}(V_+)\sim c_l\,(1-s)^{1/4},\quad s\to 1-.$$

If $k_1\le k_2$ are integers and l is a positive integer, we will let $[k_1,k_2;l]$ denote the set
$$[k_1,k_2;l]=\{kle_1:k_1\le k\le k_2\}.$$
We also write $[k_1,\infty;l),(-\infty,k_2;l],(-\infty,\infty;l)$. As before we let
$$\xi_n=\min\{j\ge 1:|S_j|\ge n\}.$$
We will say that the *half-line estimates hold* if for each fixed l, the following are true:
$$\mathbb{P}\{\xi_n<\tau_{(-\infty,\infty;l)}\}\asymp n^{-1},\qquad \mathbb{P}\{\xi_n\le\tau_{[0,\infty;l)}\}\asymp n^{-1/2}.$$
(The implicit constants in the $\asymp$ notation may depend on l.) If $p\in L_0^{3+}(\mathbb{Z}^2)$ or $p\in L_0^2(\mathbb{Z}^2)$ and symmetric then the half-line estimates hold. (We suspect that $p\in L_0^2(\mathbb{Z}^2)$ suffices.)

Lemma 18.4 *If $p \in L_0^2$ and l is a positive integer, there is a c such that for every positive integer k,*

$$c\,\mathbb{P}\{\xi_{kl} < \tau_{(-\infty,0;l]}\} \leq \mathbb{P}\{\tau_{[k,\infty;l)} < \tau_{(-\infty,0;l]}\} \leq \mathbb{P}\{\xi_{kl} < \tau_{(-\infty,0;l]}\}.$$

Proof. Fix l and allow all constants to depend on l. Since $\{\tau_{[k,\infty;l)} < \tau_{(-\infty,0;l]}\} \subset \{\xi_{kl} < \tau_{(-\infty,0;l]}\}$, the second inequality is immediate and the first inequality follows if we show that

$$\mathbb{P}\{\tau_{[k,\infty;l)} < \tau_{(-\infty,0;l]} \mid \xi_{kl} < \tau_{(-\infty,0;l]}\} \geq c.$$

Using the invariance principle and Proposition 16.7, we can see that it suffices to show that

$$\mathbb{P}\{Y_{\xi_{kl}} \geq 0 \mid \xi_{kl} < \tau_{(-\infty,0;l]}\} \geq c.$$

However, the local central limit theorem can be used to show that $\mathbb{P}\{Y_{\xi_{kl}} \geq 0\} \geq c$ (c independent of k) and translation invariance of the random walk shows that for all $m > 0$,

$$\mathbb{P}^{-me_1}\{Y_{\xi_{kl}} \geq 0\} \leq \mathbb{P}\{Y_{\xi_{kl}} \geq 0\}.$$

Hence, using the strong Markov property,

$$\mathbb{P}\{Y_{\xi_{kl}} \geq 0 \mid \xi_{kl} \geq \tau_{(-\infty,0;l]}\} \leq \mathbb{P}\{Y_{\xi_{kl}} \geq 0\},$$

$$\mathbb{P}\{Y_{\xi_{kl}} \geq 0 \mid \xi_{kl} < \tau_{(-\infty,0;l]}\} \geq \mathbb{P}\{Y_{\xi_{kl}} \geq 0\} \geq c.$$

□

Lemma 18.5 *If l is a positive integer and $p \in L_0^2(\mathbb{Z}^2)$ for which the half-line estimates hold, there is a c such that for all sufficiently large positive integers k,*

$$\mathbb{P}\{\tau_{[k,\infty;l)} < \tau_{(-\infty,0;l]}; kl \leq Y_{\tau_{[k,\infty;l)}} \leq 4kl\} \geq c\,\mathbb{P}\{\xi_{2kl} < \tau_{(-\infty,0;l]}\} \asymp k^{-1/2}.$$

Proof. By the invariance principle and Proposition 16.7, we can see that

$$\mathbb{P}\{\tau_{[k,\infty;l)} < \tau_{(-\infty,0;l]}; kl \leq Y_{\tau_{[k,\infty;l)}} \leq 4k\} \geq c\,\mathbb{P}\{\xi_{2kl} < \tau_{(-\infty,0;l]}; Y_{\xi_{2kl}} \geq 0; Z_{\xi_{2kl}} \leq 3kl\}.$$

In the previous proposition, we showed that $\mathbb{P}\{\xi_{2kl} < \tau_{(-\infty,0;l]}; Y_{\xi_{2kl}} \geq 0\} \asymp k^{-1/2}$. Hence it suffices to show that $\mathbb{P}\{\xi_{2lk} < \tau_{(-\infty,0;l]}; Z_{\xi_{2kl}} > 3kl\} = o(k^{-1/2})$. Let $\rho_k = \xi_{2kl} \wedge \tau_{(-\infty,0;l]}$. Then it suffices to show that $\mathbb{P}\{|S_{\rho_{2kl}} - S_{\rho_{2kl}-1}| \geq k\} = o(k^{-1/2})$. But,

$$\mathbb{P}\{|S_{\rho_{2kl}} - S_{\rho_{2kl-1}}| \geq k\} \leq \mathbb{E}[\rho_{2kl}]\,\mathbb{P}\{|S_1| \geq k\} = o(k^{-2})\,\mathbb{E}[\rho_{2kl}].$$

For $m \leq k^2$, $\mathbb{P}\{\rho_{2kl} \geq m\} \leq \mathbb{P}\{\tau_{(-\infty,0;l]} \geq m\} \leq c\,m^{-1/4}$. Therefore,

$$\mathbb{E}[\rho_{2kl}] \leq \sum_{m \leq k^2} \mathbb{P}\{\rho_{2kl} \geq m\} + \mathbb{P}\{\rho_{2kl} \geq n^2\} \sup_x \mathbb{E}^x[\xi_{2kl}] \leq cnk^{-3/2}.$$

□

A similar argument proves the following. Here we do not have to assume that the half-line estimates hold since we only use the whole-line estimate and this estimate only requires $p \in L_2^0(\mathbb{Z}^2)$.

Lemma 18.6 *If $p \in L_0^2(\mathbb{Z}^2)$, there is a c such that for all sufficiently large k,*

$$\mathbb{P}\{\tau_{[2k,\infty;l)} < \tau_{(-\infty,k;l]}; 3kl/2 \leq Y_{\tau_{[2k,\infty;l)}} \leq 4kl\} \asymp k^{-1}.$$

Proposition 18.7 *Suppose l is a positive integer and $p \in L_0^2(\mathbb{Z}^2)$ satisfies the half-line estimates. There exist c_1, c_2 such that if $0 < k < n$ are integers, then*

$$\frac{c_1\sqrt{k}}{\sqrt{n}} \leq \mathbb{P}\{\xi_{nl} < \tau_{(-\infty,-k;l]}\} \leq \frac{c_2\sqrt{k}}{\sqrt{n}}. \tag{103}$$

$$\frac{c_1}{\sqrt{kn}} \leq \mathbb{P}\{\xi_{nl} < \tau_{(-\infty,k;l]}\} \leq \frac{c_2}{\sqrt{kn}}. \tag{104}$$

.

