ANALYSIS ON THE PROJECTIVE OCTAGASKET

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Abstract. The existence of a self similar Laplacian on the Projective Octagasket, a non-finitely ramified fractal is only conjectured. We present experimental results using a cell approximation technique originally given by Kusuoka and Zhou. A rigorous recursive algorithm for the discrete Laplacian is given. Further, the spectrum and eigenfunctions of the Laplacian together with its symmetries are categorized and utilized in the construction of solutions to the heat equation.

Contents

1. Introduction

In a similar fashion to fractals given by Strichartz in [1] we give an overview of the construction of the Projective Octagasket, an approximation of its Laplacian and its spectrum. This fractal is not finitely ramified meaning that cells border along more than just vertices. As such analytical study upon it, and the existence of a self-similar Laplacian is only conjectured. Using a cell approximation given by Kusuoka and Zhou in [2] we construct the discrete graph Laplacian on a given level. This gives access to the spectrum where the eigenvalues and their eigenfunctions line up concerning symmetry and multiplicity. Solutions to the heat and wave equation are also given using an orthonormal basis of the eigenspace and computation of the heat kernel approximation.

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2. Construction of Projective Octagasket

To obtain a projective octagsket, we first need to obtain an octagasket. An octagasket is defined to be the unique nonempty compact set OG satisfying the following identity with contraction mapings $\{F_n\}$:

$$OG = \bigcup_{n=0}^{7} F_n(OG)$$

where

$$F_n(x,y) = \frac{1}{2+\sqrt{2}}(x + (1+\sqrt{2})\cos\frac{5-2n\pi}{8}, y + (1+\sqrt{2})\sin\frac{5-2n\pi}{8})$$

for all $x, y \in \mathbb{R}$ and all $n = 0, 1, ..., 7$.

Of course, an octagasket can obtained through a finite graph approximation. Let Γ_0 be a regular octagon centered at the origin with radius 1, V_0 be its vertices and R_0 be the origin. Then, we recursively let

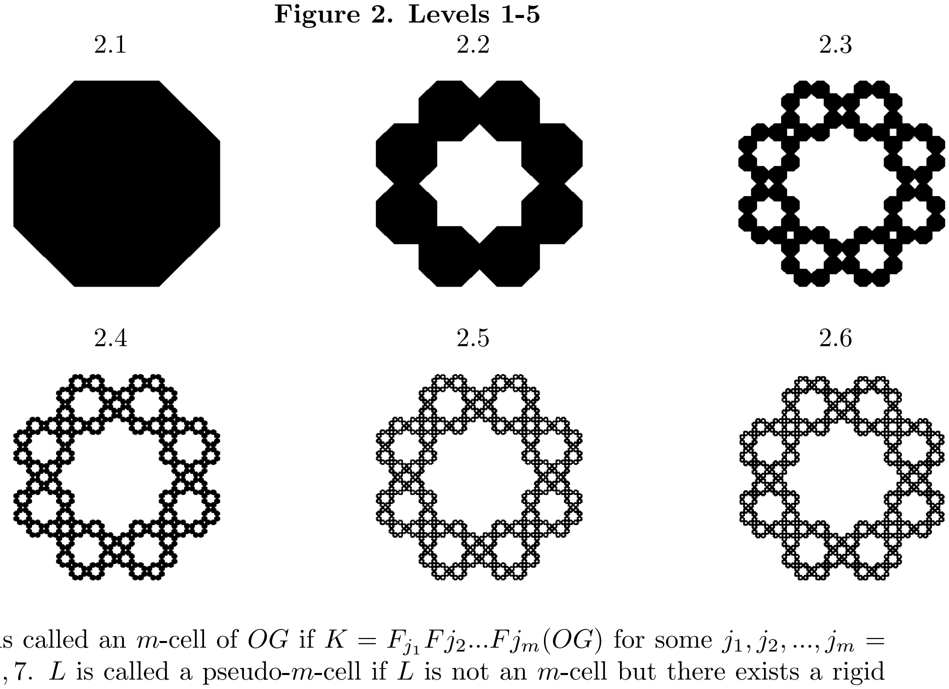
$$\Gamma_m = \bigcup_{n=0}^{7} F_n(\Gamma_{m-1}) \quad \forall m \in \mathbb{N}$$

