1 Problems in Oksendal's book

3.2.

Proof. WLOG, we assume $t = 1$, then

$$\begin{aligned} B_1^3 &= \sum_{j=1}^n (B_{j/n}^3 - B_{(j-1)/n}^3) \\ &= \sum_{j=1}^n [(B_{j/n} - B_{(j-1)/n})^3 + 3B_{(j-1)/n}B_{j/n}(B_{j/n} - B_{(j-1)/n})] \\ &= \sum_{j=1}^n (B_{j/n} - B_{(j-1)/n})^3 + \sum_{j=1}^n 3B_{(j-1)/n}^2 (B_{j/n} - B_{(j-1)/n}) \\ &\quad + \sum_{j=1}^n 3B_{(j-1)/n}(B_{j/n} - B_{(j-1)/n})^2 \\ &:= I + II + III \end{aligned}$$

By Problem EP1-1 and the continuity of Brownian motion.

$$I \le [\sum_{j=1}^n (B_{j/n} - B_{(j-1)/n})^2] \max_{1\le j\le n} |B_{j/n} - B_{(j-1)/n}| \to 0 \quad a.s.$$

To argue $II \to 3\int_0^1 B_t^2 dB_t$ as $n \to \infty$, it suffices to show $E[\int_0^1 (B_t - B_t^{(n)})^2 dt] \to 0$, where $B_t^{(n)} = \sum_{j=1}^n B_{(j-1)/n}^2 1_{\{(j-1)/n < t \le j/n\}}$. Indeed,

$$E[\int_0^1 |B_t - B_t^{(n)}|^2 dt] = \sum_{j=1}^n \int_{(j-1)/n}^{j/n} E(B_{(j-1)/n}^2 - B_t^2)^2 dt$$

We note $(B_t^2 - B_{\frac{j-1}{n}}^2)^2$ is equal to

$$(B_t - B_{\frac{j-1}{n}})^4 + 4(B_t - B_{\frac{j-1}{n}})^3 B_{\frac{j-1}{n}} + 4(B_t - B_{\frac{j-1}{n}})^2 B_{\frac{j-1}{n}}^2$$

so $E(B_{(j-1)/n}^2 - B_t^2)^2 = 3(t - (j-1)/n)^2 + 4(t - (j-1)/n)(j-1)/n$, and

$$\int_{\frac{j-1}{n}}^{\frac{j}{n}} E(B_{\frac{j-1}{n}}^2 - B_t^2)^2 dt = \frac{2j+1}{n^3}$$

Hence $E\int_0^1 (B_t - B_t^{(n)})^2 dt = \sum_{j=1}^n \frac{2j-1}{n^3} \to 0$ as $n \to \infty$.

To argue $III \to 3\int_0^1 B_t dt$ as $n \to \infty$, it suffices to prove

$$\sum_{j=1}^n B_{(j-1)/n}(B_{j/n} - B_{(j-1)/n})^2 - \sum_{j=1}^n B_{(j-1)/n}(\frac{j}{n} - \frac{j-1}{n}) \to 0 \quad a.s.$$

By looking at a subsequence, we only need to prove the L^2-convergence. Indeed,

$$
\begin{aligned}
& E\left(\sum_{j=1}^n B_{(j-1)/n}[(B_{j/n}-B_{(j-1)/n})^2-\frac{1}{n}]\right)^2 \\
= & \sum_{j=1}^n E\left(B^2_{(j-1)/n}[(B_{j/n}-B_{(j-1)/n})^2-\frac{1}{n}]^2\right) \\
= & \sum_{j=1}^n \frac{j-1}{n} E\left[(B_{j/n}-B_{(j-1)/n})^4-\frac{2}{n}(B_{j/n}-B_{(j-1)/n})^2+\frac{1}{n^2}\right] \\
= & \sum_{j=1}^n \frac{j-1}{n}(3\frac{1}{n^2}-2\frac{1}{n^2}+\frac{1}{n^2}) \\
= & \sum_{j=1}^n \frac{2(j-1)}{n^3} \to 0
\end{aligned}
$$

as $n\to\infty$. This completes our proof.

□

3.18.

Proof. If $t>s$, then

$$E\left[\frac{M_t}{M_s}|\mathcal{F}_s\right]=E\left[e^{\sigma(B_t-B_s)-\frac{1}{2}\sigma^2(t-s)}|\mathcal{F}_s\right]=\frac{E[e^{\sigma B_{t-s}}]}{e^{\frac{1}{2}\sigma^2(t-s)}}=1$$

The second equality is due to the fact B_t-B_s is independent of $\mathcal{F}_s$. □

4.4.

Proof. For part a), set $g(t,x)=e^x$ and use Theorem 4.12. For part b), it comes from the fundamental property of Ito integral, i.e. Ito integral preserves martingale property for integrands in $\mathcal{V}$. □

Comments: The power of Ito formula is that it gives martingales, which vanish under expectation.

4.5.

Proof.

$$B_t^k=\int_0^t kB_s^{k-1}dB_s+\frac{1}{2}k(k-1)\int_0^t B_s^{k-2}ds$$

Therefore,

$$\beta_k(t)=\frac{k(k-1)}{2}\int_0^t\beta_{k-2}(s)ds$$

This gives $E[B_t^4]$ and $E[B_t^6]$. For part b), prove by induction.

□

4.6. (b)

Proof. Apply Theorem 4.12 with $g(t,x) = e^x$ and $X_t = ct + \sum_{j=1}^n \alpha_j B_j$. Note $\sum_{j=1}^n \alpha_j B_j$ is a BM, up to a constant coefficient. □

5.1. (ii)

Proof. Set $f(t,x) = x/(1+t)$, then by Ito's formula, we have

$$dX_t = df(t, B_t) = -\frac{B_t}{(1+t)^2}dt + \frac{dB_t}{1+t} = -\frac{X_t}{1+t}dt + \frac{dB_t}{1+t}$$

□

(iv)

Proof. $dX_t^1 = dt$ is obvvious. Set $f(t,x) = e^t x$, then

$$dX_t^2 = df(t, B_t) = e^t B_t dt + e^t dB_t = X_t^2 dt + e^t dB_t$$

□

5.9.

Proof. Let $b(t,x) = \log(1+x^2)$ and $\sigma(t,x) = 1_{\{x>0\}}x$, then

$$|b(t,x)| + |\sigma(t,x)| \le \log(1+x^2) + |x|$$

Note $\log(1+x^2)/|x|$ is continuous on $\mathbb{R} - \{0\}$, has limit 0 as $x \to 0$ and $x \to \infty$. So it's bounded on $\mathbb{R}$. Therefore, there exists a constant C, such that

$$|b(t,x)| + |\sigma(t,x)| \le C(1 + |x|)$$

Also,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le \frac{2|\xi|}{1+\xi^2}|x-y| + |1_{\{x>0\}}x - 1_{\{y>0\}}y|$$

for some ξ between x and y. So

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le |x-y| + |x-y|$$

Conditions in Theorem 5.2.1 are satisfied and we have existence and uniqueness of a strong solution. □

5.11.

Proof. First, we check by integration-by-parts formula,

$$dY_t = \left(-a + b - \int_0^t \frac{dB_s}{1-s}\right)dt + (1-t)\frac{dB_t}{1-t} = \frac{b - Y_t}{1-t}dt + dB_t$$

Set $X_t = (1-t)\int_0^t \frac{dB_s}{1-s}$, then X_t is centered Gaussian, with variance

$$E[X_t^2] = (1-t)^2 \int_0^t \frac{ds}{(1-s)^2} = (1-t) - (1-t)^2$$

So X_t converges in L^2 to 0 as $t \to 1$. Since X_t is continuous a.s. for $t \in [0,1)$, we conclude 0 is the unique a.s. limit of X_t as $t \to 1$. □

7.8

Proof.

$$\{\tau_1 \wedge \tau_2 \le t\} = \{\tau_1 \le t\} \cup \{\tau_2 \le t\} \in \mathcal{N}_t$$

And since $\{\tau_i \ge t\} = \{\tau_i < t\}^c \in \mathcal{N}_t$,

$$\{\tau_1 \vee \tau_2 \ge t\} = \{\tau_1 \ge t\} \cup \{\tau_2 \ge t\} \in \mathcal{N}_t$$

□

7.9. a)

