EP1-1. Show that with probability one

$$\lim_{n\to\infty} \sum_{j=1}^{n} (B_{j/n} - B_{(j-1)/n})^2 = 1.$$

You may use the following fact: If $X_1, X_2, \ldots$ are i.i.d random variables with $\mathbf{E}[|X_1|^{2k}] < \infty$, then there is a $c < \infty$ such that

$$\mathbf{E}[(X_1 + \cdots + X_n)^{2k}] \leq c\, n^k.$$

EP1-2. Show that for every $\epsilon > 0$, there is a (bounded) elementary process Y_t such that

$$\mathbf{P}\left\{ \int_0^1 Y_t\, dB_t > 0 \right\} \geq 1 - \epsilon.$$

Show that for every $Y \in \mathcal{V}$,

$$\mathbf{P}\left\{ \int_0^1 Y_t\, dB_t > 0 \right\} < 1.$$

EP2-1 Suppose B^1, B^2 are independent standard Brownian motions. Let

$$Y_n = \sum_{j=1}^{n} [B^1_{j/n} - B^1_{(j-1)/n}]\, [B^2_{j/n} - B^2_{(j-1)/n}].$$

Show that with probability one $Y_n \to 0$. (Hint: Show $\mathbf{E}[Y_n] = 0, \mathbf{E}[Y_n^4] \leq c/n^2$.)

EP3-1 Let $X_1, X_2, X_3, \ldots$ be i.i.d random variables with mean zero; let $\mathcal{F}_n$ denote the σ-algebra generated by $X_1, X_2, \ldots, X_n$ ($\mathcal{F}_0$ is the trivial σ-algebra); and suppose that $Y_1, Y_2, \ldots$ are random variables such that Y_j is $\mathcal{F}_{j-1}$ measurable. The discrete stochastic integral is

$$M_0 = 0, \qquad M_n = \sum_{j=1}^{n} Y_j\, X_j.$$

Assume that $\mathbf{E}[|Y_n\, X_n|] < \infty$ for each n. Then M_n is a martingale.

(a) Suppose that $\mathbf{P}\{X_j = 1\} = \mathbf{P}\{X_j = -1\} = 1/2$. Show that if R_n is an L^2-martingale with respect to $\mathcal{F}_n$ with $R_0 = 0$, then $F_n = M_n$ w.p.1 for some $Y_1, Y_2, \ldots$ (You may wish to consider R_1 first.)

(b) Does the last result hold if $X_1, X_2, \ldots$ are $N(0,1)$?

EP5-1 Redo 7.18 (c) using the Bessel diffusion process

$$dX_t = \frac{r}{X_t}\, dt + dB_t.$$

Assume that the interval (a, b) satisfies $a > 0$.

EP6-1 Suppose $f : [0, \infty) \to \mathbb{R}^d$ is a function whose right derivative

$$f'_+(t) = \lim_{\epsilon \to 0+} \frac{f(t+\epsilon) - f(t)}{\epsilon}$$

exists everywhere.

(a) Show that if f and f'_+ are continuous, then f is differentiable for $t > 0$ with derivative f'_+.

(b) Give an example where f is continuous but f'_+ is not continuous for which f is not differentiable at some $t > 0$.

(c) Give an example where f'_+ is continuous but f is not continuous (and hence not differentiable everywhere).

EP6-2 Let S_n denote simple random walk in $\mathbb{Z}^d$. If $A \subset \mathbb{Z}^d$ we write $\partial A = \{x \in \mathbb{Z}^d : \mathrm{dist}(x, A) = 1\}$, $\overline{A} = A \cup \partial A$. We say that $f : \overline{A} \to \mathbb{R}$ is (discrete) harmonic in A if $\Delta f(x) = 0$ for all $x \in A$, where

$$\Delta f(x) = \frac{1}{2d} \sum_{|x-y|=1} [f(y) - f(x)].$$

Note that $\Delta f(x) = \mathbf{E}^x[f(S_1)] - f(x)$. Let

$$\tau_A = \min\{j \geq 0 : S_j \notin A\}.$$

(a) Suppose A is a bounded set and f is harmonic on $\overline{A}$. Assume that $S_0 \in \overline{A}$. Show that

$$M_n = f(S_{n \wedge \tau}),$$

is a martingale.

(b) Suppose A is bounded and $F : \partial A \to \mathbb{R}$ is a given function. Show that there is a unique function $f : \overline{A} \to \mathbb{R}$ satisfying:

$$f(x) = F(x), \quad x \in \partial A,$$

$$\Delta f(x) = 0, \quad x \in A.$$

(Hint: Assume that f is such a function and consider the martingale from part (a).)

(c) Suppose $d \leq 2$ and A can be unbounded. Let $F : \partial A \to \mathbb{R}$ be a given bounded function. Show that there is a unique bounded function $f : \overline{A} \to \mathbb{R}$ satisfying:

$$f(x) = F(x), \quad x \in \partial A,$$

$$\Delta f(x) = 0, \quad x \in A.$$

(d) Show by example that the result from part (c) is false if we do not require f to be bounded.

(e) Show by example that the result from part (c) is false if $d \geq 3$.

