Stochastic Loewner Evolution

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The *stochastic Loewner evolution* or *Schramm-Loewner evolution* (SLE) is a family of random curves that appear as scaling limits of curves or cluster boundaries of discrete statistical mechanical models in two dimensions at criticality. The stochastic Loewner evolution was introduced by Oded Schramm as a candidate for the limit of loop-erased random walk and the boundary of percolation clusters, and it is now believed that SLE curves appear in most planar critical systems whose scaling limit satisfies conformal invariance. The curves are defined by solving a *Loewner differential equation* with a random input.

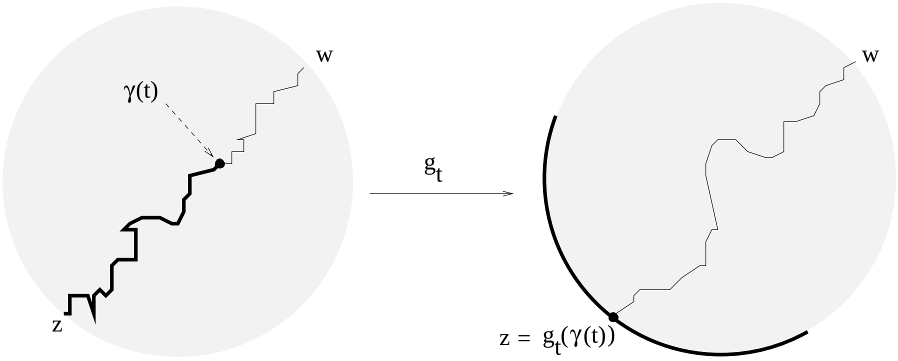
1 Definition

There are three major one-parameter families of SLE curves, *chordal, radial, and whole-plane* which correspond to curves connecting: two boundary points in a domain, a boundary point and an interior point in a domain, and two points in $\mathbb{C}$, respectively. The parameter is usually denoted $\kappa > 0$. The starting point for defining SLE is to write down the assumptions that one expects from a scaling limit, assuming that the limit is conformally invariant.

In the chordal case, we assume that there is a family of probability measures $\{\mu_D(z,w)\}$, indexed by simply connected proper domains $D \subset \mathbb{C}$ and distinct boundary points $z, w \in \partial D$, supported on continuous curves $\gamma : [0, t_\gamma] \to \overline{D}$ with $\gamma(0) = z, \gamma(t_\gamma) = w$, that satisfies:

- Conformal invariance: If $f : D \to D'$ is a conformal transformation, then the image of $\mu_D(z,w)$ under f is the same as $\mu_{D'}(f(z), f(w))$, up to a time change.
- Conformal Markov property for $\mu_D(z,w)$: Suppose $\gamma[0,t]$ is known, and let g_t be a conformal transformation of the slit domain $D\setminus\gamma[0,t]$ onto D with $g_t(\gamma(t)) = z, g_t(w) = w$. Then the conditional distribution on $g_t \circ \gamma[t, t_\gamma]$, given $\gamma[0,t]$, is the same, up to a change of parametrization, as the original distribution. (Implicit in this is the assumption that $\gamma(t)$ is on the boundary of $D \setminus \gamma(0,t]$, which will be true, e.g., if γ is non-self-intersecting and $\gamma(0, t_\gamma) \subset D$.)

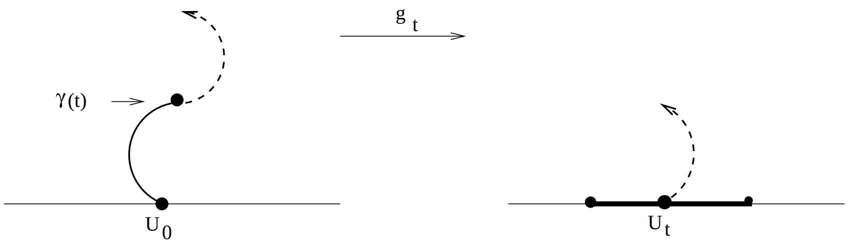
Using the Riemann mapping theorem, one can see that such a family $\{\mu_D(z,w)\}$ is determined (up to reparametrization) by $\mu_{\mathbb{H}}(0,\infty)$, where $\mathbb{H} = \{x + iy : y > 0\}$ denotes the upper half plane. Suppose $\gamma : [0,\infty) \to \mathbb{C}$ is a simple (i.e., no self-intersections) curve with



