1 An estimate

Lemma 1. *Suppose $X_1, X_2, \ldots$ are independent, identically distributed random variables with $\mathbf{E}(X_1) = 0$. Suppose that for some $a > 0$, the moment generating function $M(t) = \mathbf{E}[e^{tX_1}] < \infty$ for $|t| \le a$. Let $S_n = X_1 + \cdots + X_n$. Then there is a $c < \infty$ such that for every $b > 0$ and every integer $n \ge b^2/a^2$ and every $t > 0$,*

$$\mathbf{P}\{\max_{1\le j\le n} |S_j| \ge t\sqrt{n}\} \le 2\, e^{cb^2}\, e^{-bt}.$$

In particular, there is a $c_1 < \infty$ such that for all n,

$$\mathbf{P}\{\max_{1\le j\le n} |S_j| \ge t\sqrt{n}\} \le c_1\, e^{-at}.$$

Proof. We will show,

$$\mathbf{P}\{\max_{1\le j\le n} S_j \ge t\sqrt{n}\} \le e^{cb^2}\, e^{-bt}.$$

An identical argument using $-S_j$ gives the other inequality. Since $\mathbf{E}[X_1] = 0$, we can expand $M(t)$ about $t = 0$,

$$M(t) = 1 + \frac{\mathbf{E}[X_1^2]}{2}\, t^2 + O(t^3).$$

In particular, there is a constant c such that for all $|t| \le a$, $M(t) \le 1 + ct^2$,and hence, if $n \ge b^2/a^2$,

$$\mathbf{E}[e^{bS_n/\sqrt{n}}] = M\left(\frac{b}{\sqrt{n}}\right)^n \le \left(1 + \frac{cb^2}{n}\right)^n \le e^{cb^2}.$$

Let $Y_j = Y_{j.n,b} = e^{bS_j/\sqrt{n}}$. Note that Y_j is a positive submartingale and hence by a standard submartingale maximal inequality,

$$\mathbf{P}\{\max_{1\le j\le n} S_j \ge t\sqrt{n}\} = \mathbf{P}\{\max_{1\le j\le n} Y_j \ge e^{bt}\} \le e^{-bt}\, \mathbf{E}[Y_n] \le e^{-bt}\, e^{cb^2}.$$

Lemma 2. *Suppose $X_1, X_2, \ldots$ are independent, identically distributed random variables with $\mathbf{E}[X_i] = 0$. Then for every $\epsilon > 0$*

$$\lim_{\epsilon\to 0} \mathbf{P}\{\max_{1\le j\le n} |S_j| \ge \epsilon\, n\} = 0.$$

Proof. This is an immediate consequence of the strong law of large numbers.

2 Modulus of continuity of Brownian motion

If B_t is a standard Brownian motion and $\delta > 0$, let

$$Z_\delta = \sup\{|B_t - B_s| : 0 \le s, t \le 1, |s - t| \le \delta\}.$$

The function $\delta \mapsto Z_\delta$ is often called the *modulus of continuity* or *oscillation* of the function B_t on $[0, 1]$. Since B_t is random, Z_δ is random. Let

$$M_n = \sup\{|B_t - B_{k2^{-n}}| : k = 0, 1, \ldots, 2^n - 1,\ k2^{-n} \le t \le (k+1)2^{-n}\}.$$

Note that if $2^{-n-1} \le \delta \le 2^{-n}$, $M_{n+1} \le Z_\delta \le 3M_n$. (The first of these inequalities is trivial. For the second, note that if $\delta \le 2^{-n}$ and $s < t \le s + \delta$ we can find an integer k such that

$$\frac{k-1}{2^n} \le s < t < \frac{k+1}{2^n}.$$

If $t \le k2^{-n}$, then

$$|B_s - B_t| \le |B_s - B_{(k-1)/2^n}| + |B_t - B_{(k-1)/2^n}| \le 2\ M_n$$

and if $k2^{-n} < t \le (k+1)2^{-n}$,

$$|B_s - B_t| \le |B_s - B_{(k-1)/2^n}| + |B_{k/2^n} - B_{(k-1)/2^n}| + |B_t - B_{k/2^n}|\ \le 3\ M_n.\ )$$

We will prove the following

Theorem 1. *With probability one,*

$$\frac{M_n}{\sqrt{2^{-n+1}\ n\ \log 2}} \longrightarrow 1.$$

The denominator in the theorem should be thought of as $\sqrt{2}\ \sqrt{2^{-n}\log 2^n}$. This gives an immediate corollary.

