Math 671
Fall 2004
Final Problem Set
Due: Tuesday, December 14, 5:00 pm

- You are not to discuss these problems with anyone or show this examination to anyone before turning in the exam.
- You may use your book, notes, and any other books. You can see me or e-mail me (lawler@math.cornell.edu). If the error/clarification is significant, I will post a correction on the web page of the course and (if found soon enough) announce the change in class.
- There are a number of multi-part problems. If you cannot do a particular part, you may do latter parts of the problems assuming the result of the part that you did not do.

1. Let $X_1, X_2, \ldots$ be independent (not necessarily identically distributed) random variables with $\mathbf{E}[X_j] = 0$ and such that $\mathbf{E}[X_j^{2k}] \leq 1$ for all j, where k is a fixed positive integer. Let

$$S_n = X_1 + \cdots + X_n.$$

a. Find a $c = c(k) < \infty$, independent of the distribution of $X_1, \ldots, X_n$, such that

$$\mathbf{E}[S_n^{2k}] \leq c\, n^k.$$

b. Show that for every $\lambda > 0$,

$$\mathbf{P}\{\max_{1 \leq j \leq n} |S_j| \geq \lambda \sqrt{n}\} \leq c\, \lambda^{-2k}.$$

2. For this problem, recall that the potential kernel for two-dimensional simple random walk is defined by

$$a(x) = \lim_{n \to \infty} \sum_{j=0}^{n} [\mathbf{P}\{S_j = 0\} - \mathbf{P}\{S_j = x\}].$$

Here and below we assume that $S_0 = 0$. Recall also that a is (discrete) harmonic for $x \neq 0$, i.e.,

$$a(x) = \frac{1}{4} \sum_{|y-x|=1} a(y), \qquad x \neq 0,$$

and that there exist constants γ, c such that for all $x \neq 0$,

$$|a(x) - \frac{2}{\pi} \log |x| - \gamma| \leq \frac{c}{n}.$$

You may use these facts in the problems below.

a. Let $T_n = \min\{j \geq 1 : |S_j| \geq n\}$, $\tau = \min\{j \geq 1 : S_j = 0\}$. Find

$$\lim_{n \to \infty} (\log n)\, \mathbf{P}\{T_n < \tau\}.$$

b. Let

$$J_n = \sum_{j=0}^{T_n} 1\{S_j = 0\}$$

be the number of visits to 0 before reaching distance n. Find

$$\lim_{n\to\infty} (\log n)^{-1}\, \mathbf{E}[J_n].$$

c. Let $x_n = (n, 0)$ and let

$$K_n = \sum_{j=0}^{T_{2n}} 1\{S_j = x_n\}$$

be the number of visits to x_n before reaching distance $2n$. Find

$$\lim_{n\to\infty} \mathbf{E}[K_n].$$

3. We will construct a random Cantor set. Consider the interval $[0,1]$. If we subdivide this interval into 3^n intervals of length 3^{-n},

$$J(j,n) := \left[\frac{j}{3^n}, \frac{j+1}{3_n}\right], \quad j = 0, 1, \ldots, 3^n - 1,$$

we can label the intervals by $(k_1, \ldots, k_n)$ where $k_j \in \{0,1,2\}$. Here, $k_1 = 0$ if $J(j,n)$ is a subinterval of $[0,1/3]$; $k_1 = 1$ if $J(j,n)$ is in $[1/3, 2/3]$, and $k_1 = 2$ if the $J(j,n)$ is in $[2/3,1]$, etc. In other words the interval $J(j,n)$ is represented by $(k_1, k_2, \ldots, k_n)$ where

$$\frac{j}{3^n} = \sum_{i=1}^{n} \frac{k_i}{3^i}.$$

Suppose $0 < p < 1$ and $\{Y(j,n)\}$ are independent 0-1 random variables with $\mathbf{P}\{Y(j,n) = 1\} = p$. Call an interval $J(j,n)$, labeled $(k_1, \ldots, k_n)$, *good* if

$$Y(l_1, 1)\, Y(l_2, 2) \cdots Y(l_n, n) = 1,$$

where

$$l_1 = k_1, \quad l_2 = 3 \cdot l_1 + k_2, \; \ldots, \; l_n = 3 \cdot l_{n-1} + k_n.$$