$\gamma(0) = 0$, $\gamma(0,\infty) \subset \mathbb{H}$, and $\sup_t \mathrm{Im}[\gamma(t)] = \infty$. Let $H_t = \mathbb{H} \setminus \gamma[0,t]$. There is a unique conformal transformation $g_t : H_t \to \mathbb{H}$ whose expansion at infinity is

$$g_t(z) = z + \frac{b(t)}{z} + O(|z|^{-2}), \quad z \to \infty.$$

The coefficient $b(t)$, which is sometimes called the *half-plane capacity* of $\gamma[0,t]$ and denoted



$\mathrm{hcap}[\gamma[0,t]]$, is continuous, strictly increasing, and tending to ∞. In fact,

$$b(t) = \lim_{y\to\infty} y\, \mathbf{E}[\mathrm{Im}[X_\tau] \mid X_0 = iy],$$

where X_s denotes a complex Brownian motion and $\tau = \tau_{\gamma[0,t]}$ is the first time s such that $X_s \in \mathbb{R} \cup \gamma[0,t]$. By reparametrizing γ, we can make $b(t) = 2t$. With this parametrization, the maps g_t satisfy the Loewner differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}, \qquad g_0(z) = z,$$

where $U : [0,\infty) \to \mathbb{R}$ is a continuous function with $U_0 = 0$. In fact, $U_t = g_t(\gamma(t))$. Schramm observed that the measure $\mu_{\mathbb{H}}(0,\infty)$, at least if it were supported on simple curves and the curves were parametrized using half-plane capacity, would produce a random U_t. If the assumptions above on $\{\mu_D(z,w)\}$ are translated into assumptions on the "driving function" U_t, one shows readily that U_t must be a driftless Brownian motion, i.e., $U_t = \sqrt{\kappa}\, B_t$, for a standard one-dimensional Brownian motion B_t.

Chordal SLE_κ (in $\mathbb{H}$ connecting 0 and ∞) is defined to be the random collection of conformal maps g_t obtained by solving the initial value problem

$$\dot{g}_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \qquad g_0(z) = z, \tag{1}$$

where B_t is a standard one-dimensional Brownian motion. This equation is often given in terms of the inverse $f_t = g_t^{-1}$,

$$\dot{f}_t(z) = -f_t'(z) \, \frac{2}{z - \sqrt{\kappa} B_t}.$$

This equation describes a random evolution of conformal maps f_t from $\mathbb{H}$ into subdomains of $\mathbb{H}$. For each $z \in \overline{\mathbb{H}}$, the solution of (1) is defined up to a time $T_z \in [0, \infty]$ with $T_z > 0$ for $z \neq 0$. For fixed t, g_t is the unique conformal transformation of $H_t := \{z \in \mathbb{H} : T_z > t\}$ onto $\mathbb{H}$ with expansion

$$g_t(z) = z + \frac{2t}{z} + \cdots, \qquad z \to \infty.$$

The chordal SLE_κ path is the random curve $\gamma : [0, \infty) \to \mathbb{H}$ such that for each t, H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. It is not immediate from the definition that such a curve γ exists, but its existence has been proved. If $G_t = g_{t/\kappa}$, then we can rewrite (1) as

$$\dot{G}_t(z) = \frac{a}{G_t(z) + W_t}, \tag{2}$$

where $a = 2/\kappa$ and $W_t := -\sqrt{\kappa} B_{t/\kappa}$ is a standard Brownian motion. Then $Z_t^z := G_t(z) + W_t$ satisfies the Bessel stochastic differential equation

$$dZ_t^z = \frac{a}{Z_t^z} \, dt + dW_t, \qquad Z_0^z = z. \tag{3}$$

This equation is valid up to time κT_z, which is the first time that $Z_t^z = 0$.

Although chordal SLE_κ is defined with a particular parametrization, one generally thinks of it as a measure on curves modulo reparametrization. The scaling properties of Brownian motion imply that this measure is invariant under dilations of $\mathbb{H}$. If D is a simply connected domain and z, w are distinct boundary points of D, chordal SLE_κ in D connecting z and w is defined to be the conformal image of SLE_κ in $\mathbb{H}$ from 0 to ∞ under a conformal transformation of $\mathbb{H}$ onto D taking 0 to z and ∞ to w. There is a one-parameter family of such transformations, but the scale invariance of SLE_κ in $\mathbb{H}$ shows that the image measure is independent of the choice of transformation.