and

$$V_m = \bigcup_{n=0}^{7} F_n(V_{m-1}) \quad \forall m \in \mathbb{N}$$

One can easily see that $V_* := \bigcup_{n=0}^{\infty} V_n$ is dense in OG and $\bigcap_{n=0}^{\infty} \Gamma_n = \overline{V_*} = OG$. The following shows the graphs of $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5$ respectively.

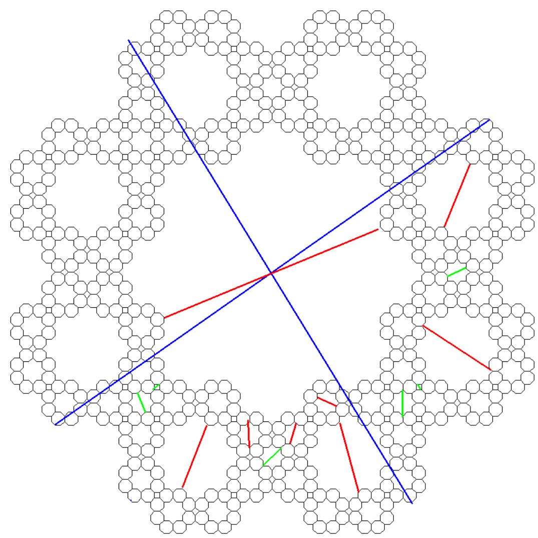
Figure 2. Levels 1-5



K is called an m-cell of OG if $K = F_{j_1}Fj_2...Fj_m(OG)$ for some $j_1, j_2, ..., j_m = 0, 1, ..., 7$. L is called a pseudo-m-cell if L is not an m-cell but there exists a rigid motion ϕ such that $\phi(L)$ is an m-cell.

Intuitively, a projective octagasket is given by identifying the boundaries of an octagasket antipodally. Two points are identified if they belong to the inner boundaries of the same m-cell and they are antipodal with respect to the centre of that m-cell; or they belong to the outer boundaries of the 0-cell and they are antipodal with respect to the origin; or they both belong to the boundary of the same pseudo-m-cell and they are antipodal with respect to the centre of that pseudo-m-cell. In the following graph, the red, blue and green lines are corresponding to some gluing of inner boundaries of an m-cell, outer boundaries of the 0-cell and inner boundaries of pseudo-m-cells on Γ_3, note the tiny squares are glued since they will become a pseudo-3-cell eventually.

Figure 2.7 Antipodal Identifications



To define the projective octagasket, we first lets E_* is the collection of all centres of all the pseudo-cells and R_* is the collection of all the centres of all the cells.

To be rigorous, we let $r = \frac{1}{2+\sqrt{2}}$, r_n be rotation by $\frac{n\pi}{4}$ anti-clock-wisely centered at the origin, R_0 be the origin,

$$P = \{(0, (1-r)\cos\frac{\pi}{8} + \sum_{i=1}^{n} \epsilon_i r^i (1-r)\sin\frac{\pi}{8}) \in \mathbb{R}^2 | \epsilon_i = \pm 1, n = 0, 1, 2, ...\}$$

$$E_0 = \bigcup_{n=0}^{7} r_n(P)$$

$$R_m = \bigcup_{n=0}^{7} F_n(R_{m-1}) \quad \forall m \in \mathbb{N}$$

$$E_m = \bigcup_{n=0}^{7} F_n(E_{m-1}) \quad \forall m \in \mathbb{N}$$

, where P is the collection of all centres of all the pseudo-cells between the 1-cells 0 and 1, E_0 is the collection of all centres of all the pseudo-cells between the 1-cells. Then, $R_* = \bigcup_{n=0}^{\infty} R_n$ and $E_* = \bigcup_{n=0}^{\infty} E_n$.