Let A_n denote the union of all the good intervals of length 3^{-n}. Note that $A_1 \supset A_2 \supset A_3 \cdots$. Let $A = \cap_n A_n$. In other words, at the first stage we keep each of the three intervals of length probability $1/3$ with probability p, with the choices independent for each interval, and we repeat this process at each level, making all choices independent.

a. Find $\mathbf{P}\{A \neq \emptyset\}$ (as a function of p).

b. Let L_n denote the Lebesgue measure of A_n. Find $\mathbf{E}[L_n]$.

c. Let $M_n = L_n / \mathbf{E}[L_n]$. Show that M_n is a martingale.

d. Show that for those values of p for which the probability in part a is positive, that there is a $c = c_p$ such that for all n, $\mathbf{E}[L_n^2] \leq c\, (\mathbf{E}[L_n])^2$.

e. For which values of p is M_n a uniformly integrable martingale?

f. Show that for those values of p for which the probability in part a is zero that

$$\sup_n \frac{\mathbf{E}[L_n^2]}{(\mathbf{E}[L_n])^2} = \infty.$$

4. Let B_t be a standard Brownian motion, and let

$$Z_t = \int_0^t B_s \, ds.$$

Since B is continuous, this Riemann integral is well defined.

a. Show that Z_1 has a normal distribution and compute its mean and variance. (Hint: it may be useful to write the integral as a limit of Riemann sums.)

b. True or false: the random vector (Z_1, Z_2) has a joint normal distribution.

c. Let $\mathcal{F}_t$ denote the filtration of the Brownian motion. True or false: Z_t is a martingale with respect to $\mathcal{F}_t$.

d. True or false: Z_t is a Markov process, i.e., for $s < t$,

$$E[Z_t \mid \mathcal{F}_s] = Z_s.$$

5. Let B_t be a standard Brownian motion. Let A denote the set of times of increase for the Brownian motion, i.e., the set of $s > 0$ such that there exist an $\epsilon > 0$ such that

$$B_t \geq B_s, \quad s \leq t \leq s + \epsilon,$$

$$B_t \leq B_s, \quad s - \epsilon \leq t \leq s.$$

The purpose of this problem is to prove that w.p.1, $A = \emptyset$.

a. Let $\tilde{A}$ be the set of times $s \in [1, 2]$ such that

$$B_t \leq B_s, \quad 0 \leq t \leq s,$$

$$B_t \geq B_s, \quad s \leq t \leq 3.$$

Show that

$$\mathbf{P}\{\tilde{A} \neq \emptyset\} = 0$$

implies $\mathbf{P}\{A = \emptyset\} = 1$.

b. Let $J(j, n) = [j/n, (j+1)/n]$. We will $J(j, n)$ an *increase interval* if

$$B_s \leq B_t, \quad 0 \leq s \leq \frac{j}{n}, \quad \frac{j+1}{n} \leq t \leq 3.$$

Let

$$Y_{n,k} = \sum_{j=n}^{k} 1\{J(j, n) \text{ is an increase interval}\}.$$

Explain why to show $\mathbf{P}\{\tilde{A} \neq \emptyset\} = 0$ it suffices to prove that $\mathbf{P}\{Y_{n,2n} > 0\} \to 0$.

c. Show that there exist constants $0 < c_1, c_2 < \infty$ such that for $j = 1, 2, \ldots, 2n$,

$$\frac{c_1}{\sqrt{jn}} \leq \mathbf{P}\{J(j,n) \text{ is an increase interval}\} \leq \frac{c_2}{\sqrt{jn}}.$$

(Hint: the gambler's ruin estimate will be useful.)

d. Show that there is a c such that $\mathbf{E}[Y_{n,3n}] \leq c$.

e.* Show that there is a c' such that $\mathbf{E}[Y_{n,3n} \mid Y_{n,2n} > 0] \geq c' \log n$. (Hint: consider

$$\mathbf{P}\{J(k,n) \text{ increase interval} \mid J(j,n) \text{ increase interval}\}$$

for $k > j$.)

f. Conclude that $\mathbf{P}\{Y_n = 0\} \to 1$ as $n \to \infty$.