EP6-3 Suppose S is a simple random walk as in EP6-2, and let $f : \mathbb{Z}^d \to \mathbb{R}$ be given. Assume that $S_0 = x \in \mathbb{Z}^d$. Show that

$$K_n = f(S_n) - \sum_{j=0}^{n-1} \Delta f(S_j)$$

is a martingale. In particular if T is a bounded stopping time for S,

$$\mathbf{E}^x[f(S_T)] = f(x) + \mathbf{E}^x\left[\sum_{j=0}^{T-1} \Delta f(S_j)\right].$$

EP6-4 Suppose A is a connected set (i.e., one can get between any two points in A by a path staying in A) and T_A is the first time the random walk leaves A. Suppose there exists $x \in A$ such that $\mathbf{E}^x[T_A] < \infty$.

(a) Show that $\mathbf{E}^y[T_A] < \infty$ for all $y \in A$.

(b) Assume A is bounded and let $f(y) = \mathbf{E}^y[T_A]$. Show that f is the unique function from $\overline{A}$ to $\mathbb{R}$ satisfying

$$\Delta f(y) = -1, \ \ y \in A; \qquad f(y) = 0, \quad y \in \partial A$$

EP7-1 Suppose D is a connected, bounded open subset of $\mathbb{R}^d$; B_t is a standard d-dimensional Brownian motion; $\tau = \inf\{t : B_t \notin D\}$; and $q : \overline{D} \to \mathbb{R}$ a continuous function. Let

$$e(x) = e(x;q) = \mathbf{E}^x\left[\exp\{-\int_0^\tau q(B_t)\,dt\}\right].$$

a. Show there is an $\epsilon = \epsilon(D) > 0$ such that if $q \geq -\epsilon$, then $e(x) < \infty$ for all $x \in D$.

b. Show that there is a $K = K(D) < \infty$ such that if $q \equiv -K$, then $e(x) = \infty$ for all $x \in D$.

c. Let $\sigma_\epsilon = \inf\{t : |B_t - B_0| \geq \epsilon\}$ and let

$$\tilde{e}(x) = \tilde{e}(x;D,\epsilon) = \mathbf{E}^x\left[\exp\{-\int_0^{\tau\wedge\sigma_\epsilon} q(B_t)\,dt\}\right].$$

Show that there is an $\epsilon = \epsilon(D,q) > 0$ such that $\tilde{e}(x) < \infty$ for all x.

d. Suppose $e(x) < \infty$ for some $x \in D$. Show that $e(y) < \infty$ for all $y \in D$.

EP7-2 Using the notation of the last problem, give an example of an unbounded D with $\mathbf{P}^x\{\tau < \infty\} = 1$ for all $x \in D$ such that for every $\epsilon > 0$,

$$\mathbf{E}^x[e^{\epsilon\tau}] = \infty.$$

EP8-1 Let $X_1, X_2, \ldots$ be independent $N(0,1)$ random variables and let f be a bounded continuous function. Let $Z_0 = 0$ and for $n > 0$,

$$Z_n = Z_{n-1} + f(Z_{n-1}) + X_n.$$

We will do the Girsanov transformation for Z_n to make Z_n a martingale (with respect to $\mathcal{F}_n$, where $\mathcal{F}_n$ is the information in $X_1, \ldots, X_n$).

(a) If a is a real number, compute $\mathbf{E}[X_1 e^{aX_1}]$. (One can do it directly, or one can differentiate the moment generating function $\mathbf{E}[e^{aX_1}]$ with respect to a.)

(b) Let $M_0 = 1$ and for $n > 0$,

$$M_n = \exp\left\{-\sum_{j=1}^n f(Z_{j-1})\,X_j - \sum_{j=1}^n \frac{f(Z_{j-1})^2}{2}\right\}.$$

Show that M_n is a martingale with respect to $\mathcal{F}_n$.

(c) Show that $M_n Z_n$ is a martingale with respect to $\mathcal{F}_n$.

(d) Show that Z_n is a $\mathbf{Q}$-martingale where $d\mathbf{Q} = M_n \, d\mathbf{P}$.

EP8-2 Let B_t be a standard one-dimensional Brownian motion with $B_0 = 1$, let $0 < \epsilon < 1$ and let

$$T_\epsilon = \inf\{t : B_t = \epsilon\}.$$

Suppose $\alpha \geq 0$.

(a) Find a process Z_t whose paths are differentiable with respect to t such that

$$M_t = B_{t \wedge T_\epsilon}^\alpha \, Z_t$$

is a continuous martingale.

(b) Suppose Q is defined by

$$dQ_t = M_t \, d\mathbf{P}.$$

Find an A_t such that if Y_t satisfies

$$dY_t = A_t \, dt + dB_t,$$

then Y_t is a Brownian motion with respect to the measure Q. Write an SDE for B_t in terms of the Brownian motion Y_t.