Proof. By Theorem 7.3.3, A restricted to $C_0^2(\mathbb{R})$ is $rx\frac{d}{dx} + \frac{\alpha^2 x^2}{2}\frac{d^2}{dx^2}$. For $f(x) = x^\gamma$, Af can be calculated by definition. Indeed, $X_t = xe^{(r-\frac{\alpha^2}{2})t + \alpha B_t}$, and $E^x[f(X_t)] = x^\gamma e^{(r-\frac{\alpha^2}{2}+\frac{\alpha^2\gamma}{2})\gamma t}$. So

$$\lim_{t \downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t} = (r\gamma + \frac{\alpha^2}{2}\gamma(\gamma-1))x^\gamma$$

So $f \in D_A$ and $Af(x) = (r\gamma + \frac{\alpha^2}{2}\gamma(\gamma-1))x^\gamma$. □

b)

Proof. We choose ρ such that $0 < \rho < x < R$. We choose $f_0 \in C_0^2(\mathbb{R})$ such that $f_0 = f$ on (ρ, R). Define $\tau_{(\rho,R)} = \inf\{t > 0 : X_t \notin (\rho, R)\}$. Then by Dynkin's formula, and the fact $Af_0(x) = Af(x) = \gamma_1 x^{\gamma_1}(r + \frac{\alpha^2}{2}(\gamma_1 - 1)) = 0$ on (ρ, R), we get

$$E^x[f_0(X_{\tau_{(\rho,R)} \wedge k})] = f_0(x)$$

The condition $r < \frac{\alpha^2}{2}$ implies $X_t \to 0$ a.s. as $t \to 0$. So $\tau_{(\rho,R)} < \infty$ a.s.. Let $k \uparrow \infty$, by bounded convergence theorem and the fact $\tau_{(\rho,R)} < \infty$, we conclude

$$f_0(\rho)(1 - p(\rho)) + f_0(R)p(\rho) = f_0(x)$$

where $p(\rho) = P^x\{X_t \text{ exits } (\rho, R) \text{ by hitting R first}\}$. Then

$$\rho(p) = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}$$

Let $\rho \downarrow 0$, we get the desired result. □

c)

Proof. We consider $\rho > 0$ such that $\rho < x < R$. $\tau_{(\rho,R)}$ is the first exit time of X from (ρ, R). Choose $f_0 \in C_0^2(\mathbb{R})$ such that $f_0 = f$ on (ρ, R). By Dynkin's formula with $f(x) = \log x$ and the fact $Af_0(x) = Af(x) = r - \frac{\alpha^2}{2}$ for $x \in (\rho, R)$, we get

$$E^x[f_0(X_{\tau_{(\rho,R)}\wedge k})] = f_0(x) + (r - \frac{\alpha^2}{2})E^x[\tau_{(\rho,R)} \wedge k]$$

Since $r > \frac{\alpha^2}{2}$, $X_t \to \infty$ a.s. as $t \to \infty$. So $\tau_{(\rho,R)} < \infty$ a.s.. Let $k \uparrow \infty$, we get

$$E^x[\tau_{(\rho,R)}] = \frac{f_0(R)p(\rho) + f_0(\rho)(1 - p(\rho)) - f_0(x)}{r - \frac{\alpha^2}{2}}$$

where $p(\rho) = P^x(X_t$ exits (ρ, R) by hitting R first). To get the desired formula, we only need to show $\lim_{\rho\to 0} p(\rho) = 1$ and $\lim_{\rho\to 0} \log \rho(1 - p(\rho)) = 0$. This is trivial to see once we note by our previous calculation in part b),

$$p(\rho) = \frac{x^{\gamma_1} - \rho^{\gamma_1}}{R^{\gamma_1} - \rho^{\gamma_1}}$$

□

7.18 a)

Proof. The line of reasoning is exactly what we have done for 7.9 b). Just replace x^γ with a general function $f(x)$ satisfying certain conditions. □

b)

Proof. The characteristic operator $\mathcal{A} = \frac{1}{2}\frac{d^2}{dx^2}$ and $f(x) = x$ are such that $\mathcal{A}f(x) = 0$. By formula (7.5.10), we are done. □

c)

Proof. $\mathcal{A} = \mu\frac{d}{dx} + \frac{\sigma^2}{2}\frac{d^2}{dx^2}$. So we can choose $f(x) = e^{-\frac{2\mu}{\sigma^2}x}$. Therefore

$$p = \frac{e^{-\frac{2\mu x}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}{e^{-\frac{2\mu b}{\sigma^2}} - e^{-\frac{2\mu a}{\sigma^2}}}$$

□

8.6

Proof. The major difficulty is to make legitimate using Feymann-Kac formula while $(x - K)^+ \notin C_0^2$. For the conditions under which we can indeed apply Feymann-Kac formula to $(x - K)^+ \notin C_0^2$, c f. the book of Karatzas & Shreve, page 366. □

8.16 a)

Proof. Let $L_t = -\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(X_s)dB_s^i$. Then L is a square-integrable martingale. Furthermore, $\langle L\rangle_T = \int_0^T |\bigtriangledown h(X_s)|^2 ds$ is bounded, since $h \in C_0^1(\mathbb{R}^n)$. By Novikov's condition, $M_t = \exp\{L_t - \frac{1}{2}\langle L\rangle_t\}$ is a martingale. We define $\bar{P}$ on $\mathcal{F}_T$ by $d\bar{P} = M_T dP$. Then

$$dX_t = \bigtriangledown h(X_t)dt + dB_t$$

defines a BM under $\bar{P}$.

$$\begin{aligned}
& E^x[f(X_t)] \\
= \; & \bar{E}^x[M_t^{-1} f(X_t)] \\
= \; & \bar{E}^x[e^{\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(X_s)dX_s^i - \frac{1}{2}\int_0^t |\bigtriangledown h(X_s)|^2 ds} f(X_t)] \\
= \; & E^x[e^{\int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(B_s)dB_s^i - \frac{1}{2}\int_0^t |\bigtriangledown h(B_s)|^2 ds} f(B_t)]
\end{aligned}$$

Apply Ito's formula to $Z_t = h(B_t)$, we get

$$h(B_t) - h(B_0) = \int_0^t \sum_{i=1}^n \frac{\partial h}{\partial x_i}(B_s)dB_s^i + \frac{1}{2}\int_0^t \sum_{i=1}^n \frac{\partial^2 h}{\partial x_i^2}(B_s)ds$$

So

$$E^x[f(X_t)] = E^x[e^{h(B_t)-h(B_0)} e^{-\int_0^t V(B_s)ds} f(B_t)]$$

□

b)

Proof. If Y is the process obtained by killing B_t at a certain rate V, then it has transition operator

$$T_t^Y(g, x) = E^x[e^{-\int_0^t V(B_s)ds} g(B_t)]$$

So the equality in part a) can be written as

$$T_t^X(f, x) = e^{-h(x)} T_t^Y(fe^h, x)$$

□

9.11 a)

Proof. First assume F is closed. Let $\{\phi_n\}_{n\geq 1}$ be a sequence of bounded continuous functions defined on ∂D such that $\phi_n \to 1_F$ boundedly. This is possible due to Tietze extension theorem. Let $h_n(x) = E^x[\phi_n(B_\tau)]$. Then by Theorem 9.2.14, $h_n \in C(\bar{D})$ and $\Delta h_n(x) = 0$ in D. So by Poisson formula, for $z = re^{i\theta} \in D$,

$$h_n(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta) h_n(e^{it})dt$$

Let $n \to \infty$, $h_n(z) \to E^x[1_F(B_\tau)] = P^x(B_\tau \in F)$ by bounded convergence theorem, and $RHS \to \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$ by dominated convergence theorem. Hence

$$P^z(B_\tau \in F) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$$

Then by $\pi - \lambda$ theorem and the fact Borel σ-field is generated by closed sets, we conclude

$$P^z(B_\tau \in F) = \frac{1}{2\pi}\int_0^{2\pi} P_r(t-\theta)1_F(e^{it})dt$$

for any Borel subset of ∂D. □

b)

Proof. Let B be a BM starting at 0. By example 8.5.9, $\phi(B_t)$ is, after a change of time scale $\alpha(t)$ and under the original probability measure P, a BM in the plane. $\forall F \in \mathcal{B}(\mathbb{R})$,

$$\begin{aligned}
& P(B \text{ exits } D \text{ from } \psi(F)) \\
= \ & P(\phi(B) \text{ exits upper half plane from } F) \\
= \ & P(\phi(B)_{\alpha(t)} \text{ exits upper half plane from } F) \\
= \ & \text{Probability of BM starting at i that exits from } F \\
= \ & \mu(F)
\end{aligned}$$

So by part a), $\mu(F) = \frac{1}{2\pi}\int_0^{2\pi} 1_{\psi(F)}(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} 1_F(\phi(e^{it}))dt$. This implies

$$\int_R f(\xi)d\mu(\xi) = \frac{1}{2\pi}\int_0^{2\pi} f(\phi(e^{it}))dt = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\phi(z))}{z}dz$$