Proof. For (103), we may assume that $k \leq n/8$.

$$\begin{aligned} cn^{-1/2} &\geq \mathbb{P}\{\xi_{nl} < \tau_{(-\infty,0;l]}\} \\ &\geq \mathbb{P}\{\tau_{[k,\infty;l)} < \tau_{(-\infty,0;l]}; kl \leq Y_{\tau_{[k,\infty;l)}} \leq 4kl\} \min_{kl \leq m \leq 4kl} \mathbb{P}^m\{\xi_{nl} < \tau_{(-\infty,0;l]}\} \\ &\geq c\,k^{-1/2} \min_{k \leq m \leq 4k} \mathbb{P}^{ml}\{\xi_{nl} < \tau_{(-\infty,0;l]}\}. \end{aligned}$$

But for $k \leq m \leq 4k$,

$$\mathbb{P}^{ml}\{\xi_{nl} < \tau_{(-\infty,0;l]}\} \geq \mathbb{P}\{\xi_{2nl} < \tau_{(-\infty,-k;l]}\}.$$

This gives the upper bound. We will now show that there is a c such that for all k sufficiently large and all $|z| < kl/2$,

$$\mathbb{P}^z\{\xi_{nl} < \tau_{(-\infty,-k;l]}\} \leq \frac{c_2\sqrt{k}}{\sqrt{n}}. \tag{105}$$

If $p \in L_0^\infty$, this follows immediately from the upper bound at $z = 0$ and the Harnack principle. More generally, if m is a positive integer less than k, we can see that

$$\mathbb{P}^{z+ml}\{\xi_{nl/2} < \tau_{(-\infty,-k;l]}\} \geq \mathbb{P}^z\{\xi_{nl} < \tau_{(-\infty,-k;l]}\}.$$

By the invariance principle and Proposition 16.7, there is a probability at least c that the random walk starting at the origin reaches

$$\{z + ml : 0 \leq m \leq k/2\}$$

before leaving $\{|w| < kl\}$. Hence,

$$c(k/n)^{1/2} \geq \mathbb{P}\{\xi_{nl/2} < \tau_{(-\infty,-k;l]} \geq c\mathbb{P}^x\{\xi_{nl} < \tau_{(-\infty,k;l]}\}.$$

This gives (105).

For any positive integer j note that

$$\mathbb{P}\{\xi_{nl} < \tau_{(-\infty,0;l]}; kl\,2^j \leq |S_{\xi_{kl}}| < kl\,2^{j+1}\} \leq$$

$$\mathbb{P}\{\xi_{kl} < \tau(-\infty,0;l]; |S_{\xi_{kl}} - S_{\xi_{kl}-1}| \geq kl2^j\} \sup_{|z|<kl\,2^{j+1}} \mathbb{P}^z\{\xi_{nl} < \tau_{(-\infty,0;l]}\}.$$

Also,

$$\begin{aligned}\mathbb{P}\{\xi_{kl} < \tau_{(-\infty,0;l]}; |S_{\xi_{kl}} - S_{\xi_{kl}-1}| \geq k2^j\} &\leq \mathbb{E}[\xi_{kl} \wedge \tau_{(-\infty,0;l]}\,\mathbb{P}\{|S_1| \geq k2^j\} \\ &\leq c\,k^{3/2}\,[k2^j]^{-2}.\end{aligned}$$

Therefore, using (105),

$$\mathbb{P}\{\xi_{nl} < \tau_{(-\infty,0;l]}; kl\,2^j \leq |S_{\xi_{kl}}| < kl\,2^{j+1}\} \leq c\,n^{-1/2}\,2^{-3j/2}.$$

But $\mathbb{P}\{\xi_{nl} < \tau_{(-\infty,0]}\} \geq \tilde{c}/\sqrt{n}$. Hence, we see that there is an R such that

$$\mathbb{P}\{\xi_{nl} < \tau_{(-\infty,0;l]}; |S_{\xi_k}| \leq R\,kl\} \geq (\tilde{c}/2)\,n^{-1/2}.$$

Finally, (105) implies

$$\mathbb{P}\{\xi_{nl} < \tau_{(-\infty,0;l]}; |S_{\xi_{kl}}| \leq R\,kl\} \leq \mathbb{P}\{\xi_{kl} < \tau_{(-\infty,0;l]}\} \sup_{|z|<Rkl} \mathbb{P}^z\{\xi_{nl} < \tau_{(-\infty,0;l]}\},$$

which gives

$$\sup_{z<Rkl} \mathbb{P}^z\{\xi_{nl} < \tau_{(-\infty,-2Rk;l]}\} \geq \sup_{z<Rkl} \mathbb{P}\{\xi_{nl} < \tau_{(-\infty,0;l}\} \geq c\,(k/n)^{1/2}.$$

Using an argument similar to that used to derive (105) we can deduce

$$\mathbb{P}\{\xi_{nl/2} < \tau_{(-\infty,-2Rk;l]} \geq c\,(k/n)^{1/2}.$$

This gives the lower bound in (103).

The lower bound in (104) follows from Lemma 18.6 and the lower bound in (103). For the upper bound we use

$$\mathbb{P}\{\xi_{kl} < \tau_{(-\infty,k;l]}; kl\,2^j \leq |S_{\xi_{kl}}| \leq kl\,2^{j+1}\} \leq c\,\mathbb{E}[\xi_{kl} \wedge \tau_{(-\infty,k;l]}]\,[k2^j]^{-2} \leq ck^{-1}\,2^{-2j}.$$

$$\mathbb{P}\{\xi_{nl} < \tau_{(-\infty,k;l]} \mid \xi_{kl} < \tau_{(-\infty,k;l]}, 2^j \leq |S_{\xi_{kl}}| \leq kl\,2^{j+1}\} \leq c\,k^{1/2}\,2^{j/2}\,n^{-1/2}.$$

Summing over j gives the upper bound. □

19 Beurling estimates

Recall that $\mathcal{J}(x) = |\Lambda^{-1}(x)|$.

Theorem 19.1 *Suppose $p \in L_0^{3+}(\mathbb{Z}^2)$ and l is a positive integer. There is a c such that if $k < n$ are positive integers and A' is a set of the form*

$$A' = \{w_k, \ldots, w_n\}$$

where

$$lj\mathcal{J}(e_1) \leq \mathcal{J}(w_j) < l(j+1)|\mathcal{J}(e_1)|, \quad j = k, k+1, \ldots, n,$$

then

$$\mathbb{P}\{\tau_{A'} > \xi_{nl}\} \leq c\,\sqrt{k/n}.$$

Proof. It will suffice to show that

$$\mathbb{P}\{\tau_{A'} > \xi_{n^2}\} \leq c\,\sqrt{k/n}\,(\log n)^{-1}.$$

The proof will be by comparision with the set

$$A = [k,n;l] = \{z_k, \ldots, z_n\},$$

with $z_j = jle_1$. We have already shown that

$$\mathbb{P}\{\tau_A > \xi_{n^2}\} \leq c\,\sqrt{k/n}\,(\log n)^{-1}.$$

We wil prove the (somewhat stronger than required) estimate

$$\mathbb{P}\{\tau_{A'} > \xi_{n^2}\} - \mathbb{P}\{\tau_A > \xi_{n^2}\} \leq c\,n^{-1/2}\,(\log n)^{-1}.$$