The geometric and fractal properties of the curve γ vary greatly as the parameter κ changes.

- If $\kappa \leq 4$, γ is a simple curve.

- If $4 < \kappa < 8$, γ has self-intersections, but is not space-filling.

- If $\kappa \geq 8$, γ is a space-filling curve.

To see this, one notes that the conformal Markov property implies that there can be double points with positive probability if and only if $T_x < \infty$ occurs with positive probability for $x > 0$. Also, the curve is space-filling if and only if $T_z < \infty$ for all z and $T_w \neq T_z$ for $w \neq z$. The problem is then reduced to a problem about the Bessel equation (3) for which the following holds:

- If $a \geq 1/2$ and $z \neq 0$, the probability that $T_z < \infty$ is zero. If $a < 1/2$, this probability equals one.

- If $1/4 < a < 1/2$, and w, z are distinct points in $\mathbb{H}$, then there is a positive probability that $T_w = T_z$.

- If $0 < a \leq 1/4$, then with probability one $T_w \neq T_z$ for all w, z.

This kind of argument is typical when studying SLE—geometric properties of the curve are established by analyzing a stochastic differential equation. The Hausdorff dimension of the path γ is given by

$$\dim[\gamma[0,\infty)] = \min\left\{1 + \frac{\kappa}{8}, 2\right\}.$$

The radial Loewner equation describes the evolution of a curve from the boundary of the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to the origin. Suppose $\gamma : [0,\infty) \to \overline{\mathbb{D}}$ is a simple curve with $\gamma(0) = 1, \gamma(0,\infty) \subset \mathbb{D} \setminus \{0\}$ and $\gamma(t) \to 0$ as $t \to \infty$. Let g_t be the unique conformal transformation of $\mathbb{D} \setminus \gamma[0,t]$ onto $\mathbb{D}$ such that $g_t(0) = 0, g_t'(0) > 0$. One can check that $g_t'(0)$ is continuous and strictly increasing in t, and hence we can parametrize γ so that $g_t'(0) = e^t$. Using this reparametrization, there is a continuous $U_t : [0,\infty) \to \mathbb{R}$ with $U_0 = 0$ such that g_t satisfies the *radial Loewner equation*

$$\dot{g}_t(z) = g_t(z)\,\frac{e^{iU_t} + g_t(z)}{e^{iU_t} - g_t(z)}, \quad g_0(z) = z.$$

If $z \neq 0$, we can define $h_t(z) = -i \log g_t(z)$ locally near z, and this equation becomes

$$\dot{h}_t(z) = \cot\left(\frac{h_t(z) - U_t}{2}\right).$$

Radial SLE_κ (connecting 1 and 0 in $\mathbb{D}$) is obtained by setting $U_t = \sqrt{\kappa}\,B_t$. If D is a simply connected domain, $z \in D, w \in \partial D$, then radial SLE_κ in D connecting w and z is obtained by conformal transformation using the unique transformation f of $\mathbb{D}$ onto D with $f(0) = z, f(1) = w$. Again, we think of this as being defined modulo time change. If $a = 2/\kappa$ and $v_t = h_{at/2}$, then

$$\dot{v}_t(z) = \frac{a}{2}\,\cot\left(\frac{v_t(z) + W_t}{2}\right). \tag{4}$$

where $W_t := -\sqrt{\kappa}\, B_{t/\kappa}$ is a standard Brownian motion. If $L_t^z = v_t(z) + W_t$, we get

$$dL_t^z = \frac{a}{2}\cot\left(\frac{L_t^z}{2}\right) dt + dW_t.$$

Radial and chordal SLE are closely related. In fact, if γ is a chordal SLE path in $\mathbb{H}$ from 0 to infinity; $\tilde\gamma$ is a radial SLE path in $\mathbb{D}$ from 1 to 0; and $\eta = -i\log\tilde\gamma$, then for small t the distribution of η is absolutely continuous to the distribution of a (random time change of) γ. Showing this involves understanding the behavior of the Loewner equation under conformal transformations. Suppose $\gamma, \tilde\gamma$ have been parametrized as in (2) and (4) with $a = 2/\kappa$. Let g_t^* be the conformal transformation of $\mathbb{H}\setminus\eta[0,t]$ onto $\mathbb{H}$ such that