Now, define an equivalence relation on OG by $x \sim y$ if $l_{xy} \cap (R_* \cup E_*) \neq \emptyset$ and $l_{xy} \cap OG = \{x, y\}$, where l_{xy} is a straight line in the $\mathbb{R}^2$ with end points x and y; or there exits l_{xy} such that $l_{xy} \cap R_0 \neq \emptyset$ with any extension $\widetilde{l_{xy}}$ of l_{xy} satifying $\widetilde{l_{xy}} \cap OG = l_{xy} \cap OG$. Then, we define the projective octagasket POG to be $OG/\sim$.

Remark 2.1. $l_{xy} \cap R_* \neq \emptyset, l_{xy} \cap E_* \neq \emptyset$ and $\widetilde{l_{xy}} \cap OG = l_{xy} \cap OG$ correspond to the identification of inner boundaries of m-cell, the identification of inner boundaries of pseudo-m-cell and the identification of outer boundary of 0-cell respectively.

3. Laplacian

The Laplacian plays a central role in constructing an analytic theory on a given fractal. In general a fractal is given as a limit of graph approximations. As such a discrete graph Laplacian is used to model the continuous Laplacian. When a renormalisation factor is found then the Laplacian at a point can be given as the limit of its discrete graph approximations. Generally the graph Laplacian is given as

$$-\triangle_m u(x) = \sum (u(y) - u(x))$$

over all vertices y that neighbor x in the m level graph approximation. The nature of the Projective Octagasket does not lend itself to addressing well and as such we decided to use a form of cell approximation given by Kusuoka and Zhou in [2]. Here the function value on an m cell A_i^m is given as the average value on its vertices. Now given some $u : V \to \mathbb{R}$ we can evaluate functions on cells and therefore we can construct a pointwise Laplacian as

$$-\triangle_m u(A_x^m) = \sum (u(A_y^m) - u(A_x^m))$$

Where every A_y^m borders A_x^m on some edge. Every m cell has 8 edges, each bordering another m cell. Thus in total every cell will have 8 adjacent cells counting multiplicities. We include the Laplacian matrix for level 1, using this we can find its eigenvalues and eigenvectors. These relay important information about the analytic structure of the fractal and allows us the possibility of approximating the heat kernel, thereby giving us solutions to the heat equation.

$$\begin{bmatrix} 8 & -1 & 0 & 0 & -6 & 0 & 0 & -1 \\ -1 & 8 & -1 & 0 & 0 & -6 & 0 & 0 \\ 0 & -1 & 8 & -1 & 0 & 0 & -6 & 0 \\ 0 & 0 & -1 & 8 & -1 & 0 & 0 & -6 \\ -6 & 0 & 0 & -1 & 8 & -1 & 0 & 0 \\ 0 & -6 & 0 & 0 & -1 & 8 & -1 & 0 \\ 0 & 0 & -6 & 0 & 0 & -1 & 8 & -1 \\ -1 & 0 & 0 & -6 & 0 & 0 & -1 & 8 \end{bmatrix}$$

Similar to what has been done on the Sierpinski gasket in [1], after constructing the Laplacian $\triangle_m$ for Γ_m, it is natural to construct the Laplacian for the projective octagasket by letting

$$\triangle u(x) = \lim_{m \to \infty} r^{-m} \triangle (A_i^m)$$

for some renormalisation factor r, where $x \in A_i^m$ for all m, where r is determined later.

4. Identification Algorithm

Now that we have constructed the Projective Octagasket in a theoretical sense it is time to turn to the labelling algorithm which lies at the core of constructing the graph approximation. We wish to construct the Laplacian for a given m level cell graph approximation. To do this we need a labelling scheme and then a recursive algorithm which allows us to find the number of edge connections between antipodally glued cells.

For Γ_1 we label the 1 cells clockwise $0-7$ starting at the top left. For Γ_2 we take the cell labelled 0 on the previous level and label its 2 cells $0-7$ as before. Now for the cell labelled 1 we rotate our labelling scheme clockwise by $\pi/4$, thus labelled $0-7$ clockwise starting on the top right. This process is repeated over Γ_2.