□

c)

Proof. By change-of-variable formula,

$$\int_R f(\xi)d\mu(\xi) = \frac{1}{\pi}\int_{\partial H} f(\omega)\frac{d\omega}{|\omega - i|^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\frac{dx}{x^2+1}$$

□

d)

Proof. Let $g(z) = u + vz$, then g is a conformal mapping that maps i to $u + vi$ and keeps upper half plane invariant. Use the harmonic measure on x-axis of a BM starting from i, and argue as above in part a)-c), we can get the harmonic measure on x-axis of a BM starting from $u + iv$. □

12.1 a)

Proof. Let θ be an arbitrage for the market $\{X_t\}_{t\in[0,T]}$. Then for the market $\{\bar{X}_t\}_{t\in[0,T]}$:
(1) θ is self-financing, i.e. $d\bar{V}_t^\theta = \theta_t d\bar{X}_t$. This is (12.1.14).
(2) θ is admissible. This is clear by the fact $\bar{V}_t^\theta = e^{-\int_0^t \rho_s ds}V_t^\theta$ and ρ being bounded.
(3) θ is an arbitrage. This is clear by the fact $V_t^\theta > 0$ if and only if $\bar{V}_t^\theta > 0$.

So $\{\bar{X}_t\}_{t\in[0,T]}$ has an arbitrage if $\{X_t\}_{t\in[0,T]}$ has an arbitrage. Conversely, if we replace ρ with $-\rho$, we can calculate X has an arbitrage from the assumption that $\bar{X}$ has an aribitrage. □

12.2

Proof. By $V_t = \sum_{i=0}^n \theta_i X_i(t)$, we have $dV_t = \theta \cdot dX_t$. So θ is self-financing. □

12.6 (e)

Proof. Arbitrage exists, and one hedging strategy could be $\theta = (0, B_1 + B_2, B_1 - B_2 + \frac{1-3B_1+B_2}{5}, \frac{1-3B_1+B_2}{5})$. The final value would then become $B_1(T)^2 + B_2(T)^2$. □

12.10

Proof. Becasue we want to represent the contingent claim in terms of original BM B, the measure Q is the same as P. Solving SDE $dX_t = \alpha X_t dt + \beta X_t dB_t$ gives us $X_t = X_0 e^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$. So

$$\begin{aligned}
& E^y[h(X_{T-t})] \\
= \; & E^y[X_{T-t}] \\
= \; & y e^{(\alpha-\frac{\beta^2}{2})(T-t)} e^{\frac{\beta^2}{2}(T-t)} \\
= \; & y e^{\alpha(T-t)}
\end{aligned}$$

Hence $\phi = e^{\alpha(T-t)}\beta X_t = \beta X_0 e^{\alpha T - \frac{\beta^2}{2}t + \beta B_t}$. □

12.11 a)

Proof. According to (12.2.12), $\sigma(t,\omega) = \sigma$, $\mu(t,\omega) = m - X_1(t)$. So $u(t,\omega) = \frac{1}{\sigma}(m - X_1(t) - \rho X_1(t))$. By (12.2.2), we should define Q by setting

$$dQ|_{\mathcal{F}_t} = e^{-\int_0^t u_s dB_s - \frac{1}{2}\int_0^t u_s^2 ds} dP$$

Under Q, $\tilde{B}_t = B_t + \frac{1}{\sigma}\int_0^t (m - X_1(s) - \rho X_1(s))ds$ is a BM. Then under Q,

$$dX_1(t) = \sigma d\tilde{B}_t + \rho X_1(t)dt$$

So $X_1(T)e^{-\rho T} = X_1(0) + \int_0^T \sigma e^{-\rho t} d\tilde{B}_t$ and $E_Q[\xi(T)F] = E_Q[e^{-\rho T}X_1(T)] = x_1$. □

b)

Proof. We use Theorem 12.3.5. From part a), $\phi(t,\omega) = e^{-\rho t}\sigma$. We therefore should choose $\theta_1(t)$ such that $\theta_1(t)e^{-\rho t}\sigma = \sigma e^{-\rho t}$. So $\theta_1 = 1$ and θ_0 can then be chosen as 0. □

2 Extra Problems

EP1-1.

Proof. According to Borel-Cantelli lemma, the problem is reduced to proving $\forall\epsilon$,

$$\sum_{n=1}^{\infty} P(|S_n| > \epsilon) < \infty$$

where $S_n := \sum_{j=1}^{n}(B_{j/n} - B_{(j-1)/n})^2 - 1$. Set

$$X_j = (B_{j/n} - B_{(j-1)/n})^2 - 1/n$$

By the hint, if we consider the i.i.d. sequence $\{X_j\}_{j=1}^n$ normalized by its 4-th moment, we have

$$P(|S_n| > \epsilon) < \epsilon^{-4}E[S_n^4] \le \epsilon^{-4}CE[X_1^4]n^2$$

By integration-by-parts formula, we can easily calculate the 2k-th moment of $N(0,\sigma)$ is of order σ^k. So the order of $E[X_1^4]$ is n^{-4}. This suffices for the Borel-Cantelli lemma to apply. □

EP1-2.

Proof. We first see the second part of the problem is not hard, since $\int_0^t Y_s dB_s$ is a martingale with mean 0. For the first part, we do the following construction. We define $Y_t = 1$ for $t \in (0, 1/n]$, and for $t \in (j/n, (j+1)/n]$ $(1 \le j \le n-1)$

$$Y_t := C_j 1_{\{B_{(i+1)/n} - B_{i/n} \le 0, \quad 0 \le i \le j-1\}}$$

where each C_j is a constant to be determined.

Regarding this as a betting strategy, the intuition of Y is the following: We start with one dollar, if $B_{1/n} - B_0 > 0$, we stop the game and gain $(B_{1/n} - B_0)$ dollars. Otherwise, we bet C_1 dollars for the second run. If $B_{2/n} - B_{1/n} > 0$, we then stop the game and gain $C_1(B_{2/n} - B_{1/n}) - (B_{1/n} - B_0)$ dollars (if the difference is negative, it means we actually lose money, although we win the second bet). Otherwise, we bet C_2 dollar for the third run, etc. So in the end our total gain/loss of this betting is

$$\begin{aligned}\int_0^t Y_s dB_s &= (B_{1/n} - B_0) + 1_{\{B_{1/n} - B_0 \le 0\}} C_1(B_{2/n} - B_{1/n}) + \cdots \\ &+ 1_{\{B_{1/n} - B_0 \le 0, \cdots, B_{(n-1)/n} - B_{(n-2)/n} \le 0\}} C_{n-1}(B_1 - B_{(n-1)/n})\end{aligned}$$

We now look at the conditions unde which $\int_0^1 Y_s dB_s \le 0$. There are several possibilities:
(1) $(B_{1/n} - B_0) \le 0$, $(B_{2/n} - B_{1/n}) > 0$, but $C_1(B_{2/n} - B_{1/n}) < |B_{1/n} - B_0|$;
(2) $(B_{1/n} - B_0) \le 0$, $(B_{2/n} - B_{1/n}) \le 0$, $(B_{3/n} - B_{2/n}) > 0$, but $C_2(B_{3/n} - B_{2/n}) < |B_{1/n} - B_0| + C_1|B_{2/n} - B_{1/n}|$;
$\cdots\cdots$;
(n) $(B_{1/n} - B_0) \le 0$, $(B_{2/n} - B_{1/n}) \le 0$, $\cdots$, $(B_1 - B_{(n-1)/n}) \le 0$.

The last event has the probability of $(1/2)^n$. The first event has the probability of

$$P(X \leq 0, Y > 0, 0 < Y < X/C_1) \leq P(0 < Y < X/C_1)$$

where X and Y are i.i.d. $N(0,1/n)$ random variables. We can choose C_1 large enough so that this probability is smaller than $1/2^n$. The second event has the probability smaller than $P(0 < X < Y/C_2)$, where X and Y are independent Gaussian random variables with 0 mean and variances $1/n$ and $(C_1^2+1)/n$, respectively, we can choose C_2 large enough, so that this probability is smaller than $1/2^n$. We continue this process untill we get all the C_j's. Then the probability of $\int_0^1 Y_t dB_t \leq 0$ is at most $n/2^n$. For n large enough, we can have $P(\int_0^1 Y_t dB_t > 0) > 1-\epsilon$ for given ϵ. The process Y is obviously bounded. □

Comments: Different from flipping a coin, where the gain/loss is one dollar, we have now random gain/loss $(B_{j/n} - B_{(j-1)/n})$. So there is no sense checking our loss and making new strategy constantly. Put it into real-world experience, when times are tough and the outcome of life is uncertain, don't regret your loss and estimate how much more you should invest to recover that loss. Just keep trying as hard as you can. When the opportunity comes, you may just get back everything you deserve.