Corollary 3. *With probability one,*

$$1 \le \liminf_{\delta \to 0+} \frac{Z_\delta}{\sqrt{\delta\ \log(1/\delta)}} \le \limsup_{\delta \to 0+} \frac{Z_\delta}{\sqrt{\delta \log(1/\delta)}} \le 6.$$

The proof of Theorem 1 uses some estimates. In fact, the estimates themselves are more important to us than the theorem so we will state the estimates separately.

Lemma 4. *If n is a positive integer and $t > 0$,*

$$\mathbf{P}\{M_n \geq t\, 2^{-n/2}\} \leq 16 \cdot 2^n\ e^{-t^2/2}.$$

More generally, if

$$M_{n,m} = \sup\{|B_t - B_{k2^{-n}}| : k = 0, 1, \ldots, m \cdot 2^n - 1,\ k2^{-n} \leq t \leq (k+1)2^{-n}\},$$

then

$$\mathbf{P}\{M_{n,m} \geq t\, 2^{-n/2}\} \leq 16m \cdot 2^n\ e^{-t^2/2}.$$

Proof. Using the reflection principle and our estimate on the normal distribution function we get,

$$\begin{aligned}
\mathbf{P}\{M_{n,m} \geq t\, 2^{-n/2}\} &\leq m \cdot 2^n\, \mathbf{P}\{\sup_{0\leq t\leq 2^{-n}} |B_t| \geq t2^{-n/2}\} \\
&\leq 2\, m \cdot 2^n\, \mathbf{P}\{\sup_{0\leq t\leq 2^{-n}} B_t \geq t2^{-n/2}\} \\
&\leq 4\, m \cdot 2^n\, \mathbf{P}\{B_{2^{-n}} \geq t2^{-n/2}\} \\
&= 4\, m \cdot 2^n\, \mathbf{P}\{B_1 \geq t\} \\
&\leq 16\, m \cdot 2^n\, e^{-t^2/2}.
\end{aligned}$$

Corollary 5. *If $\delta > 0$ and $t > 0$,*

$$\mathbf{P}\{Z_\delta \geq t\sqrt{\delta}\} \leq 16\ \delta^{-1}\ e^{-t^2/36}.$$

More generally, if for positive integer m,

$$Z_{\delta,m} = \sup\{|B_t - B_s| : 0 \leq s, t \leq m, |s-t| \leq \delta\},$$

then

$$\mathbf{P}\{Z_{\delta,m} \geq t\sqrt{\delta}\} \leq 16\, m\, \delta^{-1}\ e^{-t^2/36}.$$

Proof. Choose n so that $2^{-n-1} \leq \delta \leq 2^{-n}$. Then

$$\begin{aligned}
\mathbf{P}\{Z_{\delta,m} \geq t\sqrt{\delta}\} &\leq \mathbf{P}\{Z_{2^{-n},m} \geq t\sqrt{\delta}\} \\
&\leq \mathbf{P}\{Z_{2^{-n},m} \geq (t/\sqrt{2})2^{-n}\} \\
&\leq \mathbf{P}\{M_{n,m} \geq (t/3\sqrt{2})2^{-n}\} \\
&\leq 16\, m \cdot 2^n e^{-t^2/36} \\
&\leq 16\, m\, \delta^{-1}\ e^{-t^2/36}.
\end{aligned}$$