For Γ_m simply take the labelling on Γ_{m-1} and to label cell number k simply rotate the $0-7$ clockwise labelling scheme by $k\pi/4$

To be rigorous, w is an m-cell in Γ_{m+k} if $w = F_{j_1}F_{j_2}...F_{j_m}(\Gamma_k)$ for some $j_1, j_2, ..., j_m = 0, 1, ..., 7$ and we write $w = j_1(j_2 - j_1)...(j_m - j_{m-1})$ and $j_1(j_2 - j_1)...(j_m - j_{m-1})$ is called the address of w, with the arithmetic in $\mathbb{Z}_8$. In addition, ν is called a pseudo-m-cell in Γ_{m+k} if there exists some rigid motion ϕ such that $\phi(\nu)$ is an m-cell but ν is not an m-cell for $k > 0$; or ν is a square for $k = 0$.

Through out this paper, any arithmetic on the address will be conducted in $\mathbb{Z}_8$.

The following shows the address on Γ_1, Γ_2 and Γ_3 repectively:

Figure 4.1 Level 1

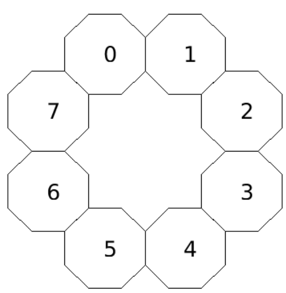


Figure 4.2 Level 2

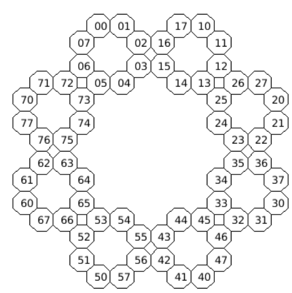
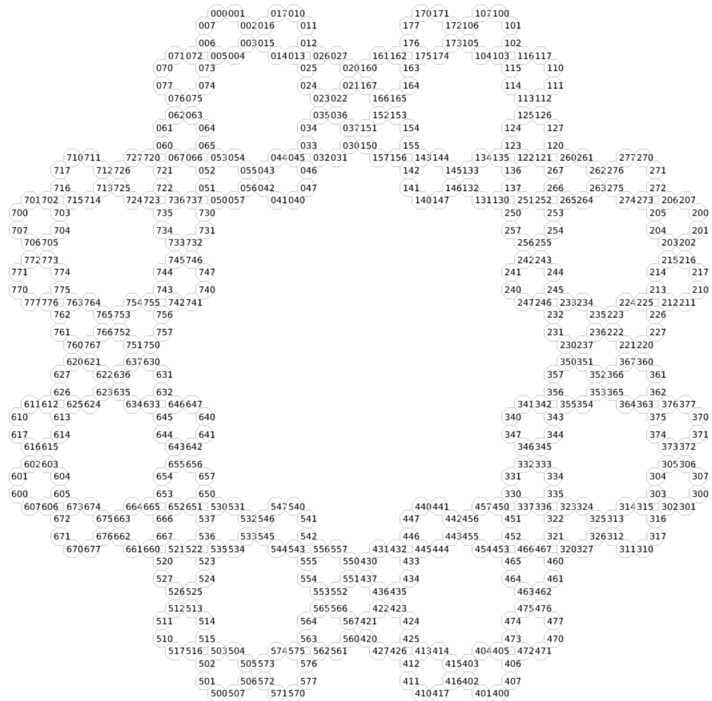


Figure 4.3 Level 3



The number of edges shared between cells is crucial to the Laplacian matrix of our approximation. Let A_i^m be an m-cell. We initialize a $8^m \times 8^m$ matrix L where row i will store the value of $-\triangle A_i^m$ in terms of the other m cells. Thus $L(i,i) = 8$ for all i and $L(i,j)$ will be negative of the number of edges between the two cells i and j. Recall that our labelling algorithm gives an m cell as a m tuple of the references. Thus each cell can be represented in base 8. Now in order to reference a cell in the Laplacian matrix we simply translate it to base 10. Thus some (a,b,c) will correspond to row $64a + 8b + c$ in the matrix L.