EP2-1.

Proof. This is another application of the fact hinted in Problem EP1-1. $E[Y_n] = 0$ is obvious. And

$$\begin{aligned} & E[(B^1_{j/n} - B^1_{(j-1)/n})^4 (B^2_{j/n} - B^2_{(j-1)/n})^4] \\ = \; & (3E[(B^1_{j/n} - B^1_{(j-1)/n})^2]^2)^2 \\ = \; & \frac{9}{n^4} \\ := \; & a_n \end{aligned}$$

We set $X_j = [B^1_{j/n} - B^1_{(j-1)/n}][B^2_{j/n} - B^2_{(j-1)/n}]/a_n^{\frac{1}{4}}$, and apply the hint in EP1-1,

$$E[Y_n^4] = a_n E(X_1 + \cdots + X_n)^4 \leq \frac{9}{n^4} cn^2 = \frac{9c}{n^2}$$

for some constant c. This implies $Y_n \to 0$ with probability one, by Borel-Cantelli lemma. □

Comments: This following simple proposition is often useful in calculation. If X is a centered Gaussian random variable, then $E[X^4] = 3E[X^2]^2$. Furthermore, we can show $E[X^{2k}] = C_k E[X^{2k-2}]^2$ for some constant C_k. These results can be easily proved by integration-by-part formula. As a consequence, $E[B_t^{2k}] = Ct^k$ for some constant C.

EP3-1.

Proof. *A short proof:* For part (a), it suffices to set

$$Y_{n+1} = E[R_{n+1} - R_n | X_1, \cdots, X_{n+1} = 1]$$

(*What does this really mean, rigorously?*). For part (b), the answer is NO, and $R_n = \sum_{j=1}^n X_j^3$ gives the counter example.

A long proof:

We show the analysis behind the above proof and point out if $\{X_n\}_n$ is i.i.d. and symmetrically distributed, then Bernoulli type random variables are the only ones that have martingale representation property.

By adaptedness, $R_{n+1} - R_n$ can be represented as $f_{n+1}(X_1, \cdots, X_{n+1})$ for some Borel function $f_{n+1} \in \mathcal{B}(\mathbb{R}^{n+1})$. Martingale property and $\{X_n\}_n$ being i.i.d. Bernoulli random variables imply

$$f_{n+1}(X_1, \cdots, X_n, -1) = -f_{n+1}(X_1, \cdots, X_n, 1)$$

This inspires us set Y_{n+1} as

$$f_{n+1}(X_1, \cdots, X_n, 1) = E[R_{n+1} - R_n | X_1, \cdots, X_{n+1} = 1].$$

For part b), we just assume $\{X_n\}_n$ is i.i.d. and symmetrically distributed. If $(R_n)_n$ has martingale representation property, then

$$f_{n+1}(X_1, \cdots, X_{n+1})/X_{n+1}$$

must be a function of $X_1, \cdots, X_n$. In particular, for $n = 0$ and $f_1(x) = x^3$, we have X_1^2 =constant. So Bernoulli type random variables are the only ones that have martingale representation theorem.

□

EP5-1.

Proof. $\mathcal{A} = \frac{r}{x}\frac{d}{dx} + \frac{1}{2}\frac{d^2}{dx^2}$, so we can choose $f(x) = x^{1-2r}$ for $r \neq \frac{1}{2}$ and $f(x) = \log x$ for $r = \frac{1}{2}$. □

EP6-1. (a)

Proof. Assume the claim is false, then there exists $t_0 > 0$, $\epsilon > 0$ and a sequence $\{t_k\}_{k\geq 1}$ such that $t_k \uparrow t_0$, and

$$\left| \frac{f(t_k) - f(t_0)}{t_k - t_0} - f'_+(t_0) \right| > \epsilon$$

WLOG, we assume $f'_+(t_0) = 0$, otherwise we consider $f(t) - tf'_+(t_0)$. Because f'_+ is continuous, there exists $\delta > 0$, such that $\forall t \in (t_0 - \delta, t_0 + \delta)$,

$$|f'_+(t) - f'_+(t_0)| = |f'_+(t)| < \frac{\epsilon}{2}$$

Meanwhile, there exists infinitely many t_k's such that

$$\frac{f(t_k) - f(t_0)}{t_k - t_0} > \epsilon \text{ or } \frac{f(t_k) - f(t_0)}{t_k - t_0} < -\epsilon$$

By considering f or $-f$ and taking a subsequence, we can WLOG assume for all the t_k's, $t_k \in (t - \delta, t + \delta)$, and

$$\frac{f(t_k) - f(t_0)}{t_k - t_0} - f'_+(t_0) > \epsilon$$

Consider $h(t) = \epsilon(t-t_0) - [f(t) - f(t_0)] = (t-t_0)\left[\epsilon - \frac{f(t)-f(t_0)}{t-t_0}\right]$. Then $h(t_0) = 0$, $h'_+(t) = \epsilon - f'_+(t) > \epsilon/2$ for $t \in (t_0-\delta, t_0+\delta)$, and $h(t_k) > 0$. On one hand,

$$\int_{t_k}^{t_0} h'_+(t)dt > \frac{\epsilon}{2}(t_0 - t_k) > 0$$

On the other hand, if h is monotone increasing, then

$$\int_{t_k}^{t_0} h'_+(t)dt \leq h(t_0) - h(t_k) = 0 - h(t_k) < 0$$

Contradiction.

So it suffices to show h is monotone increasing on $(t_0-\delta, t_0+\delta)$. This is easily proved by showing h cannot obtain local maximum in the interior of $(t_0-\delta, t_0+\delta)$. □

(b)

Proof. $f(t) = |t-1|$. □

(c)

Proof. $f(t) = 1_{\{t\geq 0\}}$. □

EP6-2. (a)

Proof. Since A is bounded, $\tau < \infty$ a.s..

$$\begin{aligned} E^x[M_{n+1} - M_n|\mathcal{F}_n] &= E^x[f(S_{n+1}) - f(S_n)|\mathcal{F}_n]1_{\{\tau\geq n+1\}} \\ &= (E^{S_n}[f(S_1)] - f(S_n))1_{\{\tau\geq n+1\}} \\ &= \Delta f(S_n)1_{\{\tau\geq n+1\}} \end{aligned}$$

Because $S_n \in A$ on $\{\tau \geq n+1\}$ and f is harmonic on $\bar{A}$, $\Delta f(S_n)1_{\{\tau\geq n+1\}} = 0$. So M is a martingale. □

(b)

Proof. For existence, set $f(x) = E^x[F(S_\tau)]$ $(x \in \bar{A})$, where $\tau = \inf\{n \geq 0 : S_n \notin A\}$. Clearly $f(x) = F(x)$ for $x \in \partial A$. For $x \in A$, $\tau \geq 1$ under P^x, and we have

$$\begin{aligned} \Delta f(x) &= E^x[f(S_1)] - f(x) \\ &= E^x[E^{S_1}[F(S_\tau)]] - f(x) \\ &= E^x[E^x[F(S_\tau)\circ\theta_1|S_1]] - f(x) \\ &= E^x[F(S_\tau)\circ\theta_1] - f(x) \\ &= E^x[F(S_\tau)] - f(x) \\ &= 0 \end{aligned}$$

For the 5th equality, we used the fact under P^x, $\tau \geq 1$ and hence $S_\tau \circ \theta_1 = S_\tau$.

For uniqueness, by part a), $f(S_{n\wedge\tau})$ is a martingale, so use optimal stopping time, we have

$$f(x) = E^x[f(S_0)] = E^x[f(S_{n\wedge\tau})]$$

Becasue f is bounded, we can use bounded convergence theorem and let $n \uparrow \infty$,

$$f(x) = E^x[f(S_\tau)] = E^x[F(S_\tau)]$$

□

(c)

Proof. Since $d \leq 2$, the random walk is recurrent. So $\tau < \infty$ a.s. even if A is bounded. The existence argument is exactly the same as part b). For uniqueness, we still have $f(x) = E^x[f(S_{n\wedge\tau})]$. Since f is bounded, we can let $n \uparrow \infty$, and get $f(x) = E^x[F(S_\tau)]$. □

(d)

Proof. Let $d = 1$ and $A = \{1, 2, 3, ...\}$. Then $\partial A = \{0\}$. If $F(0) = 0$, then both $f(x) = 0$ and $f(x) = x$ are solutions of the discrete Dirichlet problem. We don't have uniqueness. □

(e)

Proof. $A = \mathbb{Z}^3 - \{0\}$, $\partial A = \{0\}$, and $F(0) = 0$. $T_0 = \inf\{n \geq 0 : S_n \geq 0\}$. Let $c \in \mathbb{R}$ and $f(x) = cP^x(T_0 = \infty)$. Then $f(0) = 0$ since $T_0 = 0$ under P^0. f is clearly bounded. To see f is harmonic, the key is to show $P^x(T_0 = \infty | S_1 = y) = P^y(T_0 = \infty)$. This is due to Markov property: note $T_0 = 1 + T_0 \circ \theta_1$. Since c is arbitrary, we have more than one bounded solution.