$$g_t^*(z) = z + \frac{a^*(t)}{z} + \cdots, \quad z\to\infty,$$

and let U_t^* be the Loewner driving function such that

$$\dot g_t^*(z) = \frac{\dot a^*(t)}{g_t^*(z) - U_t^*}.$$

Here $a^*(t) = \mathrm{hcap}[\eta[0,t]]$. If we consider a time change σ such that $a^*(\sigma(t)) = at$ and let $U_t = \tilde U_{\sigma(t)}$ be the time-changed driving function, Itô's formula can be used to show that

$$dU_t = \frac{1}{2}(1-3a)\, F_t\, dt + d\tilde W_t, \tag{5}$$

where the F_t in the drift term depends on $\gamma[0,t]$ and is independent of a, and $\tilde W$ is a standard Brownian motion. Girsanov's Theorem implies that Brownian motions with the same variance but different drifts have absolutely continuous distributions. In particular, qualitative properties such as existence of double points or Hausdorff dimension of paths are the same for radial and chordal SLE. U_t is a driftless Brownian motion if $a = 1/3, \kappa = 6$.

Whole-plane SLE_κ from 0 to ∞ is a path $\gamma : (-\infty,\infty)\to\mathbb{C}$ with $\gamma(-\infty) = 0, \gamma(\infty) = \infty$, such that given $\gamma(-\infty, t]$, the distribution of $\gamma(t,\infty)$ is that of radial SLE_κ from boundary point $\gamma(t)$ to interior point ∞ in the domain $\mathbb{C}\setminus\gamma[-\infty,t]$. One can define whole-plane SLE_κ connecting two distinct points in $\mathbb{C}$ by conformal transformation.

2 Locality and restriction

There are two special values of κ: $\kappa = 6, a = 1/3$ that satisfies the *locality* property and $\kappa = 8/3, a = 3/4$ that satisfies the *restriction* property. Suppose γ is a chordal SLE_κ curve from 0 to ∞ in $\mathbb{H}$ parametrized as in (2). Suppose $\Phi : \mathcal{N}\to H$ is a conformal map taking a neighborhood $\mathcal{N}$ of 0 in $\mathbb{H}$ to $\Phi(\mathcal{N})$ and that locally maps $\mathbb{R}$ into $\mathbb{R}$. Let $\tilde\gamma(t) = \Phi\circ\gamma(t)$, which is defined for sufficiently small t. Let g_t^* be the conformal transformation of $\mathbb{H}\setminus\tilde\gamma[0,t]$ onto $\mathbb{H}$ with

$$g_t^*(z) = z + \frac{a^*(t)}{z} + \cdots,$$

and let $\tilde{U}_t$ be the driving function such that

$$\dot{g}_t^*(z) = \frac{\dot{a}^*(t)}{g_t^*(z) - \tilde{U}_t}.$$

Here $a^*(t) = \text{hcap}[\tilde{\gamma}[0,t]]$. If we change time, $\gamma_t^* = \tilde{\gamma}_{\sigma(t)}$, so that $a^*(\sigma(t)) = at$, then an application of Itô's formula shows that $U_t^* := \tilde{U}_{\sigma(t)}$ satisfies

$$dU_t^* = \frac{1}{2}(3a-1)\frac{\Phi''_{\sigma(t)}(W_{\sigma(t)})}{\Phi'_{\sigma(t)}(W_{\sigma(t)})}\,dt + d\tilde{W}_t.$$

Here $\tilde{W}_t$ is a standard Brownian motion, $\Phi_t = g_t^* \circ \Phi \circ g_t^{-1}$, and g_t is the conformal map associated to γ. In particular, if $a = 1/3, \kappa = 6$, U_t^* is a standard Brownian motion, and hence, $\tilde{\gamma}^*$ has the distribution of SLE_6. The locality property for SLE_6 can be stated as "the conformal image of SLE_6 is (a time change of) SLE_6." Intuitively, the SLE_6 path in a restricted domain does not feel the boundary of the domain until it reaches it. Radial SLE_6 satisfies a similar locality property. Moreover, (2) shows that the image of chordal SLE_6 under the exponential map is the same (for small time t) as radial SLE_6. The locality property explains why SLE_6 is a natural candidate for the boundary of percolation clusters.