The idenfication for Γ_1 is simple, for cell number i there is an edge shared between both $(i-1)$ and $(i+1)$. There are 6 edges connecting it to $(i+4)$, 2 lie on the inner boundary and 4 lie on the outer boundary. This allows one to construct the Laplacian matrix easily.

To understand the structure of Γ_{m+1} for $m > 0$, we will need to understand when two cells share an edge. We have the following algorithm to create Γ_{m+1} from Γ_m:

(1) Remove the outer boundary identification of Γ_m.
(2) Make 8 copies of Γ_m.
(3) Identify the adjacent edges of the Γ_m and the inner boundary of the pseudo-cell antipodally.
(4) Identify the inner boundary of the 0-cell antipodally

(5) Identify the outer boundary of the 0-cell antipodally.

(1) **Remove outer boundary identification**
For $m > 1$, we simply omit step (5) for the previous construction. For $m = 1$, it is easy.
(2) **Create 8 copies of Γ_m**
Simply take the disjoint union of 8 copies of Γ_m, label them according to the previous labelling scheme.
(3) **Adjacent Edges and Pseudo-cell Identifications**
This is the most difficult part of the identification algorithm. We show the explicit construction for level 2,3,4 and a recursive generalization for higher levels. On Γ_2 one can see the small pseudo cells, they are the inner boundary of four adjacent cells. Notice that for any two adjacent 2 cells these pseudo-cell boundaries always lie between $X2, X3$ and $(X+1)5, (X+1)6$. We will start our explanation here. Notice that for each subsequent level the number of pseudo-cells doubles. Also notice that if any cell has n edges along the boundary then on the next level those n edges will be replaced by $n+1$ cells.
Level 2
Initially we take the adjacency gluing of Γ_2, this is where the four cells are glued along a square antipodally. Given an arbitrary 1 cell denoted X from 0-7 we construct the gluing below. Here every edge denotes 1 edge connection between cells.

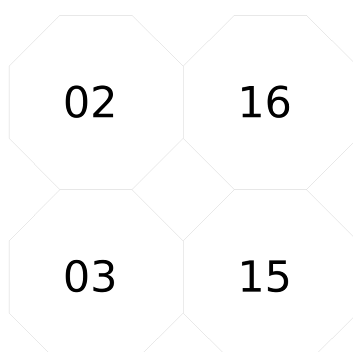


Figure 4.4

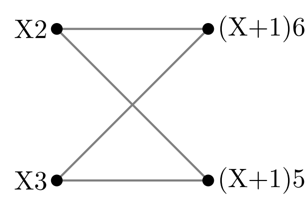


Figure 4.5

Level 3

Recall that for a given cell with n edges along the boundary on the next level that cell will turn into $n+1$ finer cells. To construct the next level we observe how many edge connections each cell has. In Γ_2 each has 2, so we take 2+1 copies of each cell and append to it the repeating string $\overline{01267}$ where we start at 0 and continue with this pattern throughout the cell. When changing over to a next cell a 0 will follow a 0 and a 6 follows a 2. This process is for labelling the left side, the right side is identical except with the order reversed. The gluing process is copied twice from Γ_2 here we have $X20$ together with $X21$ paired with $X60$ and $X67$ in the cross formation as well as $X37$ and $X30$ paired with $X51$ and $X50$ as shown in black. The central 4 cells are connected antipodally, as shown in red. All of these central edge connections correspond to 2 edges between cells.

