Solutions to Assignment 7

Barr 3.2, Page 200: 8.

8.

(a) $\ln(5) \approx 1.60944$.

So, $\ln(5^{10000}) = 10{,}000 \ln(5) = 160{,}944$.

(b) log2 (3) $\approx$ 1.58496.

So, $\log_2(3^{-84371}) = -84371 \cdot (1.58496) = -133{,}725$.

(c) $2^x < 10^7 \Rightarrow x \ln 2 < 7 \ln 10 \Rightarrow x < (7 \ln 10) / (\ln 2) = 23.2535$. Therefore, the largest possible integer value of x is 23.

(d) $3^{13} \leq 4^x \Rightarrow 13 \ln 3 \leq x \ln 4 \Rightarrow x \geq (13 \ln 3) / (\ln 4) = 10.3023$. Therefore, the smallest possible integer value of x is 11.

Barr 3.3, Pages 208: 2*, 5, 6.

2.* Explain why $O(\log_b(n)) = O(\ln(n))$ for all $b > 1$.

Using the logarithm base conversion formula, $\log_a x = (\log_b x) / (\log_b a)$ for any $b > 0$.

So, $O(\log_b(n)) = O((\log_e n) / (\log_e b)) = O(\ln(n) / \ln(b))$. Notice, $\ln(b)$ is a constant $\neq 0$ since $b > 1$, and thus,

$O(\log_b(n)) = O(\ln(n) / \ln(b)) = O(\ln(n))$.

5.

$O(10^{100}) \subset O(\ln(n)) = O(\log_{10}(n)) \subset O(n^3) = O(n^3 + n^2) \subset O(n^{100}) \subset O(1.1^n) \subset O(n2^n) \subset O(3^n) \subset O(n!)$.

6. If $\mu \in O(a^n)$, then there are constants M and N such that $|\mu(n)| \leq Ma^n < Mb^n$ for all $n \geq N$. Thus, if $\mu \in O(a^n)$, $\mu \in O(b^n)$.

Example: $\mu(n) = b^n$ is not in $O(a^n)$: if it were, there would be M and N such that $b^n \leq Ma^n$ for all $n \geq N$. This would mean that $(b/a)^n \leq M$ for all $n \geq N$, which is impossible since $(b/a) > 1$ and the sequence $(b/a)^n$ goes to ∞ as $n \to \infty$.

Barr 3.4, Page 220: 3(b)*.

3(b)*. $b_4 \leftarrow b_3 + b_2 + b_1$, Initial Condition 1000. Using the procedure described on Pages 213-215, we get:

t	b_4	b_3	b_2	b_1
0	1	0	0	0
1	0	1	0	0
2	1	0	1	0
3	1	1	0	1
4	0	1	1	0
5	0	0	1	1
6	0	0	0	1
7	1	0	0	0

Notice that the shift register bits have returned to their initial values at t = 7, so the pattern of b_1 starts to repeat. Thus, the output is 0001011.

G1.* Given register contents DCBA, output = (A+C) mod 2, C’ = D, B’ = C, A’ = B, and D’ = (A+B) mod 2.

(a) Starting with DCBA = 1011, we obtain the following:

t	D	C	B	A	Output
0	1	0	1	1	1
1	0	1	0	1	0
2	1	0	1	0	0
3	1	1	0	1	0
4	1	1	1	0	1
5	1	1	1	1	0
6	0	1	1	1	0
7	0	0	1	1	1
8	0	0	0	1	1
9	1	0	0	0	0
10	0	1	0	0	1
11	0	0	1	0	0
12	1	0	0	1	1
13	1	1	0	0	1
14	0	1	1	0	1
15	1	0	1	1	1

Notice that the bits return to their initial value at t = 15. So, the output is 100010011010111.

(b) Recall that an *ordinary* four bit shift register is the following (Page 213):

If $b_k, \ldots, b_4, b_3, b_2, b_1$ are the bits in the register at a given time and $b_k', \ldots, b_4', b_3', b_2', b_1'$ are the bits at the next time, then

$$b_1' \leftarrow b_2$$
$$b_2' \leftarrow b_3$$
$$b_3' \leftarrow b_4$$
$$\vdots$$
$$b_k' \leftarrow (c_k \cdot b_k + c_{k-1} \cdot b_{k-1} + \ldots + c_3 \cdot b_3 + c_2 \cdot b_2 + c_1 \cdot b_1) \bmod 2.$$

So, given DCBA we get A' = B, B' = C, C' = D, and letting D' = (A + B) mod 2 with starting value 0001, we get:

t	D	C	B	A
0	0	0	0	1
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	1	0	0	1
5	1	1	0	0
6	0	1	1	0
7	1	0	1	1
8	0	1	0	1
9	1	0	1	0
10	1	1	0	1
11	1	1	1	0
12	1	1	1	1

13	0	1	1	1
14	0	0	1	1
15	0	0	0	1

Notice that the bits return to their initial value at $t = 15$. So, the output (given by A) is 100010011010111 which is the same output we got in (a).

G2.*

$x_{n+1} = Ax_n + b$ (Mod M)

$A = 3$, $B = 2$, $M = 17$, $x_0 = 2$.

So,

$x_1 = 3x_0 + 2$ (Mod 17) $= 3(2) + 2 = 8$.

Repeating this procedure we get:

$x_2 = 9$, $x_3 = 12$, $x_4 = 4$, $x_5 = 14$, $x_6 = 10$, $x_7 = 15$, $x_8 = 13$, $x_9 = 7$, $x_{10} = 6$, $x_{11} = 3$, $x_{12} = 11$, $x_{13} = 1$, $x_{14} = 5$, $x_{15} = 0$ and $x_{16} = 2$. Notice that $x_{16} = 2 = x_0$, so $x_{17} = x_1$, $x_{18} = x_2$, and so on. Thus, the length of the cycle is 16.