If $\kappa \leq 4$, SLE_κ paths are simple, i.e., have no self-intersections. Suppose $A \subset \overline{\mathbb{H}} \setminus \{0\}$ is a compact set such that $\mathbb{H} \setminus A$ is simply connected. Let γ denote a chordal SLE_κ in $\mathbb{H}$ connecting 0 and ∞ and let E_A be the event $E_A = \{\gamma(0,\infty) \cap A = \emptyset\}$. Let $\Phi_A : \mathbb{H} \setminus A \to \mathbb{H}$ be the unique conformal transformation with $\Phi_A(0) = 0, \Phi_A(\infty) = \infty, \Phi'_A(\infty) = 1$. On the event E_A, we can define $\tilde{\gamma}(t) = \Phi_A \circ \gamma(t)$. Chordal SLE_κ is said to satisfy the restriction property if the conditional distribution of $\tilde{\gamma}$ given E_A is the same as (a time change of) γ. The only $\kappa \leq 4$ that satisfies this property is $\kappa = 8/3$. The proof of this fact also establishes the formula: if γ is a chordal $SLE_{8/3}$ curve in $\mathbb{H}$ from 0 to ∞, then

$$\mathbf{P}\{\gamma(0,\infty) \cap A = \emptyset\} = \Phi'_A(0)^{5/8}. \tag{6}$$

There is a similar formula for radial $SLE_{8/3}$ which establishes a radial restriction property. Suppose $A \subset \overline{\mathbb{D}} \setminus \{0,1\}$ is a compact set such that $\mathbb{D} \setminus A$ is simply connected. Let Ψ_A be the unique conformal transformation of $\mathbb{D} \setminus A$ onto $\mathbb{D}$ with $\Psi_A(0), \Psi'_A(0) > 0$. Then, if γ is a radial $SLE_{8/3}$ curve from 1 to 0 in $\mathbb{D}$,

$$\mathbf{P}\{\gamma(0,\infty) \cap A = \emptyset\} = \Psi'_A(0)^{5/48}\,|\Psi'_A(1)|^{5/8}.$$

The restriction property makes $SLE_{8/3}$ the candidate for the scaling limit of self-avoiding walks.

3 Relation to conformal field theory

The Schramm-Loewner evolution is one of the tools used to prove rigorously predictions made using powerful, yet nonrigorous, arguments of conformal field theory. In conformal field theory, there is a parameter c, called the *central charge*, which classifies theories. To each $c \leq 1$, there corresponds a $\kappa \leq 4$ and a "dual" $\kappa' = 16/\kappa \geq 4$,

$$c = c_\kappa = \frac{(8-3\kappa)(\kappa-6)}{2\kappa}.$$

In particular $\kappa = 8/3, \kappa' = 6$ corresponds to central charge zero. It is expected, and has been proved in a number of cases, that SLE_κ or $SLE_{\kappa'}$ curves will appear in scaling limits of systems with central charge c_κ. These systems can also be parametrized by the boundary *scaling exponent* or *conformal weight*

$$\alpha = \alpha_\kappa = \frac{6-\kappa}{2\kappa}.$$

For $\kappa = 8/3$, $\alpha = 5/8$ which is the exponent in (6).

In studying the relationship between SLE and conformal field theories, two other probabilistic objects arise, *restriction measures* and the *(Brownian) loop soup*. An $\mathbb{H}$-hull (connecting 0 and ∞) is an unbounded, connected, closed set $K \subset \overline{\mathbb{H}}$ with $K \cap \mathbb{R} = \{0\}$ and such that $\mathbb{H} \setminus K$ consists of two connected components, one whose boundary includes the positive reals and the other whose boundary includes the negative reals. A (chordal) restriction measure on hulls K is a probability measure with the property that for any A as in (6), the distribution of $\Phi_A \circ K$ given $\{K \cap A = \emptyset\}$ is the same as the original measure. The (Brownian) loop measure is a measure on unrooted loops derived from Brownian bridges. It is the scaling limit of the measure on random walk loops that gives each unrooted simple random walk loop of length $2n$ measure 4^{-2n}. The loop measure in a bounded domain is obtained by restricting to loops that stay in that domain. We can consider this as a measure on "hulls" by filling in the bounded holes (so that the complement of the hull is connected). By doing this we get a family of infinite measures on hulls, indexed by domains D, and this family satisfies conformal invariance and the restriction property. The loop soup with parameter λ is a Poissonian realization from this measure with parameter λ.