Before proceeding with the proof, it will be useful to collect the facts we will use. First, for $|z|, |w| \leq nl$,

$$G_{\mathcal{B}_{n^2}}(z,w) = 2\,C_2\,[\log n^2 - \log \mathcal{J}(z-w)] + O\left(\frac{1}{|z-w|+1}\right). \tag{106}$$

The other estimate concerns the escape probability from the line: if $k \leq j \leq n$,

$$\mathbb{P}^{z_j}\{\tau_{[k,n;l]} > \xi_{n^2}\} \leq c\,(j-k+1)^{-1/2}\,(n+1-j)^{-1/2}\,(\log n)^{-1}. \tag{107}$$

We will write just g for $G_{\mathcal{B}_{n^2}}$. A last-exit decomposition tells us that

$$\mathbb{P}\{\tau_A < \xi_{n^2}\} = \sum_{j=k}^{n} g(0,z_j)\,\mathbb{P}^{z_j}\{\tau_A > \xi_{n^2}\},$$

and similarly for A'. Therefore,

$$\mathbb{P}\{\tau_{A'} > \xi_{n^2}\} - \mathbb{P}\{\tau_A > \xi_{n^2}\} \leq$$

$$\sum_{j=k}^{n} \mathbb{P}^{z_j}\{\tau_A > \xi_{n^2}\}\,[g(0,z_j) - g(0,w_j)] + \sum_{j=k}^{n} g(0,w_j)\,[\mathbb{P}^{z_j}\{\tau_A > \xi_{n^2}\} - \mathbb{P}^{w_j}\{\tau_{A'} > \xi_{n^2}\}]. \quad (108)$$

Using (106), we can see that $|g(0,z_j) - g(0,w_j)| \leq c/j$. Hence, using (107) we can see that the absolute value of the first sum in (108) is bounded above by a constant times

$$\frac{1}{\log n}\sum_{j=k}^{n} j^{-1}(j+1-k)^{-1/2}(n+1-k)^{1/2} \leq c\, n^{-1/2}\,(\log n)^{-1}.$$

To bound the second sum in (108), note that this sum is $F(0)$ where

$$F(y) = \sum_{j=k}^{n} g(y,w_j)\,[\mathbb{P}^{z_j}\{\tau_A > \xi_{n^2}\} - \mathbb{P}^{w_j}\{\tau_{A'} > \xi_{n^2}\}].$$

Note that F is harmonic on $\mathcal{B}_{n^2} \setminus A'$ and is zero on $\mathbb{Z}^2 \setminus \mathcal{B}_{n^2}$. Hence, either F is nonpositive everywhere or its maximum is obtained on A'. Hence it suffices to show that there is a c such that for every $w_m \in A'$, $F(w_m) \leq c\, n^{-1/2}\,(\log n)^{-1}$. Using a last-exit decomposition again, we can see that

$$\sum_{j=k}^{n} g(w_m,w_j)\,\mathbb{P}^{w_j}\{\tau_{A'} > \xi_{n^2}\} = 1 = \sum_{j=k}^{\infty} g(z_m,z_j)\,\mathbb{P}^{z_j}\{\tau_A > \xi_{n^2}\}.$$

Hence,

$$F(w_m) = \sum_{j=k}^{n} \mathbb{P}^{z_j}\{\tau_A > \xi_{n^2}\}\,[g(z_m,z_j) - g(w_m,w_j)].$$

Note that $J(w_j - w_m) \geq J(z_j - z_m) - l$. Hence

$$g(z_m - z_j) - g(w_m,w_j) \leq \frac{c}{|m-j|+1}$$

(note that we are not taking absolute values on the left-hand side). Using (108) again, we get that

$$F(w_m) \leq \frac{c}{\log n}\sum_{j=k}^{n}(j-k+1)^{-1/2}\,(n-j+1)^{-1/2}\,(|m-j|+1)^{-1} \leq c\, n^{-1/2}\,(\log n)^{-1}.$$

$\square$

Theorem 19.2 (Beurling estimate) *If $p \in L_0^{3+}(\mathbb{Z}^2)$, there is a c such that if $[\omega_0,\omega_1,\ldots,\omega_m]$ is a nearest neighbor path in $\mathbb{Z}^2$ with $|\omega_0| \leq k$, $|\omega_m| \geq n$, then*

$$\mathbb{P}\{\tau_{\{\omega_0,\ldots,\omega_n\}} > \xi_n\} \leq c\,\sqrt{k/m}.$$

Proof. We can find an l (depending only on p) and a subset of $\{\omega_0,\omega_1,\ldots,\omega_m\}$ satisfying the conditions of the previous theorem. $\square$

20 Skorohod embedding

The invariance principle is one statement about the convergence of random walk to Brownian motion. In many applications, it is useful to have stronger convergence statements.

Definintion. A *Brownian motion/random walk coupling* is an ordered pair of processes (B_t, S_n) defined on the same probability space $(\Omega, \mathbb{P}, \mathcal{F})$ where B_t is a d-dimensional Brownian motion with covariance matrix Γ and S_n is a random walk with increment distribution $p \in L_0^2(\mathbb{Z}^2)$ with covariance matrix Γ

If S_n is a discrete time random walk, we define $S_t, t \in [0,\infty)$ by linear interpolation. If (B_t, S_n) is a Brownian motion/random walk coupling, we define

$$\Delta_t = \sup_{0\leq s\leq t} |B_s - S_s|.$$

Note that if B, S are independent, we would expect $\Delta_t \approx t^{1/2}$. A good coupling would be one for which Δ_t is of smaller magnitude. In this section, we discuss one such coupling given by Skorohod embedding.

We will start by considering $d = 1$; assume that $p \in L_0^2(\mathbb{Z})$. Suppose first that p takes on only two values, k_1 and $-k_2$. Since p has mean zero, $p(k_1) = k_2/(k_1+k_2), p(-k_2) = k_1/(k_1+k_2)$. Let B_t be a standard Brownian motion and let

$$T = T_{k_1,k_2} = \inf\{t : B_t = k_1 \text{ or } B_t = -k_2\}.$$

Since $M_t = B_{t\wedge T}$ is a bounded martingale, the optional sampling theorem implies that

$$0 = \mathbb{E}[M_T] = \mathbb{P}\{B_T = k_1\}\,k_1 - \mathbb{P}\{B_T = -k_2\}\,k_2,$$

which implies that $\mathbb{P}\{B_T = k_1\} = p(k_1)$. If we let $T_0 = 0, T_1 = T$ and for $m > 1$,

$$T_j = \inf\{t > T_{j-1} : B_t - B_{T_{j-1}} = k_1 \text{ or } B_t - B_{T_{j-1}} = -k_2\},$$

Then $S_j := B_{T_j}$ has the distribution of a random walk with increment distribution p. This is an example of *Skorohod embedding*. In order to analyze this, the following lemma will be useful.