Proof of Theorem 1. Let V_n be the event

$$V_n = \{M_n \geq 2^{-n/2}\sqrt{2n\log 2 + 4\log n}\}.$$

By the lemma, $\mathbf{P}(V_n) \leq 16n^{-2}$ and hence $\sum \mathbf{P}(V_n) < \infty$. By the Borel-Cantelli lemma,

$$\mathbf{P}\{M_n \geq 2^{-n/2}\sqrt{2n\log 2 + 4\log n} \text{ i.o.}\} = 0,$$

and hence with probability one

$$\limsup_{n\to\infty} \frac{M_n}{2^{-n/2}\sqrt{2n\log 2}} \leq 1.$$

To get the estimate in the opposite direction we first state an easy lower bound on the normal distribution function,

$$\mathbf{P}\{|B_1| \geq t\} = 2\int_t^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx \geq \frac{1}{\sqrt{\pi}}\int_t^{t+1} e^{-x^2/2}\,dx \geq \frac{1}{\sqrt{\pi}}\, e^{-(t+1)^2/2}.$$

Hence

$$\begin{aligned}
\mathbf{P}\{M_n \leq t2^{-n/2}\} &\leq \mathbf{P}\{|B_{k2^{-n}} - B_{(k-1)2^{-n}}| \leq t2^{-n/2},\ \ k = 1,\dots 2^n\} \\
&\leq [\mathbf{P}\{|B_{2^{-n}}| \leq t2^{-n/2}\}]^{2^n} \\
&\leq [\mathbf{P}\{B_1 \leq t\}]^{2^n} \\
&\leq [1 - \frac{1}{\sqrt{\pi}} e^{-(t+1)^2/2}]^{2^n}.
\end{aligned}$$

If we choose $t = \sqrt{2n\log 2 - \log n} - 1$, we get

$$\mathbf{P}\{M_n \leq 2^{-n/2}[\sqrt{2n\log 2 - \log n} - 1]\} \leq [1 - \frac{1}{\sqrt{\pi}} 2^{-n} n^{1/2}]^{2^n} \sim e^{-n^{1/2}/\sqrt{\pi}}.$$

In particular,

$$\sum_{n=1}^{\infty} \mathbf{P}\{M_n \leq 2^{-n/2}[\sqrt{2n\log 2 - \log n} - 1]\} < \infty,$$

and by the Borel-Cantelli Lemma for all n sufficiently large,

$$M_n \geq 2^{-n/2}[\sqrt{2n\log 2 - \log n} - 1].$$

Therefore, with probability one,

$$\liminf_{n\to\infty} \frac{M_n}{2^{-n/2}\sqrt{2n\log 2}} \geq 1.$$

3 Strong approximation of Brownian motion

Let B_t be a standard Brownian motion. Let $\tau_0 = 0$ and for $j > 0$, let

$$\tau_j = \inf\{t > \tau_{j-1} : |B_t - B_{\tau_{j-1}}| = 1\}.$$

Let $\Delta_j = \tau_j - \tau_{j-1}$. Then the strong Markov property tells us that $\Delta_1, \Delta_2, \ldots$ are independent, identically distributed random variables. Let $\Delta = \Delta_1$.

Lemma 6. $\mathbf{E}(\Delta) = 1$. *Moreover, there is an* $a > 0$ *such that* $\mathbf{E}[e^{a\Delta}] < \infty$.

Proof. For the second assertion note that

$$\mathbf{P}\{\Delta \le k+1 \mid \Delta \le k\} \le \mathbf{P}\{|B_{k+1} - B_k| \le 2\} = \rho < 1,$$

where $\rho = \mathbf{P}\{|B_1| \le 2\}$. Therefore, $\mathbf{P}\{\Delta \le k\} \le \rho^k$ and $\mathbf{E}[e^{a\Delta}] < \infty$ provided that $e^a < 1/\rho$. For the first assertion, we will use a continuous analogue of the optional sampling theorem. Note that $Y_t = B_t^2 - t$ is a martingale. The exponential bound shows that it is uniformly integrable and hence,

$$0 = \mathbf{E}[Y_0] = \mathbf{E}[Y_{\tau_1}] = \mathbf{E}[B_{\tau_1}^2] - \mathbf{E}[\tau_1] = 1 - \mathbf{E}[\tau_1].$$