(c) For which α is it the case that

$$Q\{T_{1/2} = \infty\} > 0. \quad ?$$

EP9-1 Let D be a domain (open connected subset) in $\mathbb{R}^d$. Call two distinct points $z, w \in D$ *adjacent* if

$$|z - w| < \frac{1}{2} \max\{\operatorname{dist}(z, \partial D), \operatorname{dist}(w, \partial D)\}.$$

(a) Define $\rho_D(z, w)$ by: $\rho_D(z, z) = 0$, and if $z \neq w$ then $\rho_D(z, w)$ is the minimum integer k such that there exist a finite sequence

$$z = x_0, x_1, \ldots, x_k = w$$

of points in D such that for each j, x_j is adjacent to x_{j+1}. Show that $\rho_D(z, w) < \infty$ for every $z, w \in D$ and that ρ_D is a metric on D.

(b) Suppose $z_0 \in D$ and define

$$U_k = U_k(z_0, D) = \{w : \rho_D(z_0, w) \leq k\}.$$

Show that for $k \geq 1$, U_k is an open set.

(c) Show that there is a $c < \infty$ such that if D is any domain and $h : D \to (0, \infty)$ is a positive harmonic function, then for all $z, w \in D$,

$$h(z) \leq c^{\rho_D(z,w)} \, h(w).$$

(d) Show that if $K \subset D$ is a compact set, then there exists a $c = c(K, D) < \infty$ such that if $h : D \to (0, \infty)$ is harmonic, then

$$h(z) \leq c\, h(w), \quad z, w \in K.$$

EP9-2

(a) Show that there exist $0 < \alpha, c < \infty$ such that the following is true. Let B_t be a Brownian motion in the unit disk $D \subset \mathbb{R}^2$ and let E be the event that $B[0, \tau_D]$ does not disconnect the origin from the unit circle. Then,

$$\mathbf{P}^x(E) \leq c\,|x|^\alpha.$$

(Hint: Suppose $|x| < 2^{-n}$ and let σ_n be the first time that the Brownian motion reaches the circle of radius 2^{-n}. Then

$$E \subset \bigcap_{j=1}^{n} E_j$$

where E_j is the event that $[B_{\sigma_j}, B_{\sigma_{j-1}}]$ does not make a closed loop about the origin.)

(b) Show that if D is a domain in $\mathbb{R}^2$ whose boundary is connected and larger than a single point, then every point on ∂D is regular for D.

EP9-3. Suppose D is a domain and h_n is a sequence of harmonic functions such that for every compact $K \subset D$,

$$\sup_{n \geq 1} \sup_{z \in K} |h_n(z)| < \infty.$$

(a) Show that the sequence of functions $\{h_n\}$ is *equicontinuous* on K, i.e., for every $\epsilon > 0$, there is a $\delta > 0$ such that if $z, w \in K$ with $|z - w| < \delta$ then

$$|h_n(z) - h_n(w)| < \delta \text{ for all } n.$$

(Hint: use derivative estimates for harmonic functions.)

(b) Show that there exists a subsequence h_{n_j} and a harmonic function h on D such that $h_{n_j}(z) \to h(z)$ for all $z \in D$. (Hint: You may use the Arzela-Ascoli Theorem.)

EP10-1

(a) Show that there is a $c > 0$ such that if B_t is a one-dimensional Brownian motion and $x > 0$,

$$\mathbf{P}^x\{B_1 \geq 1; B_t > 0 \text{ for all } 0 \leq t \leq 1\} \geq c\,x.$$

(b) Show there is a $c > 0$ such that if B_t is a two-dimensional Brownian motion; D is the unit disk; and $\tau = \tau_D$, then for all $z \in D$,

$$\mathbf{P}^z\{|B_1| \leq 1/2 \mid \tau > 1\} \geq c.$$

EP10-2. Let B_t be a two-dimensional Brownian motion with $|B_0| = 1$ and let $T_n = \inf\{t : |B_t| \geq n\}$. Let E_n be the event that $B[0, T_n]$ does not make a closed loop about the origin. Show that there is an $\alpha \in (0, 1]$ such that $\mathbf{P}(E_n) \approx n^{-\alpha}$ in the sense that

$$\lim_{n \to \infty} \frac{\log \mathbf{P}(E_n)}{\log n} = -\alpha.$$

(Hint: Let $F(n) = \mathbf{P}(E_{2^n})$ and show that $F(n+m) \leq F(n)\,F(m)$.).

EP10-3 Suppose $p_{j,k}$, $j,k = 1,2,\ldots$ are positive numbers such that there exist $0 < \epsilon \leq K < \infty$ such that for all j,

$$\sum_{k=1}^{\infty} p_{j,k} \leq K,$$

$$p_{j,1} \geq \epsilon.$$

Let $f_0(1) = 1$ and $f_0(k) = 0$ for all $k \geq 2$ and for $n \geq 1$,

$$f_n(k) = \sum_{j=1}^{\infty} p_{j,k}\, f_{n-1}(j).$$

a. Show that there exist $0 < c_1 < c_2 < \infty$ and $\alpha \in [\epsilon, K]$ such that for all k, n,

$$c_1\,\alpha^n \leq f_n(k) \leq c_2\,\alpha^n.$$

b. (*) Show that for every k, the limit

$$c(k) = \lim_{n\to\infty} \alpha^{-n}\, f_n(k),$$

exists.