Lemma 20.1 *There is a c such that if B_t is a standard Brownian motion, $a, b > 0$, $T_{a,b} = \inf\{t : B_t = a \text{ or } B_t = -b\}$, then*

$$\mathbb{P}\{T_{a,b} \geq t\} \leq c\,\frac{a\wedge b}{a+b}\,\exp\left\{-\frac{t\,\pi^2}{2\,(a+b)^2}\right\}$$

Proof. Note that

$$f(t,x) = \frac{1}{2}\,e^{-t/2}\,\sin x, \quad 0 \leq x \leq \pi,$$

satisfies the heat equation $\partial_t f = (1/2) f''$ with boundary conditions $f(t,0) = f(t,\pi) = 0$. This implies that if we start a standard Brownian motion on $[0,\pi]$ with initial density $(1/2)\sin x$, then the probability that it has stayed in the interval $[0,\pi]$ is exactly $e^{-t/2}$; in fact, the density of the Brownian motion killed upon reaching $\{0,\pi\}$ is $(1/2)\, e^{-t}\sin x$ for $0 < x < \pi$. The density of B_1 (without killing, and hence, also with killing) is bounded above by $1/\sqrt{2\pi}$, regardless of the value of B_0. Hence,

$$\mathbb{P}^x\{B_s \in (0,\pi), 0 \le s \le t\} \le c_0\, e^{-t/2}, \quad 0 \le x \le \pi,$$

where $c_0 = (2e/\pi)^{1/2}$. The probability that a Brownian motion starting at x stays in $(0,\pi)$ up through times $1/2$ is bounded above by a constant times x (this can be checked, for example, using the reflection principle for Brownian motion). Hence, we can improve the estimate above to

$$\mathbb{P}^x\{B_s \in (0,\pi), 0 \le s \le t\} \le c_1\, x\, e^{-t/2}, \quad 0 \le x \le \pi/2.$$

For $\pi/2 \le x \le \pi$, symmetry tells us that the estimate is true with $\pi - x$ replacing x on the right-hand side. Using translation and Brownian scaling, we can see that this implies

$$\begin{aligned}\mathbb{P}^0\{B_s \in (-b,a), 0 \le s \le t\} &= \mathbb{P}^b\{B_s \in (0,a+b), 0 \le s \le t\} \\ &= \mathbb{P}^{b\pi/(a+b)}\{B_s \in (0,\pi), 0 \le s \le t\,\pi^2/(a+b)^2\} \\ &\le c\frac{a\wedge b}{a+b}\,\exp\{-\frac{t\pi^2}{2(a+b)^2}\}.\end{aligned}$$

□

It is easy to check that $B_t^2 - t$ is a martingale. The estimate in the lemma shows that $M_t = B^2_{t\wedge T_{a,b}} - (t \wedge T_{a,b})$ is a uniformly integrable martingale. Hence, another use of the optional sampling theorem gives

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_{T_{a,b}}] = \mathbb{E}[B^2_{T_{a,b}}] - \mathbb{E}[T_{a,b}].$$

But $\mathbb{E}[B^2_{T_{a,b}}] = a^2[b/(a+b)] + b^2[a/(a+b)] = ab$. Hence

$$\mathbb{E}[T_{a,b}] = ab = \sigma^2,$$

where σ^2 denotes the variance of a random variable with distribution p.

Theorem 20.2 *Suppose $p \in L^2_0(\mathbb{Z})$ and let*

$$\sigma^2 = \sum_{k=-\infty}^{\infty} k^2\, p(k),$$

be the variance. There is a Brownian motion B_t adapted to a filtration $\mathcal{F}_t$ and a stopping time T with respect to this filtration such that B_T has increment distribution p. Moreover, $\mathbb{E}[T] = \sigma^2$.

Proof. We will give a proof that uses some "extra randomness". We will construct a probability distribution ν on $\{0,1,2,\cdots\} \times \{0,1,2,\cdots\}$ such that: $\nu(0,0) = p(0)$ and for k_1, k_2 positive integers,

$$p(k_1) = \sum_{j_2=1}^{\infty} \nu(k_1,j_2)\,\frac{j_2}{k_1+j_2}, \qquad p(-k_2) = \sum_{j_1=1}^{\infty} \nu(j_1,k_2)\,\frac{j_1}{j_1+k_2}.$$

Once we have constructed such a ν exists, we will let $K = (K_1,K_2)$ be a random variable with distribution ν independent of B_t and such that K is measurable with respect to $\mathcal{F}_0$. If $K = (0,0)$, then $T = 0$. Otherwise, we set $T = T_{K_1,K_2}$ as defined in Lemma 20.1. Note that if k_1 is a positive integer,

$$\mathbb{P}\{B_T = k_1\} = \sum_{j_2=1}^{\infty} \mathbb{P}\{K = (k_1,j_2)\}\mathbb{P}\{T_{(k_1,j_2)} = K_1\} = \sum_{j_2=1}^{\infty} \nu(k_1,j_2)\,\frac{j_2}{k_1+j_2} = p(k_1).$$

Similarly, $\mathbb{P}\{B_T = -k_2\} = p(-k_2)$ for nonpositive intetgers.

Let us now show how to construct such a ν. First, we set $\nu(0,0) = p(0)$ and we let p_0 be the subprobability measure equal to p restricted to the nonzero integers. Note that for $l = 0$,

$$\sum_{k_1=1}^{\infty} k_1 p_l(k_1) = \sum_{k_2=1}^{\infty} k_2\, p_l(-k_2). \tag{109}$$

We now describe a recursive procedure. Suppose p_l is given.

- Let k_1 be the smallest positive integer with $p_l(k_1) > 0$ and k_2 the smallest positive integer with $p_l(-k_2) > 0$.
- If $k_1 p_l(k_1) \le k_2 p_l(k_2)$, let
  $$\nu(k_1,k_2) = \frac{k_2\,(k_1+k_2)}{p_l(k_1)}.$$
  If $k_2 p_l(k_2) \le k_1 p_l(k_1)$, let
  $$\nu(k_1,k_2) = \frac{k_1\,(k_1+k_2)}{p_l(k_2)}.$$
  Note that these definitions agree if $k_1 p_l(k_1) = k_2 p_l(k_2)$.
- Let $p_{l+1}(k) = p_l(k)$ for $k \ne k_1, -k_2$ and let
  $$p_{l+1}(k_1) = p_l(k_1) - \nu(k_1,-k_2)\,\frac{k_2}{k_1+k_2},$$
  $$p_{l+1}(-k_2) = p_l(k_2) - \nu(k_1,-k_2)\,\frac{k_1}{k_1+k_2}.$$
  Note that either $p_{l+1}(k_1) = 0$ or $p_{l+1}(-k_2) = 0$ (or both). Also $p_{l+1}(k) \ge 0$ for all k.

- Since $k_1\,[p_l(k_1)-p_{l+1}(k_1)] = k_2\,[p_l(-k_2)-p_{l+1}(-k_2)]$, p_{l+1} satisfies (109) and we can repeat this procedure. Also note that the choice of ordered pair (k_1,k_2) is different for each l.