The set of all restriction measures is parametrized by $\alpha \geq 5/8$; the α-restriction measure has the property that

$$\mathbf{P}\{K \cap A \neq \emptyset\} = \Phi_A'(0)^\alpha.$$

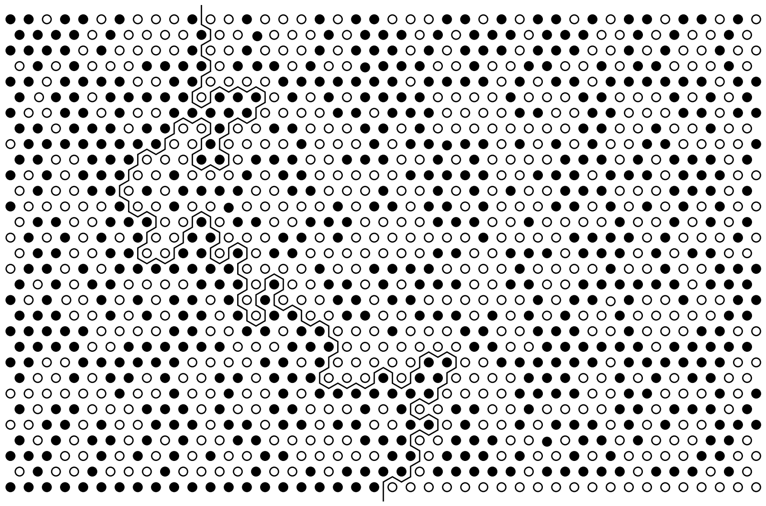
For $\alpha = 5/8$, K is given by the path of $SLE_{8/3}$. For integer α, the hull K can be constructed by taking α independent Brownian excursions in $\mathbb{H}$ (Brownian motions starting at 0 conditioned to stay in $\mathbb{H}$ for all times), and letting K be the hull obtained by taking the union of the paths and filling in the bounded holes. If $\kappa \leq 8/3, c_\kappa \leq 0$, then the restriction measure with exponent $\alpha_\kappa \geq 5/8$ can also be constructed as follows: take a chordal SLE_κ path and an independent realization of the loop soup with intensity $\lambda_\kappa = -c_\kappa$; add to the SLE path all the loops in the soup that intersect the SLE_κ curve; and then fill in all the bounded hulls. The limiting case $\alpha = 5/8, \lambda = 0$ gives the only measure supported on simple curves that is also a restriction measure, $SLE_{8/3}$.

For $8/3 < \kappa \leq 4, 0 < c_\kappa \leq 1$, it is conjectured, and proved for small c_κ, that SLE_κ curves can be found by taking a loop soup with parameter $\lambda = c_\kappa$ and looking at connected curves in the fractal set given by the complement of the union of all the hulls generated by the loops.

4 Examples

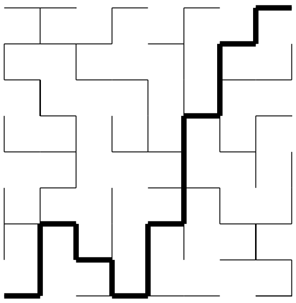
The scaling limit of simple random walk, Brownian motion, is known to be conformally invariant. A two-dimensional Brownian bridge or loop is a Brownian motion, $B_t, 0 \leq t \leq 1$, conditioned so that $B_0 = B_1$. The *frontier* or *outer boundary* of the Brownian motion is the boundary of the unbounded component of the complement. Benoit Mandelbrot first observed numerically that the outer boundary of Brownian motion had fracal dimension about 4/3. Gregory Lawler, Oded Schramm, and Wendelin Werner used SLE to prove that the boundary has Hausdorff dimension 4/3. In fact, the outer boundary can be considered as an $SLE_{8/3}$ loop.

SLE_6 and $SLE_{8/3}$ arise in the scaling limit of critical percolation on the triangular lattice. Suppose that each vertex in the upper half-plane triangular lattice is colored black or white each with l probability 1/2. Suppose the real line is give a boundary condition of black on the negative real line and white on the positive real line. Then if we represent the vertices in the lattice as hexagons as in the figure, a curve is formed which is the boundary between the black and white clusters. This curve is called the *percolation exploration process.* Stanislav



Smirnov proved that the scaling limit of this curve is conformally invariant, and from this it can be concluded that the curve is chordal SLE_6. In particular, the Hausdorff dimension is 7/4 and the scaling limit has double points. In the scaling limit, the "outer boundary" of this curve has Hausdorff dimension 4/3 and its dimension is absolutely continuous with respect to that of $SLE_{8/3}$. While this result is expected for other critical percolation model, such as bond percolation in $\mathbb{Z}^2$ with critical probability 1/2, it has only been proved for the triangular lattice. Percolation has central charge 0 and the "locality" property can be seen in the lattice model. The outer boundary of the curve has the same distribution as the outer boundary of a Brownian motion that is reflected at angle $\pi/3$ off the real line. Locally, the outer boundary of percolation, the outer boundary of complex Brownian motion, and $SLE_{8/3}$ all look the same, and it is expected that this will also be true for the scaling limit of self-avoiding walks.