□

EP6-3.

Proof.

$$\begin{aligned}
E^x[K_n - K_{n-1}|\mathcal{F}_{n-1}] &= E^x[f(S_n) - f(S_{n-1})|\mathcal{F}_{n-1}] - \Delta f(S_{n-1}) \\
&= E^{S_{n-1}}[f(S_1)] - f(S_{n-1}) - \Delta f(S_{n-1}) \\
&= \Delta f(S_{n-1}) - \Delta f(S_{n-1}) \\
&= 0
\end{aligned}$$

Applying Dynkin's formula is straightforward. □

EP6-4. (a)

Proof. By induction, it suffices to show if $|y - x| = 1$, then $E^y[T_A] < \infty$. We note $T_A = 1 + T_A \circ \theta_1$ for any sample path starting in A. So

$$E^x[T_A 1_{\{S_1\}}] = E^x[T_A | S_1 = y] P^x(S_1 = y) = E^y[T_A - 1] P^x(S_1 = y)$$

Since $E^x[T_A 1_{\{S_1\}}] \leq E^x[T_A] < \infty$ and $P^x(S_1 = y) > 0$, $E^y[T_A] < \infty$. □

(b)

Proof. If $y \in \partial A$, then under P^y, $T_A = 0$. So $f(y) = 0$. If $y \in A$,

$$\begin{aligned}
\Delta f(y) &= E^y[f(S_1)] - f(y) \\
&= E^y[E^y[T_A \circ \theta_1 | S_1]] - f(y) \\
&= E^y[E^y[T_A - 1 | S_1]] - f(y) \\
&= E^y[T_A] - 1 - f(y) \\
&= -1
\end{aligned}$$

To see uniqueness, use the martingale in EP6-3 for any solution f, we get

$$E^x[f(S_{T_A \wedge K})] = f(x) + E^x[\sum_{j=0}^{T_A - 1} \Delta f(S_j)] = f(x) - E^x[T_A]$$

Let $K \uparrow \infty$, we get $0 = f(x) - E^x[T_A]$.

□

EP7-1. a)

Proof. Since D is bounded, there exists $R > 0$, such that $D \subset\subset B(0, R)$. Let $\tau_R := \inf\{t > 0 : |B_t - B_0| \geq R\}$, then $\tau \leq \tau_R$. If $q \geq -\epsilon$

$$e(x) = E^x[e^{\epsilon\tau}] \leq E^x[e^{\epsilon\tau_R}] = E^x[\int_0^{\tau_R} \epsilon e^{\epsilon t} dt + 1] = 1 + \int_0^\infty P^x(\tau_R > t)\epsilon e^{\epsilon t} dt$$

For any $n \in \mathbb{N}$, $P^x(\tau_R > n) \leq P^x(\cap_{i=1}^n \{|B_k - B_{k-1}| < 2R\}) = a^n$, where $a = P^x(|B_1 - B_0| < 2R) < 1$. So $e(x) \leq 1 + \epsilon e^\epsilon \sum_{n=1}^\infty (ae^\epsilon)^{n-1}$. For ϵ small enough, $ae^\epsilon < 1$, and hence $e(x) < \infty$. Obviously, ϵ is only dependent on D. □

c)

Proof. Since q is continuous and $\bar{D}$ is compact, q attains its minimum M. If $M \geq 0$, then we have nothing to prove. So WLOG, we assume $M < 0$. Then similar to part a),

$$\tilde{e}(x) \leq E^x[e^{-M(\tau \wedge \sigma_\epsilon)}] \leq E^x[e^{-M\sigma_\epsilon}] = 1 + \int_0^\infty P^x(\sigma_\epsilon > t)(-M)e^{-Mt} dt$$

Note $P^x(\sigma_\epsilon > t) = P^x(\sup_{s \leq t} |B_s - B_0| < \epsilon) = P^0(\sup_{s \leq t} |\epsilon B_{s/\epsilon^2}| < \epsilon) = P^x(\sigma_1 > t/\epsilon^2)$. So $\tilde{e}(x) = 1 + \int_0^\infty P^x(\sigma_1 > u)(-M\epsilon^2)e^{-M\epsilon^2 u} du = E^x[e^{-M\epsilon^2 \sigma_1}]$. For ϵ small enough, $-M\epsilon^2$ will be so small that, by what we showed in the proof of part a), $E^x[e^{-M\epsilon^2\sigma_1}]$ will be finite. Obviously, ϵ is dependent on M and D only, hence q and D only. □

d)

Proof. Cf. Rick Durrett's book, *Stochastic Calculus: A Practical Introduction*, page 158-160. □

b)

Proof. From part d), it suffices to show for a give x, there is a $K = K(D, x) < \infty$, such that if $q = -K$, then $e(x) = \infty$. Since D is open, there exists $r > 0$, such that $B(x, r) \subset\subset D$.

Now we assume $q = -K < 0$, where K is to be determined. We have

$$e(x) = E^x[e^{K\tau}] \geq E^x[e^{K\tau_r}].$$

Here $\tau_r := \inf\{t > 0 : |B_t - B_0| \geq r\}$. Similar to part a), we have

$$E^x[e^{K\tau_r}] \geq 1 + \sum_{n=1}^{\infty} P^x(\tau_r \geq n)e^{kn}(1 - e^{-k})$$

So it suffices to show there exists $\delta > 0$, such that $P^x(\tau_r \geq n) \geq \delta^n$.

Note

$$P^x(\tau_r > n) = P^x(\max_{t \leq n} |B_t - B_0| < r) \geq P^x(\max_{t \leq n} |B_t^i - B_0^i| < C(d)r, i \leq d),$$

where B^i is the i-th coordinate of B and $C(d)$ is a constant dependent on d. Set $a = C(d)r$, then by independence

$$P^x(\tau_r > n) \geq P^0(\max_{t \leq n} |W_t| < a)^d$$

Here W is a standard one-dimensional BM. Let

$$\delta = \inf_{-\frac{a}{2} < x < \frac{a}{2}} P^x(\max_{t \leq 1} |W_t| < a, |W_0| < a/2, |W_1| < a/2)(> 0)$$

then we have

$$\begin{aligned}
& P^0(\max_{t \leq n} |W_t| < a) \\
\geq\ & P^0(\cap_{k=1}^{n}\{\max_{k-1 \leq t \leq k} |W_t| < a, |W_{k-1}| < \frac{a}{2}, |W_k| < \frac{a}{2}\}) \\
=\ & P^0(\{\max_{n-1 \leq t \leq n} |W_t| < a, |W_{n-1}| < \frac{a}{2}, |W_n| < \frac{a}{2}\} | \cap_{k=1}^{n-1} \\
& \{\max_{k-1 \leq t \leq k} |W_t| < a, |W_{k-1}| < \frac{a}{2}, |W_k| < \frac{a}{2}\}) \\
& \times P^0(\cap_{k=1}^{n-1}\{\max_{k-1 \leq t \leq k} |W_t| < a, |W_{k-1}| < \frac{a}{2}, |W_k| < \frac{a}{2}\}) \\
\geq\ & \delta P^0(\cap_{k=1}^{n-1}\{\max_{k-1 \leq t \leq k} |W_t| < a, |W_{k-1}| < \frac{a}{2}, |W_k| < \frac{a}{2}\})
\end{aligned}$$

The last line is due to Markov property. By induction we have

$$P^0(\max_{t \leq n} |W_t| < a) > \delta^n,$$

and we are done. □

EP7-2.