- Note that if k_1 is a positive integer
$$p(k_1) = \sum_{l=0}^{\infty}[p_l(k_1)-p_{l+1}(k_1)] = \sum_{k_2=1}^{\infty}\nu(k_1,k_2)\,\frac{k_2}{k_1+k_2},$$
and similarly
$$p(-k_2) = \sum_{k_1=1}^{\infty}\nu(k_1,k_2)\,\frac{k_1}{k_1+k_2},$$

Since K is independent of B, we can compute
$$\begin{aligned}
\mathbb{E}[T] &= \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\nu(k_1,k_2)\,\mathbb{E}[T_{k_1,k_2}] \\
&= \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\nu(k_1,k_2)\;k_1k_2 \\
&= \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\nu(k_1,k_2)\;\left[\frac{k_1^2\,k_2}{k_1+k_2}+\frac{k_2^2\,k_1}{k_1+k_2}\right] \\
&= \sum_{k_1=1}^{\infty}k_1^2\,p(k_1)+\sum_{k_2=1}^{\infty}k_2^2\,p(-k_2) = \sigma^2.
\end{aligned}$$

□

Definition. Suppose $p \in L_0^2(\mathbb{Z})$ and B_t is a standard Brownian motion. Let $K^{(1)}, K^{(2)},\ldots$ be i.i.d. random variables talking values in $\{0,1,2,\ldots\}\times\{0,1,2,\ldots\}$, defined on the same probability space and independent of B, whose distribution is that of ν in the previous proof. Let
$$\tau_1 = \inf\{t : B_t = K_1^{(1)} \text{ or } B_t = -K_2^{(1)}\},$$
and for $j>1$,
$$\tau_j = \inf\{t : B_t - B_{\tau_{j-1}} = K_1^{(j)} \text{ or } B_t - B_{\tau_{j-1}} = -K_2^{(j)}\}.$$
Let $S_j = B_{\tau_j}$. If σ^2 is the variance of p, then $\hat{B}_t := B_{\sigma^2/t}$ is a Brownian motion with variance σ^2 and $S_j = \hat{B}_{\tau_j/\sigma^2 j}$. We will call this the *Skorohod embedding* of S_j in B or $\hat{B}$.

We will be interested in moment conditions satisfied by the stopping time T.

Proposition 20.3 *Suppose $p \in L_0^2(\mathbb{Z})$ and $T=\tau_1$ is the stopping time of the Skorohod embedding.*

- *If $p \in L_0^\infty$, then there is an $r>0$ such that $\mathbb{E}[e^{rT}]<\infty$.*
- *if m is a positive integer and $p\in L_0^{2m}$, then $\mathbb{E}[T^m]<\infty$.*

Proof. If $\mathbb{P}\{|X_1|\le a\}=1$, then $T\le T_{a,a}$ where $T_{a,b}$ is as defined in Lemma 20.1. The first assertion follows from the exponential estimate in that lemma. We have already shown the second assertion for $m=1$. Let us fix a positive integer $m\ge 2$ and allow constants in the remainder of this proof to depend on m. The exponential estimate implies that for every positive integer m, there is a $c_m\ge 1$ such that if $a+b=1$,
$$\mathbb{E}[T_{a,b}^m] \le c\;[\,\mathbb{E}[(T_{a,b}\wedge 1)^m]+\mathbb{E}[(T_{a,b}-T_{a,b}\wedge 1)^m]\,] \le c\;\left[\mathbb{E}[T_{a,b}]+\frac{a\wedge b}{a+b}\right].$$
By Brownian scaling, we see that this implies for all $a,b>0$,
$$\mathbb{E}[T_{a,b}^m] \le c\;\left[\mathbb{E}[T_{a,b}]+(a+b)^{2m}\,\frac{a\wedge b}{a+b}\right] \le c\;\left[ab+(a^{2m}+b^{2m})\,\frac{a\wedge b}{a+b}\right].$$
Hence,
$$\begin{aligned}
\mathbb{E}[T^m] &= \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\nu(k_1,k_2)\,\mathbb{E}[T_{k_1,k_2}^m] \\
&\le c\;\left[\mathbb{E}[T]+\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\nu(k_1,k_2)\,[k_1^{2m}+k_2^{2m}]\frac{k_1\wedge k_2}{k_1+k_2}\right] \\
&\le c\;\left[\mathbb{E}[T]+\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}k_1^{2m}\;\nu(k_1,k_2)\,\frac{k_2}{k_1+k_2}+\sum_{k_2=1}^{\infty}\sum_{k_1=1}^{\infty}k_2^{2m}\;\nu(k_1,k_2)\,\frac{k_1}{k_1+k_2}\right] \\
&\le c\;\left[\mathbb{E}[T]+\mathbb{E}[S_1^{2m};S_1>0]+\mathbb{E}[|S_1|^{2m};S_1<0]\;\right]<\infty.
\end{aligned}$$

□

We will now try to give estimate for $\mathbb{P}\{\Delta_n\ge x\}$ for the Skorohod embedding. Note that $|S_n-B_{\sigma^2 n}| = |B_{\tau_n}-B_{\sigma^2 n}|$. Note that
$$\tau_n = \tau_1+(\tau_2-\tau_1)+\cdots+(\tau_n-\tau_{n-1}),$$
is the sum of i.i.d. random variables each with the distribution of T as in Theorem 20.2. If $p\in L_0^4$, then $\mathbb{E}[T^2]<\infty$ and the central limit theorem tells us roughly that
$$\tau_n = \sigma^2\,n+O(n^{1/2}).$$
Hence by properties of Brownian motion, we would expect roughly that
$$B_{\tau_n}-B_{\sigma^2 n}\approx n^{1/4}.$$
We will now make this more precise.

Theorem 20.4 *Suppose $p \in L_0^4(\mathbb{Z})$, (B_t, S_t) are defined as above using the Skorohod embedding and*

$$\Delta_t = \sup_{0 \le s \le t} |B_{\sigma^2 s} - S_s|.$$

- *If m is a positive integer and $p \in L_0^{4m}(\mathbb{Z})$, then there is a c such that for all $1 \le \beta \le \alpha$ and all positive integers n,*

$$\mathbb{P}\{\Delta_n \ge 3\,\alpha\, n^{1/4}\} \le c \left[\alpha^{-4m}\, n^{1-4m} + \beta^{-2m} + + n\, e^{-\alpha^2/(2\beta)} \right].$$

- *If $p \in L_0^\infty(\mathbb{Z})$, then there is a c such that for all $1 \le \beta \le \alpha$ and all positive integers n,*

$$\mathbb{P}\{\Delta_n \ge 3\,\alpha\, n^{1/4}\} \le c \left[e^{-\beta} + n\, e^{-\alpha^2/(2\beta)} \right].$$

In particular, if $\beta = a, \alpha = \sqrt{2}a$,

$$\mathbb{P}\{\Delta_n \ge 3\sqrt{2}\, a\, n^{1/4}\} \le c\,(n+1)\, e^{-a}.$$

Proof. One can check that it suffices to prove the result for all n sufficiently large.