There are three models derived in some way from simple ranodm walk that have been proved to have scaling limits of SLE_κ. The *loop-erased random walk (LERW)* in a finite subset V of $\mathbb{Z}^2$ connecting two distinct points is obtained by taking a simple random walk from one point to the other and erasing loops chronologically. The LERW is closely related to uniform spanning trees; in fact, if one chooses a spanning tree of V from the uniform distribution on all spanning trees, then the distribution of the unique path connecting the two points is exactly that of the LERW. Another description of the LERW is as the *Laplacian*

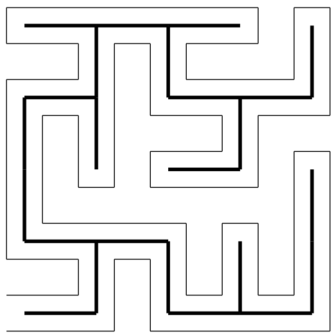


random walk: the LERW from z to w in V chooses a new step weighted by the value of the function that is harmonic on the complement of w and the path up to that point with boundary values 0 on the path and 1 on w. The LERW in the discrete upper half-plane can be obtained from erasing loops from a simple random walk excursion. The LERW and the uniform spanning tree are systems with central charge $c = -2$. It has been proved that the scaling limit of the LERW is SLE_2, and hence the paths have Hausdorff dimension 5/8.

There is another path associated to spanning trees given by the one-to-one correspondence between spanning trees and Hamiltonian walks on a corresponding directed (Manhattan) lattice on the dual graph. If the spanning trees, or equivalently the Hamiltonian walks, are chosen using the uniform distribution, then the scaling limit of this walk is the space-filling curve SLE_8. Note that 2 and 8 are the dual values of κ associated to $c = -2$.

Another discrete process derived from simple random walk, the *harmonic explorer*, has a scaling limit of SLE_4. There is a particular property of SLE_κ that leads to the definition of this discrete process. Consider a chordal SLE_κ curve, let $z \in \mathbb{H}$, and let Z_t^z be as in (3) with $a = 2/\kappa$. Itô's formula shows that $\Theta_t := \arg(Z_t^z)$ satisfies

$$d\Theta_t = \left(\frac{1}{2} - a\right) \frac{\sin(2\Theta_t)}{|Z_t^z|^2}\, dt - \frac{\sin \Theta_t}{|Z_t^z|}\, dW_t.$$



In particular, Θ_t is a martingale if and only if $a = 1/2, \kappa = 4$. The probability that a complex Brownian motion starting at $z \in \mathbb{H}$ first hits $\mathbb{R}$ on the negative half-line can be shown to be $\arg(z)$. If $\kappa \leq 4$, we can see that Θ_∞ equals 0 or π, depending on whether z is on the right or left side of the path $\gamma(0, \infty)$. For the martingale case $\kappa = 4$, Θ_t represents the probability that z is on the left side of $\gamma(0, \infty)$ given $\gamma(0, t]$. The harmonic explorer is a process on the hexagonal lattice defined to have this property. In a way similar to the percolation process, the walk is defined as the boundary between black and white hexagons on the triangular lattice. However, when an unexplored hexagon is reached in the harmonic explorer, it is colored black with probability q where q is the probability that a simple random walk on the triangular lattice starting at that hexagon (considered as a vertex in the triangular lattice) hits a black hexagon before hitting a white hexagon. It is not difficult to show that this process has the property that for z away from the curve, the "probability of z ending on the left given the curve of n steps" is a martingale.