Proof. Consider the case of dimension 1. $D = \{x : x > 0\}$. Then for any $x > 0$, $P^x(\tau < \infty) = 1$. But by $P^x(\tau \in dt) = \frac{x}{2\pi t^3} e^{-\frac{x^2}{2t}} dt$, we can calculate that $E^x[\tau] = \infty$. So for every $\epsilon > 0$, $E^x[e^{\epsilon\tau}] \geq e^{\epsilon E[\tau]} = \infty$. □

EP8-1. a)

Proof.

$$E[e^{aX_1}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}+ax} dx = e^{\frac{a^2}{2}}$$

So $E[X_1 e^{aX_1}] = ae^{\frac{a^2}{2}}$. □

b)

Proof. We note $Z_n \in \mathcal{F}_n$ and X_{n+1} is independent of $\mathcal{F}_n$, so we have

$$\begin{aligned} & E[\frac{M_{n+1}}{M_n}|\mathcal{F}_n] \\ = & E[e^{-f(Z_n)X_{n+1}-\frac{1}{2}f^2(Z_n)}|\mathcal{F}_n] \\ = & E[e^{-f(z)X_{n+1}-\frac{1}{2}f^2(z)}]|_{z=Z_n} = e^{\frac{1}{2}f^2(Z_n)-\frac{1}{2}f^2(Z_n)} = 1 \end{aligned}$$

So $(M_n)_{n\geq 0}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq 0}$. □

c)

Proof.

$$\begin{aligned} & E[M_{n+1}Z_{n+1} - M_n Z_n|\mathcal{F}_n \\ = & M_n E[\frac{M_{n+1}}{M_n} Z_{n+1} - Z_n|\mathcal{F}_n] \\ = & M_n E[\frac{M_{n+1}}{M_n}(Z_n + f(Z_n) + X_{n+1}) - Z_n|\mathcal{F}_n] \\ = & M_n E[Z_n + f(Z_n) - Z_n + E[\frac{M_{n+1}}{M_n} X_{n+1}|\mathcal{F}_n]] \\ = & M_n[f(Z_n) + E[X_{n+1}e^{-f(Z_n)X_{n+1}-\frac{1}{2}f^2(Z_n)}|\mathcal{F}_n]] \\ = & M_n[f(Z_n) - f(Z_n)] \\ = & 0 \end{aligned}$$

So $(M_n Z_n)_{n\geq 0}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\geq 0}$. □

d)

Proof. $\forall A \in \mathcal{F}_n$, $E^Q[Z_{n+1}; A] = E^P[M_{n+1}Z_{n+1}; A] = E^P[M_n Z_n; A] = E^Q[Z_n; A]$. So $E^Q[Z_{n+1}|\mathcal{F}_n] = Z_n$, that is, Z_n is a Q-martingale. □

EP8-2. a)

Proof. Let $Z_t = \exp\{\int_0^{t\wedge T_\epsilon} \frac{\alpha(\alpha-1)}{2B_s^2} ds\}$. Note $B^\alpha_{t\wedge T_\epsilon} = (\int_0^t 1_{\{s\le T_\epsilon\}} dB_s)^\alpha$, we have

$$dB^\alpha_{t\wedge T_\epsilon} = \alpha B^{\alpha-1}_{t\wedge T_\epsilon} 1_{\{t\le T_\epsilon\}} dB_t + \frac{\alpha(\alpha-1)}{2} B^{\alpha-2}_{t\wedge T_\epsilon} 1_{\{t\le T_\epsilon\}} dt$$

So $M_t = B^\alpha_{t\wedge T_\epsilon} Z_t$ satisfies

$$dM_t = B^\alpha_{t\wedge T_\epsilon} dZ_t + Z_t \alpha B_t^{\alpha-1} 1_{\{t\le T_\epsilon\}} dB_t + Z_t \frac{\alpha(\alpha-1)}{2} B_t^{\alpha-2} 1_{\{t\le T_\epsilon\}} dt$$

Meanwhile, $dZ_t = \frac{\alpha(\alpha-1)}{2B_t^2} 1_{\{t\le T_\epsilon\}} e^{\int_0^t \frac{\alpha(1-\alpha)}{2B_s^2} ds} dt$. So

$$B^\alpha_{t\wedge T_\epsilon} dZ_t + \frac{\alpha(\alpha-1)}{2} 1_{\{t\le T_\epsilon\}} B_t^{\alpha-2} Z_t dt = 0$$

Hence $dM_t = Z_t \alpha B_t^{\alpha-1} 1_{\{t\le T_\epsilon\}} dB_t$. To check M is a martingale, we note we actually have

$$E[\int_0^T Z_t^2 \alpha^2 B_t^{2\alpha-2}] 1_{\{t\le T_\epsilon\}} dt < \infty.$$

Indeed, $Z_t^2 1_{\{t\le T_\epsilon\}} \le e^{\frac{\alpha|1-\alpha|}{2\epsilon^2} T}$. If $\alpha \le t$, $B_t^{2\alpha-2} 1_{\{t\le T_\epsilon\}} \le \epsilon^{2\alpha-2}$; if $\alpha > 1$, $E[B_t^{2\alpha-2} 1_{\{t\le T_\epsilon\}}] \le t^{\alpha-1}$. Hence M is martingale. □

b)

Proof. Under Q, $Y_t = B_t - \int_0^t \frac{1}{M_s} d\langle M, B\rangle_s$ is a BM. We take $A_t = -\frac{\alpha}{B_t} 1_{\{t\le T_\epsilon\}}$. The SDE for B in terms of Y_t is

$$dB_t = dY_t + \frac{\alpha}{B_t} 1_{\{t\le T_\epsilon\}} dt$$

□

c)

Proof. Under Q, B satisfies the Bessel diffusion process before it hits $\frac{1}{2}$. That is, up to the time $T_{\frac{1}{2}}$, B satisfies the equation

$$dB_t = dY_t + \frac{\alpha}{B_t} dt$$

This line may sound fishy as we haven't defined what it means by an SDE defined up to a random time. Actually, a rigorous theory can be built for this notion. But we shall avoid this theoretical issue at this moment.

We choose $b > 1$, and define $\tau_b = \inf\{t > 0 : B_t \notin (\frac{1}{2}, b)\}$. Then $Q^1(T_{\frac{1}{2}} = \infty) = \lim_{b\to\infty} Q^1(B_{\tau_b} = b)$. By the results in EP5-1 and Problem 7.18 in Oksendal's book, we have

(i) If $\alpha > 1/2$, $\lim_{b\to\infty} Q^1(B_{\tau_b} = b) = \lim_{b\to\infty} \frac{1-(\frac{1}{2})^{1-2\alpha}}{b^{1-2\alpha}-(\frac{1}{2})^{1-2\alpha}} = 1 - (\frac{1}{2})^{2\alpha-1} > 0$. So in this case, $Q^1(T_{\frac{1}{2}} = \infty) > 0$.

(ii) If $\alpha < 1/2$, $\lim_{b\to\infty} Q^1(B_{\tau_b} = b) = \lim_{b\to\infty} \frac{1-(\frac{1}{2})^{1-2\alpha}}{b^{1-2\alpha}-(\frac{1}{2})^{1-2\alpha}} = 0$. So in this case, $Q^1(T_{\frac{1}{2}} = \infty) = 0$.

(iii) If $\alpha = 1/2$, $\lim_{b\to\infty} Q^1(B_{\tau_b} = b) = \lim_{b\to\infty} \frac{0-\log\frac{1}{2}}{\log b - \log\frac{1}{2}} = 0$. So in this case, $Q^1(T_{\frac{1}{2}} = \infty) = 0$.

□

EP9-1. a)

Proof. Fix $z \in D$, consider $A = \{\omega \in D : \rho_D(z,\omega) < \infty\}$. Then A is clearly open. We show A is also closed. Indeed, if $\omega_k \in A$ and $\omega_k \to \omega_* \in D$, then for k sufficiently large, $|\omega_k - \omega_*| < \frac{1}{2}\mathrm{dist}(\omega_*, \partial D)$. So ω_k and ω_* are adjacent. By definition, $\rho_D(\omega_*, z) < \infty$, i.e. $\omega_* \in A$.

Since D is connected, and A is both closed and open, we conclude $A = D$. By the arbitrariness of z, $\rho_D(z,\omega) < \infty$ for any z, $\omega \in D$.