Note that for positive integer n,

$$\Delta_n \le \max_{j=1,\ldots,n} |B_{\sigma^2 j} - S_j| + \max_{j=1,\ldots,n} |S_j - S_{j-1}| + \max_{j=1,\ldots,n} \sup_{0 \le s \le \sigma^2} |B_{\sigma^2 j} - B_{\sigma^2 j - s}|.$$

Since $S_j = B_{\tau_j}$, for every $b, x > 0$ with $x^2 \ge b$,

$$\mathbb{P}\{\max_{j=1,\ldots,n} |B_{\sigma^2 j} - S_j| \ge x\} \le$$

$$\mathbb{P}\{\max_{j=1,\ldots,n} |\tau_j - \sigma^2 j| \ge b\} + \mathbb{P}\{\max_{1 \le j \le n} \sup_{-b \le s \le b} |B_{\sigma^2 j + s} - B_{\sigma^2 j}| \ge x\}.$$

The reflection principle for one-dimensional Brownian motion gives

$$\mathbb{P}\{\sup_{0 \le t \le \sigma^2} |B_t| \ge x\} \le 2\,\mathbb{P}\{|B_{\sigma^2}| \ge x\} = 2\,\mathbb{P}\{|B_1| \ge x/\sigma\}.$$

Similarly,

$$\begin{aligned}
\mathbb{P}\{\sup_{-b \le s \le b} |B_{\sigma^2 j + s} - B_{\sigma^2 j}| \ge x\} &\le 2\,\mathbb{P}\{\sup_{0 \le s \le b} |B_s| \ge x\} \\
&\le 4\,\mathbb{P}\{|B_b| \ge x\} \\
&= 4\,\mathbb{P}\{|B_1| \ge x/\sqrt{b}\} \le c\, e^{-x^2/(2b)}.
\end{aligned}$$

The last estimate is a standard estimate for the normal distribution. Combining these estimate we get for all $b, r, r' > 0$,

$$\mathbb{P}\{\Delta_n \ge r(1 + \sigma\, b^{-1/2}) + r'\} \le$$

$$n\,\mathbb{P}\{|S_1| \ge r'\} + 2\,\mathbb{P}\{\max_{j=1,\ldots,n} |\tau_j - \sigma^2 j| \ge b\} + 6\, n\, e^{-r^2/(2b)}.$$

We will choose $r = r' = \alpha\, n^{1/4}$ and $b = \beta\, n^{1/2}$, with $\alpha \ge \beta^2 \ge 1$. Note that for this choce of r, r', b, if $n \ge \sigma^2$ the probability on the left is greater than $\mathbb{P}\{\Delta_n \ge 3\,\alpha\, n^{1/4}\}$.

Let $Z_j = \tau_j - \tau_{j-1} - \sigma^2$, and let

$$N_j = \tau_j - \sigma^2\, j = \sum_{i=1}^{j} Z_i.$$

Note that N_j is an i.i.d. sum of mean zero random variables and hence is a martingale. Let $m \ge 1$ be an integer. If $p \in L_0^{4m}$, then $\mathbb{E}[Z_1^{2m}] < \infty$ and hence, by standard estimates, $\mathbb{E}[N_n^{2m}] \le c\, n^m$. Applying the maximal inequality to the submartingle $Y_n = N_n^{2m}$, we get

$$\mathbb{P}\{\max_{j=1,\ldots,n} N_j \ge b\} \le b^{-2m}\mathbb{E}[N_n^{2m}] \le c\, n^m\, b^{-2m}.$$

In particular,

$$\mathbb{P}\{\max_{j=1,\ldots,n} |\tau_j - \sigma^2 j| \ge \beta\, n^{1/2}\} \le c\, \beta^{-2m}.$$

Since $p \in L_0^{4m}$, a simple estimate shows that

$$n\mathbb{P}\{|S_1| \ge \alpha n^{1/4}\} \le c\, n^{1-4m}\, \alpha^{-4m}.$$

If $p \in L_0^\infty$, then $\mathbb{P}\{|S_1| \ge \alpha\, n^{1/4}\} = 0$ for n sufficiently large. Also, $\mathbb{E}[e^{\beta Z_1}] < \infty$ for some $\beta > 0$. Since $\mathbb{E}[Z_1] = 0$, we know that

$$\mathbb{E}[e^{\beta Z_1}] = 1 + O(\beta^2), \qquad \beta \to 0.$$

In particular, there is a c such that for all n sufficiently large,

$$\mathbb{E}[e^{N_n/\sqrt{n}}] = (\mathbb{E}[e^{Z_1/\sqrt{n}}])^n = [1 + O(\frac{1}{n})]^n \le c.$$

For fixed n sufficiently large, consider the submartingale $Y_j = e^{Z_j/\sqrt{n}}$. Using a maximal inequality again,

$$\mathbb{P}\{\max_{j=1,\ldots,n} N_j \ge b\} \le e^{-b/\sqrt{n}}\mathbb{E}[e^{N_n/\sqrt{n}}] \le c\, e^{-b/\sqrt{n}}.$$

We can do the same argument with $-N_j$ replacing N_j. Hence,

$$\mathbb{P}\{\max_{j=1,\ldots,n} |N_j| \ge b\} \le e^{-b/\sqrt{n}}\mathbb{E}[e^{N_n/\sqrt{n}}] \le c\, e^{-b/\sqrt{n}}.$$

20.1 Higher dimensions

If $d > 1$, one cannot *embed* a d-dimensional random walk in a Brownian motion since with probability one a d-dimensional Brownian motion does not visit $\mathbb{Z}^d$ after time 0. However, in some cases one can use the Skorohod embedding to give a Brownian motion/random walk coupling. For example, suppose p is such that the random walk $S_j = (S_j^{(1)}, \ldots, S_j^{(d)})$ has independent components with variances $\sigma_1^2, \ldots, \sigma_d^2$. Then we can take independent Brownian motions $B^1, \ldots, B^d$ with variance parameters $\sigma_1^2, \ldots, \sigma_d^2$; define $S^{(1)}, \ldots, S^{(d)}$ using the Skorohod embedding so that $S_n^{(j)} = B^j_{\tau_n^j}$; and then set

$$S_n = (S_n^{(1)}, \ldots, S_n^{(d)}) = (B^1_{\tau_n^1}, \ldots, B^d_{\tau_n^d}).$$

Note that the stopping times $\tau_n^1, \ldots, \tau_n^d$ are different so the random walk S is not embedding in B.