There are many other models for which SLE_κ curves are expected in the limit but it has not been established. The hardest part is to show the existence of a limit that is conformally invariant. One example is the self-avoiding walk (SAW). It is an open problem to establish that there exists a scaling limit of the uniform measure on SAWs and to establish conformal invariance of the limit. However, the nature of the discrete model is such that if the limit exists, it must satisfy the restriction property. Hence, under the assumption of conformal invariance, the only possible limit is $SLE_{8/3}$. Numerical simulations strongly support the conjecture that $SLE_{8/3}$ is the limit of SAWs, and this gives strong evidence for the conformal invariance conjecture for SAWs. Critical exponents for SAWs (as well as critical exponents for many other models) can be predicted nonrigorously from rigorous scaling exponents for the corresponding SLE paths.

5 Generalizations

One of the reasons that the theory of SLE is nice for simply connected domains is that a simply connected domain with an arc connected to the boundary of the domain removed is again simply connected. For non-simply connected domains, it is more difficult to describe because the conformal type of the slit domain changes as time evolves. In the case of a curve crossing an annulus this can be done with an added parameter referring to the conformal type of the annulus (two annuli of the form $\{z : r_j < |z| < s_j\}$ are conformally equivalent if and only if $r_1/s_1 = r_2/s_2$). It is not immediately obvious what the correct definition of SLE should be in general domains, and more generally, on Riemann surfaces. One possibility for $\kappa \leq 4$ is to consider a configurational (equilibrium statistical mechnics) view of SLE. Consider a family of measures $\{\mu_D(z,w)\}$ where D ranges over domains and z, w are distinct boundary points at ∂D is locally analytic, supported on simple curves from z to w (modulo time change). Let $\mu_D^\#(z,w) = \mu_D(z,w)/|\mu_D(z,w)|$ be the corresponding probability measures, which may be defined even if ∂D is not smooth at z, w. Then the following axioms should hold:

- Conformal invariance: if $f : D \to D'$ is a conformal transformation, $f \circ \mu_D^\#(z,w) = \mu_{D'}^\#(f(z), f(w))$.
- Conformal Markov property.
- Perturbation of domains: Suppose $D_1 \subset D$ and ∂D_1, ∂D agree near z, w. Let Y denote the Radon-Nikodym derivative of $\mu_{D_1}(z,w)$ with respect to $\mu_D(z,w)$. Then

$$Y(\gamma) = 1\{\gamma(0, t_\gamma) \subset D_1\} \, J(\gamma, D, D_1)^{-c},$$

where $J(\gamma, D, D_1)$ denotes the probability that there is a loop in the Brownian loop soup in D that intersects both γ and $D \setminus D_1$. (There is no problem defining the loop soup in non-simply connected domains.) Here $c = c_\kappa$. The case $c = 0$ corresponds to the restriction property.

- Conformal covariance: If f is as above, $\partial D, \partial D'$ are smooth near z, w and $f(z), f(w)$, respectively, then

$$f \circ \mu_D(z,w) = |f'(z)|^\alpha \, |f'(w)|^\alpha \, \mu_{D'}(f(z), f(w)).$$

Here $\alpha = \alpha_\kappa$ is the boundary scaling exponent.

References

[1] Kager W, Nienhuis B (2004), A guide to Stochastic Loewner evolution and its applications, J. Stat. Phys. **115**, 1149–1229.

[2] Lawler G (2004), Conformally invariant processes in the plane. In: School and Conference on Probability Theory, ICTP Lecture Notes **17**, 305–351.

[3] Lawler G (2005), *Conformally Invariant Processes in the Plane*, Amer. Math. Soc.

[4] Lawler G, Schramm O, Werner W (2004), Conformal restriction, the chordal case, J. Amer. Math. Soc. **16**, 917–955.

[5] Lawler G, Schramm O, Werner W (2004) Conformal invariance of planar loop-erased random walks and uniform spanning trees (2004), Annals of Probab. **32**, 939–995.

[6] Lawler G, Werner W (2004), The Brownian loop soup, Probab. Theory Rel. Fields **128**, 565–588.

[7] Schramm O (2000), Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math **118**, 221–288.

[8] Smirnov S (2001), Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits, C. R. Acad. Sci. Paris S. I Math **333**, 239–244.

[9] Werner W (2004), Random planar curves and Schramm-Loewner evolutions. In: Ecole d'Eté de Probabilités de Saint-Flour XXXII - 2002, Lecture Notes in Mathematics **1840**, 113–195, Springer-Verlag.

[10] Werner W (2003), SLEs as boundaries of clusters of Brownian loops, C. R. Ac. Sci. Paris Ser. I Math. 337, 481-486.