Let $X_j = \Delta_j - 1$ so that $\mathbf{E}[X_j] = 0$ and

$$X_1 + \cdots + X_n = \tau_n - n.$$

Corollary 7. *There exist constant* $0 < a, C < \infty$ *such that for all* n *and all* $t > 0$,

$$\mathbf{P}\{\max_{1 \le k \le n} |\tau_k - k| \ge t\sqrt{n}\} \le c\, e^{-at}.$$

Let $S_0 = 0$ and $S_n = B_{\tau_n}$. Then S_n is a simple random walk defined on the same probability space as B_t.

Proposition 8. *There exist constants* b, C *such that for any* $\alpha \le 1$,

$$\mathbf{P}\{\max_{1 \le k \le n} |S_k - B_k| \ge \alpha\sqrt{n}\} \le C\, n^{1/2}\, \exp\{-b\alpha n^{1/4}\}.$$

Proof. The result is trivial if $\alpha \le n^{-1/4}$ so we will assume that $\alpha \ge n^{-1/4}$. For any β the event

$$\{\max_{1 \le k \le n} |S_k - B_k| \ge \alpha\sqrt{n}\}$$

is contained in the event

$$\{\max_{1\le k\le n}|\tau_k-k|\ge\beta\sqrt{n}\}\cup\{\sup\{|B_t-B_s|:s\le n,|t-s|\le\beta\sqrt{n}\}\ge\alpha\sqrt{n}\}.$$

By Corollary 7,

$$\mathbf{P}\{\max_{1\le k\le n}|\tau_k-k|\ge\beta\sqrt{n}\}\le c\,\exp\{-a\beta\}.$$

By Brownian scaling and Corollary 3, assuming $\beta\le\sqrt{n}$,

$$\begin{aligned}\mathbf{P}\{\sup\{|B_t-B_s|:s\le n,|t-s|\le\beta\sqrt{n}\}\ge\alpha\sqrt{n}\}&\\ =\ \mathbf{P}\{\sup\{|B_t-B_s|:0\le s\le 1,|t-s|\le\beta n^{-1/2}\}\ge\alpha\}&\\ \le\ \mathbf{P}\{Z_{\beta n^{-1/2},2}\ge(\alpha\beta^{-1/2}n^{1/4})\sqrt{\beta n^{-1/2}}\}&\\ \le\ 32\,n^{1/2}\beta^{-1}\,\exp\{-\frac{\alpha^2\beta^{-1}n^{1/2}}{36}\}&\end{aligned}$$

Let $\beta=\alpha n^{1/4}/\sqrt{36a}$ and let $b=\sqrt{a/36}$. If $\alpha\ge n^{-1/3}$, then $\beta^{-1/2}\le c'\,n^{1/2}$ and hence the proposition follows from these estimates.

In particular,

$$\sum_{n=1}^{\infty}\{\max_{1\le k\le n}|S_k-B_k|\ge n^{7/16}\}<\infty. \tag{1}$$

Now consider a similar construction except that we let

$$\tau_{j,n}=\inf\{t>\tau_{j-1,n}:|B_t-B_{\tau_{j-1,n}}|=1/\sqrt{n}\}.$$

Let $W_t^{(n)}$ be defined by $W_{k/n}^{(n)}=B_{\tau_{k,n}}$ and by linear interpolation otherwise. Then by using (1) we can see that for,

$$\sum_{n=1}^{\infty}\mathbf{P}\{\|W^{(n)}-B\|\ge 2n^{1/16}\}=0,$$

where $\|\cdot\|$ denotes the supremum norm in $C[0,1]$. Therefore (using the Borel-Canelli Lemma) with probability one $W^{(n)}\to B$ where $\to$ denotes convergence in $C[0.1]$ with the supremum norm.