Another case where we can use Skorohod embedding is for symmetric random walks. We will restrict our discussion here to symmetric walks with finite range where this technique is most useful. To be specific, suppose $\mathcal{A} = \{y_1, \ldots, y_k\}$ is a set of points in $\mathbb{Z}^d$ and q is a probability distribution on $\mathcal{A}$. Let $R_1, R_2, \ldots$ be i.i.d random variables distributed according to q and let $J_1, J_2, \ldots$ be independent random variables, $\mathbb{P}\{J_j = 1\} = \mathbb{P}\{J_j = -1\} = 1/2$. Then if we set $S_0 = 0$ and

$$S_n = J_1 R_1 + J_2 R_2 + \cdots + J_n R_n,$$

we have a symmetric random walk with finite range. Let $L_1, L_2, \ldots$ denote a multinomial process with parameters $q(y_1), \ldots, q(y_k)$, i.e., $L_n = H_1 + \cdots + H_n$ where $H_1, H_2, \ldots$ are i.i.d. $\mathbb{Z}^k$ random variables with $\mathbb{P}\{H_1 = e_j\} = q(y_j), j = 1, \ldots, k$. Let $B^1, \ldots, B^k$ be independent one-dimensional Brownian motions with variance parameters $|y_1|^2, \ldots, |y_k|^2$, respectively, and let $S^{(1)}, \ldots, S^{(k)}$ denote the random walks obtained by Skorohod embedding using the stopping times

$$\tau_l^j = \inf\{t < \tau_{l-1}^j : B_t^j - B^j_{\tau^j_{l-1}} = \pm |y_j|\}.$$

Let $u_j = y_j/|y_j|$ Then set

$$B_t = B_t^1 \, u_1 + B_t^2 \, u_2 + \cdots B_t^k \, u_k,$$
$$S_n = S^{(1)}_{L_n^1} \, u_1 + S^{(2)}_{L_n^2} \, u_2 + + S^{(k)}_{L_n^k} \, u_k.$$

Here we write $L_n = (L_n^1, \ldots, L_n^k)$.

21 Dyadic Coupling

Here we will describe a more complicated coupling method which leads to a much better approximation of a random walk by a Brownian motion. We will restrict ourselves to simple one-dimensional random walk, although the methods can be extended to a wider class of walks (the original paper considered one-dimensional walks with an exponential moment).

Let $S_n = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are i.i.d. random variables with $\mathbb{P}\{X_j = 1\} = \mathbb{P}\{X_j = -1\} = 1/2$. Let n be a positive integer and recall that

$$\mathbb{P}\{S_{2n} = 2k\} = 2^{-2n} \binom{2n}{n+k} = \frac{1}{\sqrt{\pi \, n}} \, e^{-k^2/n} \exp\left\{ O\left(\frac{1}{n}\right) + O\left(\frac{k^4}{n^3}\right) \right\}.$$

We will only use the weaker estimate

$$\mathbb{P}\{S_{2n} = 2k\} = 2^{-2n} \binom{2n}{n+k} = \frac{1}{\sqrt{\pi \, n}} \, e^{-k^2/n} \exp\left\{ O\left(\frac{1}{n}\right) + O\left(\frac{k^3}{n^2}\right) \right\}.$$

Definition. Suppose Y is a continuous random variable with strictly increasing distribution function F and p is an increment distribution on $\mathbb{Z}$ with distribution function

$$F_p(j) = \sum_{k \le j} p(k).$$

Let Z be the integer-valued random variable

$$Z = j \quad \text{if} \quad F^{-1}(F_p(j-1)) < Y \le F^{-1}(F_p(j)).$$

We will say that Z the random varible with distribution p that is *quantile-coupled* with Y.

Proposition 21.1 *There is a constant c such that if Y is a normal random variable with mean 0, variance $2n$, and Z is the random variable with the distribution of S_{2n} that is quantile-coupled with Y, then*

$$|Z - Y| \le c\,[1 + \frac{Z^2}{n}], \qquad \textit{for} \ \ |Z| < n.$$

Proof. One can check that it suffices to find an $\epsilon > 0$ so that the result holds for $|Z| < \epsilon n$. Let $F = F_n$ denote the distribution function for Y and $\tilde F = \tilde F_n$ the distribution function for S_{2n}. Note that for $y > 0$,

$$F(-2y) = \int_{2y}^{\infty} \frac{1}{\sqrt{(2\pi)(2n))}} \, e^{-x^2/(2(2n))} \, dx = \int_y^\infty \frac{1}{\sqrt{\pi \, n}} \, e^{-x^2/n} \, dx.$$

It suffices to find a c such that

$$F(2j - c[1 + \frac{j^2}{n}]) \le \tilde F(2j) \le F(2j + c[1 + \frac{j^2}{n}]).$$

For ease we will assume $j < 0$. By symmetry,

$$\sum_{j=1}^{\infty} \mathbb{P}\{S_{2n} = 2j\} = \frac{1}{2}\,[1 - \mathbb{P}\{S_{2n} = 0\}] = \frac{1}{2} - \frac{1}{2\sqrt{\pi \, n}} + O\left(\frac{1}{n^{3/2}}\right).$$

If $j \leq \sqrt{n}$,

$$\sum_{k=1}^{j} \mathbb{P}\{S_{2n} = -2k\} = \sum_{k=1}^{j} \frac{1}{\sqrt{\pi\, n}}\, e^{-k^2/n}\, [1 + O\left(\frac{1}{n}\right)] = F(0) - F(-2j) + O\left(\frac{1}{\sqrt{n}}\right)$$

Therefore,

$$\tilde F(-2j) = \sum_{k\leq j} \mathbb{P}\{S_{2n} = -2k\} = F(-2j) + O\left(\frac{1}{\sqrt{n}}\right).$$

Since $\mathbb{P}\{S_{2n} = -2k\} \geq c/\sqrt{n}$ for $k \leq 2\sqrt{n}$, we can see that there is a c such that

$$F(-2j - c) \leq \tilde F(-2j) \leq F(-2j + c).$$

For $j \geq \sqrt{n}$ we can ignore the $O(1/n)$ correction term and write just

$$\mathbb{P}\{S_{2n} = -2j\} = \frac{1}{\sqrt{\pi\, n}} \exp\left\{-\frac{j^2}{n} + O\left(\frac{j^4}{n^3}\right)\right\} = \frac{1}{\sqrt{\pi\, n}} \exp\left\{-\frac{1}{n}\,[j + O(\frac{j^2}{n})]^2\right\}.$$

In fact, we are only using the weaker estimate for $j \geq \sqrt{n}$:

$$\mathbb{P}\{S_{2n} = -2j\} = \frac{1}{\sqrt{\pi\, n}} \exp\left\{-\frac{j^2}{n} + O\left(\frac{j^3}{n^2}\right)\right\}.$$

To be more precise: there is a c_1 such that for all positive integers j with $\sqrt{n} \leq j \leq n$,

$$\exp\left\{-\frac{j^2}{n} - \frac{c_1 j^3}{n^2}\right\} \leq \sqrt{\pi\, n}\, \mathbb{P}\{S_{2n} = -2j\} \leq \exp\left\{-\frac{j^2}{n} + \frac{c_1 j^3}{n^2}\right\}.$$

Therefore, if $j \geq \sqrt{n}$,

$$\tilde F(-2j) \geq \int_j^n \frac{1}{\sqrt{\pi n}} \exp\left\{-\frac{x^2}{n} - \frac{c_1\, x^3}{n^2}\right\} dx = \int_{j/\sqrt{n}}^{\sqrt{n}} \frac{1}{\sqrt{\pi}} \exp\left\{-y^2 - \frac{c_1 y^3}{\sqrt{n}}\right\} dy.$$