To see ρ_D is a metric on D, note $\rho_D(z,z) = 0$ by definition and $\rho(z,\omega) \geq 1$ for $z \neq \omega$. So $\rho_D(z,\omega) = 0$ iff $z = \omega$. If $\{x_k\}$ is a finite adjacent sequence connecting z_1 and z_2, and $\{y_l\}$ is a finite adjacent sequence connecting z_2 and z_3, then $\{x_k, z_2, y_l\}_{k,l}$ is a finite adjacent sequence connecting z_1 and z_3. So $\rho_D(z_1,z_3) \leq \rho_D(z_1,z_2) + \rho_D(z_2,z_3)$. Meanwhile, it's clear that $\rho_D(z,\omega) \geq$ and $\rho_D(z,\omega) = \rho_D(\omega,z)$. So ρ_D is a metric. □

b)

Proof. $\forall z \in U_k$, then $\rho_D(z_0,z) \leq k$. Assume $z_0 = x_0$, x_1, $\cdots$, $x_k = z$ is a finite adjacent sequence. Then $|z - x_{k-1}| < \frac{1}{2}\max\{\mathrm{dist}(z,\partial D), \mathrm{dist}(x_{k-1},\partial D)\}$. For ω close to z,

$$|\omega - x_{k-1}| \leq |z-\omega| + |z - x_{k-1}| < \frac{1}{2}\max\{\mathrm{dist}(\omega,\partial D), \mathrm{dist}(x_{k-1},\partial D)\}.$$

Indeed, if $\mathrm{dist}(x_{k-1}, D) > \mathrm{dist}(z,\partial D)$, then for ω close to z, $\mathrm{dist}(\omega,\partial D)$ is also close to $\mathrm{dist}(z,\partial D)$, and hence $< \mathrm{dist}(x_{k-1},\partial D)$. Choose ω such that $|z-\omega| < \frac{1}{2}\mathrm{dist}(x_{k-1},\partial D) - |z - x_{k-1}|$, we then have

$$\begin{aligned}
& |\omega - x_{k-1}| \\
\leq\ & |z-\omega| + |z - x_{k-1}| \\
<\ & \frac{1}{2}\mathrm{dist}(x_{k-1},\partial D) \\
=\ & \frac{1}{2}\max(\mathrm{dist}(x_{k-1},\partial D), \mathrm{dist}(\omega,\partial D))
\end{aligned}$$

If $\mathrm{dist}(x_{k-1},\partial D) \leq \mathrm{dist}(z,\partial D)$, then for ω close to z, $\frac{1}{2}\max\{\mathrm{dist}(\omega,\partial D), \mathrm{dist}(x_{k-1},\partial D)\}$ is very close to $\frac{1}{2}\max\{\mathrm{dist}(z,\partial D), \mathrm{dist}(x_{k-1},\partial D)\} = \frac{1}{2}\mathrm{dist}(z,\partial D)$. Hence, for ω close to z,

$$|\omega - x_{k-1}| \leq |z-\omega| + |z - x_{k-1}| < \frac{1}{2}\max(\mathrm{dist}(x_{k-1},\partial D), \mathrm{dist}(\omega,\partial D))$$

Therefore ω and x_{k-1} are adjacent. This shows $\rho_D(z_0,\omega) \leq k$, i.e. $\omega \in U_k$. □

c)

Proof. By induction, it suffices to show there exists a constant $c > 0$, such that for adjacent z, $\omega \in D$, $h(z) \le ch(\omega)$. Indeed, let $r = \frac{1}{4}\min\{\text{dist}(z, \partial D), \text{dist}(\omega, \partial D)\}$, then by mean-value property, $\forall y \in B(\omega, r)$, we have $B(y, r) \subset B(\omega, 2r)$, so

$$h(\omega) = \frac{\int_{B(\omega,2r)} h(x)dx}{V(B(\omega, 2r))} \ge \frac{\int_{B(y,r)} h(x)dx}{V(B(\omega, 2r))} = \frac{V(B(y, r))}{V(B(\omega, 2r))} h(y) = \frac{h(y)}{2^d}$$

By using a sequence of small balls connecting ω and z, we are done. □

d)

Proof. Since K is compact and $\{U_1(x)\}_{x\in U}$ is an open covering of K, we can find a finite sub-covering $\{U_{n_i}(x)\}_{i=1}^N$ of K. This implies $\forall z, \omega \in K$, $\rho_D(z, \omega) \le N$. By the result in part c), we're done.

□

EP9-2. a)

Proof. We first have the following observation. Consider circles centered at 0, with radius r and $2r$, respectively. Let B be a BM on the plane and $\sigma_{2r} = \inf\{t > 0 : |B_t| = 2r\}$.

$\forall x \in \partial B(0, r)$, $P^x([B_0, B_{\sigma_{2r}}]$ doesn't loop around 0) is invariant for different x's on $\partial B(0, r)$, by the rotational invariance of BM. $\forall \theta > 0$, we define $\bar{B}_t = B_{\theta t}$, and $\bar{\sigma}_{2r} = \inf\{t > 0 : |\bar{B}_t| = 2r\}$. Since $\bar{B}$ and B have the same trajectories,

$$\begin{aligned}
& P^x([B_0, B_{\sigma_{2r}}] \text{ doesn't loop around } 0) \\
= \;& P([B_0, B_{\sigma_{2r}}] + x \text{ doesn't loop around } 0) \\
= \;& P([\bar{B}_0, \bar{B}_{\bar{\sigma}_{2r}}] + x \text{ doesn't loop around } 0) \\
= \;& P(\tfrac{1}{\sqrt{\theta}}[\bar{B}_0, \bar{B}_{\bar{\sigma}_{2r}}] + \tfrac{x}{\sqrt{\theta}} \text{ doesn't loop around } 0)
\end{aligned}$$

Define $W_t = \frac{\bar{B}_t}{\sqrt{\theta}} = \frac{B_{\theta t}}{\sqrt{\theta}}$, then W is a BM under P. If we set $\tau = \inf\{t > 0 : |W_t| = \frac{2r}{\sqrt{\theta}}\}$, then $\tau = \bar{\sigma}_{2r}$. So

$$\begin{aligned}
& P(\tfrac{1}{\sqrt{\theta}}[\bar{B}_0, \bar{B}_{\bar{\sigma}_{2r}}] + \tfrac{x}{\sqrt{\theta}} \text{ doesn't loop around } 0) \\
= \;& P([W_0, W_\tau] + \tfrac{x}{\sqrt{\theta}} \text{ doesn't loop around } 0) \\
= \;& P^{\frac{x}{\sqrt{\theta}}}([W_0, W_\tau] \text{ doesn't loop around } 0)
\end{aligned}$$

Note $\frac{x}{\sqrt{\theta}} \in \partial B(0, \frac{r}{\sqrt{\theta}})$, we conclude for different r's, the probability that BM starting from $\partial B(0, r)$ exits $B(0, 2r)$ without looping around 0 is the same.

Now we assume $2^{-n-1} \le |x| < 2^{-n}$ and $\sigma_n = \inf\{t > 0 : |B_t| = 2^{-n}\}$. Then for $E_j = \{[B_{\sigma_j}, B_{\sigma_{j-1}}]$ doesn't loop around $0\}$, $E \subset \cap_{j=1}^n E_j$. From what we observe above, $P^{B_{\sigma_j}}([B_0, B_{\sigma_{j-1}}]$ doesn't loop around 0) is a constant, say β. Use strong Markov property and induction, we have

$$P^x(\cap_{j=1}^n E_j) = P^x(\cap_{j=2}^n E_j; P^x(E_1|\mathcal{F}_{\sigma_1})) = \beta P^x(\cap_{j=2}^n E_j) = \beta^n = 2^{n \log \beta}$$

Set $-\log \beta = \alpha$, we have $P^x(E) \le 2^{-\alpha n} = 2^\alpha (2^{-n-1})^\alpha \le 2^\alpha |x|^\alpha$. Clearly $\beta \in (0,1)$. So $\alpha \in (0, \infty)$.

The above discussion relies on the assumtion $|x| < 1/2$. However, when $1/2 \le |x| < 1$, the desired inequality is trivial. Indeed, in this case $2^\alpha |x|^\alpha \ge 1$.

□

b)

Proof. $\forall x \in \partial D$, WLOG, we assume $x = 0$. $\forall \epsilon > 0$, let $\bar{B}_t = \epsilon B_{t/\epsilon^2}$, $\sigma = \inf\{t > 0 : |B_t| = 1\}$ and $\bar{\sigma} := \bar{\sigma}_\epsilon = \inf\{t > 0 : |\bar{B}_t| = \epsilon\}$, then $\bar{\sigma} = \epsilon^2 \sigma$. Hence $P^0\{[\bar{B}_0, \bar{B}_{\bar{\sigma}}] \text{ loops around } 0\} = P^0\{[B_0, B_\sigma] \text{ loops around } 0\}$. By part a), $P\{[B_0, B_\sigma] \text{ loops around } 0\} = 1$. So,

$$P^0(\bar{B} \text{ loops around } 0 \text{ before exiting } B(0,\epsilon)) = 1.$$

This means $P(\tau_D < \bar{\sigma}_\epsilon) = 1$, $\forall \epsilon > 0$. This is equivalent to x being regular. □

EP9-3. a)

Proof. We first establish a derivative estimate for harmonic functions. Let h be harmonic in D. Then $\frac{\partial h}{\partial z_i}$ is also harmonic. By mean-value property and integration-by-parts formula, $\forall z_0 \in D$ and $\forall r > 0$ such that $B(z_0, r) \subset U$, we have

$$\left|\frac{\partial h}{\partial z_i}(z_0)\right| = \left|\frac{\int_{B(z_0,r/2)} \frac{\partial h}{\partial z_i} dz}{V(B(z_0, r/2))}\right| = \left|\frac{\int_{\partial B(z_0,r/2)} h v_i dz}{V(B(z_0, r/2))}\right| \le \frac{2d}{r} \|h\|_{L^\infty(\partial B(z_0,r/2))}$$

Now fix K. There exists $\eta > 0$, such that when K is enlarged by a distance of η, the enlarged set is contained in the interior of a compact subset K' of U. Furthermore, if η is small enough, $\forall z, \omega \in K$ with $|z - \omega| < \eta$, we have $\cup_{\xi \in [z,\omega]} B(\xi, \eta) \subset K'$. Denote $\sup_n \sup_{z \in K'} |h_n(z)|$ by C, then by the above derivative estimate, for $z, \omega \in K$ with $|z - \omega| < \eta$,

$$|h_n(z) - h_n(\omega)| \le \frac{2d}{\eta} C |z - \omega|$$

This clearly shows the desired δ exists.

□

b)

Proof. Let K be a compact subset of D, then by part a) and Arzela-Ascoli theorem, $\{h_n\}_n$ is relatively compact in $C(K)$. So there is a subsequence $\{h_{n_j}\}$ such that $h_{n_j} \to h$ uniformly on K. Furthermore, by mean-value property, h must be also harmonic in the interior of K. By choosing a sequence of compact subsets $\{K_n\}$ increasing to D, and choosing diagonally subsequences, we can find a subsequence of $\{h_n\}$ such that it converges uniformly on any compact subset of D. This will consistently define a function h in D. Since harmonicity is a local property, h is harmonic in D.

□

EP10-1. a)

Proof. First, we note that

$$P^x(B_1 \geq 1; B_t > 0, \forall t \in [0,1]) = P^x(B_1 \geq 1) - P^x(\inf_{0 \leq s \leq 1} B_s \leq 0, B_1 \geq 1)$$

Let τ_0 be the first passage time of BM hitting 0, then by strong Markov property

$$\begin{aligned}
& P^x(\inf_{s \leq 1} B_s \leq 0, B_1 \geq 1) \\
= \;& P^x(\tau_0 \leq 1, P^x(B_1 \geq 1 | \mathcal{F}_{\tau_0})) \\
= \;& P^x(\tau_0 \leq 1, P^{B_{\tau_0}}(B_u \geq 1)|_{u=1-\tau_0}) \\
= \;& P^x(\tau_0 \leq 1, P^0(B_u \geq 1)|_{u=1-\tau_0}) \\
= \;& P^x(\tau_0 \leq 1, P^0(B_u \leq -1)|_{u=1-\tau_0}) \\
= \;& P^x(\tau_0 \leq 1, B_1 \leq -1) \\
= \;& P^x(B_1 \leq -1)
\end{aligned}$$

So

$$\begin{aligned}
& P^x(B_1 \geq 1; B_t > 0, \forall t \in [0,1]) \\
= \;& P^x(B_1 \geq 1) - P^x(B_1 \leq -1) \\
= \;& \int_{1-x}^{1+x} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\
\geq \;& 2x \frac{e^{-2}}{\sqrt{2\pi}}
\end{aligned}$$

where the last inequality is due to $x < 1$. □

EP10-2.

Proof. Let $F(n) = P(E_{2^n})$ and let DLA be the shorthand for "doesn't loop around" then

$$\begin{aligned}
F(n+m) &= P(E_{2^{n+m}}) \\
&= P([B_0, B_{T_{2^{n+m}}}] \text{ DLA } 0) \\
&\leq P([B_0, B_{T_{2^n}}] \text{ DLA } 0; P([B_{T_{2^n}}, B_{T_{2^{n+m}}}] \text{ DLA } 0 | \mathcal{F}_{T^{2^n}})) \\
&= P([B_0, B_{T_{2^n}}] \text{ DLA } 0; P^{B_{T_{2^n}}}([B_0, B_{T_{2^{n+m}}}] \text{ DLA } 0))
\end{aligned}$$

By rotational invariance of BM $P^x([B_0, B_{T_{2^{n+m}}}] \text{ DLA } 0)$ is a constant for any $x \in \partial B(0, 2^n)$. By scaling, we have

$$P^x([B_0, B_{T_{2^{n+m}}}] \text{ DLA } 0) = P^{\frac{x}{2^n}}([B_0, B_{T_{2^m}}] \text{ DLA } 0) = P(E_{2^m}) = F(m)$$

So $F(n+m) \leq F(n)F(m)$. By the properties of submultiplicative functions, $\lim_{n\to\infty} \frac{\log F(n)}{n}$ exists. We set this limit by $-\alpha$. $\forall m \in \mathbb{N}$, for m large enough, we can find n, such that $2^n \leq m < 2^{n+1}$, then $P(E_{2^n}) \geq P(E_m) \geq P(E_{2^{n+1}})$. So

$$\frac{\log P(E_{2^n})}{\log 2^n} \frac{\log 2^n}{\log m} \geq \frac{\log P(E_m)}{\log m} \geq \frac{\log P(E_{2^{n+1}})}{\log 2^{n+1}} \frac{\log 2^{n+1}}{\log m}$$

Let $m \to \infty$, then $\log 2^n / \log m \to 1$ as seen by $\log 2^n \leq \log m < \log 2 + \log 2^n$. So $\lim_m \frac{\log P(E_m)}{\log m}$ exists and equals to $-\alpha$. To see $\alpha \in (0, 1]$, note $F(1) < 1$ and $F(n) \leq F(1)^n$. So $\alpha > 0$. Furthermore, we note

$$\begin{aligned} & P^x([B_0, B_{T_n}] \text{ DLA } 0) \\ \geq \; & P^x(B^1 \text{ exits } (0, n) \text{ by hitting n}) \\ = \; & \frac{1}{n} \end{aligned}$$

So $\log P(E_n)/\log n \geq -1$. Hence $\alpha \leq 1$. □

EP10-3. a)

Proof. We assume $f_0(k) = 1$, $\forall k$ and $j, k = 1, \cdots, N$. We let P be the $N \times N$ matrix with $P_{jk} = p_{j,k}$. Then if we regard f_n as a row vector, we have $f_n = f_{n-1}P$. Define $M_n = \max_{k \leq N} f_n(k)$, then

$$f_{n+m} = f_0 P^{n+m} = f_0 P^m P^n = f_m P^n \leq M_m f_0 P^n = M_m f_n \leq M_m M_n f_0$$

So $M_{n+m} \leq M_n M_m$. By properties of submultiplicative functions, $\lim_n \frac{\log M_n}{n}$ exists and equals $\inf_n \frac{\log M_n}{n}$. Meanwhile, $\delta := \min_{j,k \leq N} p_{j,k} > 0$. So

$$M_n \geq f_n(k) \geq \delta \sum_{j=1}^{N} f_{n-1}(j) \geq \delta M_{n-1}$$

By induction, $M_n \geq \delta^n$. Hence $\inf_n \frac{\log M_n}{n} \geq \log \delta > -\infty$. Let $\beta = \inf_n \frac{\log M_n}{n}$, then $M_n \geq e^{\beta n}$. We set $\alpha = e^{\beta}$. Then $M_n \geq \alpha^n$. Meanwhile, there exists constant $C \in (0, \infty)$, such that for $m_n = \min_{k \leq N} f_n(k)$, $M_n \leq C m_n$. Indeed, for $n = 1$, $M_1 = m_1$, and for $n > 1$, $f_n(k) = \sum_j p_{j,k} f_{n-1}(j) \leq K \sum_j f_{n-1}(j)$ and $f_n(k) \geq \delta \sum_j f_{n-1}(j)$. So $M_n \leq \frac{K}{\delta} m_n$. Let $C = \frac{K}{\delta} \vee 1$, then

$$f_n(k) \geq m_n \geq \frac{M_n}{C} \geq \frac{\alpha^n}{C}$$

Similarly, we can show m_n is supermultiplicative and similar argument gives us the upper bound. □