Let $\epsilon = (2c_1)^{-1}$. Then, if $\sqrt{n} \leq j \leq \epsilon n$,

$$\begin{aligned}\tilde F(-2j-2) &\leq \tilde F(-2\epsilon n) + \int_j^{\epsilon n} \frac{1}{\sqrt{\pi n}} \exp\left\{-\frac{x^2}{n} + \frac{c_1\, x^3}{n^2}\right\} dx \\ &\leq O(e^{-rn}) + \int_{j/\sqrt{n}}^{\epsilon\sqrt{n}} \frac{1}{\sqrt{\pi}} \exp\left\{-y^2 + \frac{c_1\, y^3}{\sqrt{n}}\right\} dy.\end{aligned}$$

The last inequality uses a simple large deviation estimate (r depends on ϵ). We will use the fact that

$$\int_j^\infty \frac{1}{\sqrt{\pi\, n}}\, e^{-x^2/n} \asymp \frac{\sqrt{n}}{j}\, e^{-j^2/n}, \qquad j \geq \sqrt{n}.$$

In particular, there exist c_3, c_4 such that if $\sqrt{n} \leq x \leq n$,

$$F(-2x - c_3\frac{x^2}{n}) \exp\{\frac{c_4\, x^3}{n^2}\} \leq F(-2x) \leq F(-2x + c_3\frac{x^2}{n}) \exp\{-\frac{c_4\, x^3}{n^2}\}. \tag{110}$$

We will do two cases separately: $j \geq n^{2/3}$ and $n^{1/2} \leq j \leq n^{2/3}$. Note that for $j \geq n^{2/3}$, then (at least for n sufficiently large)

$$\int_j^{j+1} e^{-x^2/n}\, dx \geq \int_{2j}^\infty e^{-x^2/n}\, dx.$$

This can be seen easily by noting that the first integral is greater than $e^{-(j+1)^2/n}$ while the second is bounded above by

$$\int_{2j}^n e^{-2xj/n}\, dx = \frac{n}{2j}\, e^{-2j^2/n}.$$

For the lower bound we have (for n sufficiently large)

$$\begin{aligned}\int_j^n \frac{1}{\sqrt{\pi n}} \exp\left\{-\frac{x^2}{n} - \frac{c_1\, x^3}{n^2}\right\} dx &\geq e^{-8c_1 j^3/n^2} \int_j^{2j} \frac{1}{\sqrt{\pi n}}\, e^{x^2/n}\, dx \\ &\geq e^{-8c_1 j^3/n^2} \int_{j+1}^\infty \frac{1}{\sqrt{\pi n}}\, e^{-x^2/n}\, dx \\ &\geq \int_{j+c(j^2/n)}^\infty \frac{1}{\sqrt{\pi n}}\, e^{-x^2/n}\, dx,\end{aligned}$$

for some c. The last inequality uses (110). Similarly, for the upper bound,

$$\begin{aligned}\int_j^{\epsilon n} \frac{1}{\sqrt{\pi n}} \exp\left\{-\frac{x^2}{n} + \frac{c_1\, x^3}{n^2}\right\} dx &\leq \int_{j-1}^{2j} \frac{1}{\sqrt{\pi n}} \exp\left\{-\frac{x^2}{n} + \frac{c_1\, x^3}{n^2}\right\} dx \\ &\leq e^{8c_1 j^3/n^2} \int_{j-1}^\infty \frac{1}{\sqrt{\pi n}}\, e^{-x^2/n}\, dx \\ &\leq \int_{j-c(j^2/n)}^\infty \frac{1}{\sqrt{\pi n}}\, e^{-x^2/n}\, dx.\end{aligned}$$

For $\sqrt{n} \leq j \leq n^{2/3}$ we can use a Taylor series approximation for the exponential. For the upper bound, we have

$$\begin{aligned}\int_j^n \frac{1}{\sqrt{\pi n}} \exp\left\{-\frac{x^2}{n} - \frac{c_1\, x^3}{n^2}\right\} dx &\geq \int_j^{2n^{2/3}} \frac{1}{\sqrt{\pi n}} \exp\left\{-\frac{x^2}{n} - \frac{c_1\, x^3}{n^2}\right\} dx \\ &\geq \int_j^{2n^{2/3}} \frac{1}{\sqrt{\pi n}} - [\text{error}]\end{aligned}$$

where the error is bounded above in absolute value by a constant times

$$\frac{1}{n^2}\int_j^\infty \frac{x^3}{\sqrt{n}}\, e^{-x^2/n}\, dx \leq \frac{c\, j^3}{n^2}\int_j^\infty e^{-x^2/2}\, dx.$$

Hence,

$$\begin{aligned}\int_j^n \frac{1}{\sqrt{\pi n}}\exp\left\{-\frac{x^2}{n}-\frac{c_1\, x^3}{n^2}\right\} dx &\geq \exp\{-c\frac{j^3}{n^2}\}\int_j^\infty \frac{1}{\sqrt{\pi n}}\, e^{-x^2/n}\, dx \\ &\geq \int_{j+c(j^2/n)}^\infty \frac{1}{\sqrt{\pi n}}\, e^{-x^2/n}\, dx.\end{aligned}$$

The lower bound is done similarly.

We will now describe the construction in the quadratic coupling. For ease suppose that $n = 2^m$ and B_t is a standard Brownian motion.

- Define W_n to be the the random variable with distribution S_n that is optimally coupled with B_n. Note that
$$|W_n - B_n| \leq c\,[1+\frac{W_n^2}{n}].$$
In particular, for a typical value, this difference is $O(1)$.

- We will now construct $W_{n/2}$. Given B_n, $B_{n/2}$ is a normal random variable with mean $B_n/2$ and variance $n/4$. In other words, if $N = (B_n/2) - B_{n/2}$, $\tilde{N} = (B_n/2) + B_{n/2}$, then N and $\tilde{N}$ are independent normal random variables, mean zero, variance $n/2$. For each integer $j \in [-n/2, n/2]$ let $q_{n,j}$ denote the conditional distribution of $S_{n/2}$ given $S_n = 2j$. On the event $\{S_n = 2j\}$ define $W_{n/2}$ to he the random variable with distribution $q_{n,j}$ optimally coupled with $-N + (W_n/2)$. Note that
$$|B_{n/2} - W_{n/2}| \leq |W_{n/2} - (-N + W_n/2)| + \frac{1}{2}|W_n - B_n|.$$
Also the construction shows that $(W_{n/2}, W_n)$ has the same distribution as $(S_{n/2}, S_n)$.

Theorem 21.2 *For the dyadic coupling, there exist c, a, β such that for all n,*

$$\mathbb{E}[e^{a\Delta_n}] \leq c\, n^\beta.$$

Proof. TO BE DONE LATER

Corollary 21.3 *For the dyadic coupling, there is a c, α, β such that*

$$\mathbb{P}\{\Delta_n \geq c_1 \log n\} \leq c\, n^{\beta - c_1\alpha}.$$

Proof.

$$\mathbb{P}\{\Delta_n \geq c_1 \log n\} = \mathbb{P}\{e^{\alpha\Delta_n} \geq n^{c_1\alpha}\} \leq n^{-c_1\alpha}\mathbb{E}[e^{\alpha\Delta_n}] \leq cn^{\beta - c_1\alpha}.$$