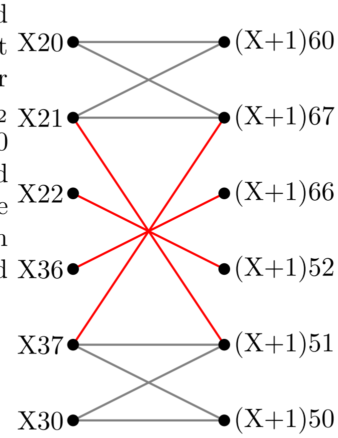
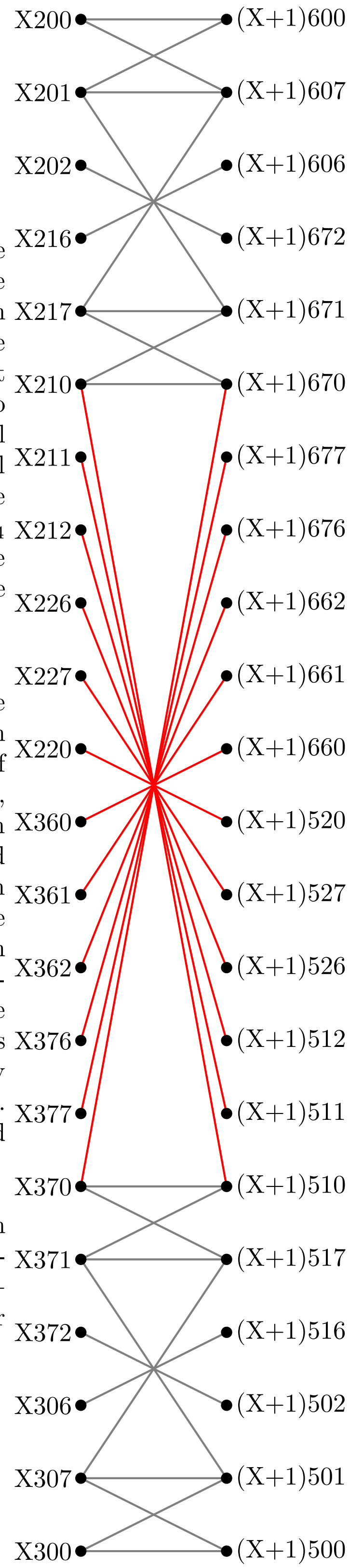


Figure 4.7

Figure 4.6



Level 4
For Γ_4 we again take $n+1$ copies of each cell where n is the number of edge connections of the cell in the step (3) of previous level. So a cell like $X21$ from before has two edges each of size 1 and an edge of size 2, thus there will be 5 copies of $X21Y$ on the next level. The gluing structure of Γ_3 is copied over to the top and bottom of the Γ_4 boundary. The central 12 cells are glued antipodally. On the previous level we had all central edge connections equal to 2, we had a sequence $2, 2, 2, 2$ of edge connections. On Γ_4 we replace every 2 in this sequence with the sequence 242, thus getting $2, 4, 2, 2, 4, 2, 2, 4, 2, 2, 4, 2$ as the edge connections across the center.

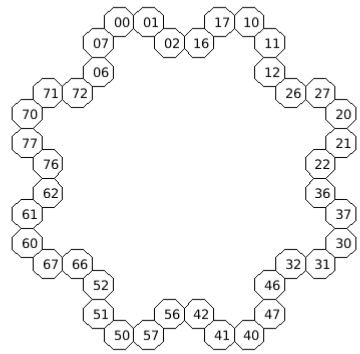
Level m
To give a recursive argument we assume we have the adjacency labelling along Γ_{m-1} now each cell has an edge connection n associated with it in the step (3) of previous level. For Γ_m take $n+1$ copies of each cell, list everything in the same order as before and begin labelling with $\overline{01267}$. Whenever a new cell is reached we restart the labelling at 0 if the previous one ends in 0 and a 6 if the previous one ends in 2. Now copy the exact gluing structure of Γ_{m-1} on the top and bottom parts of Γ_m. The central portion is connected antipodally as before and has a edge connection sequence associated with it. This sequence from Γ_{m-1} consists of 2's and 4's, whenever there is a 2 we replace by $2, 4, 2$ whenever there is a 4 we replace by $2, 4, 4, 4, 2$. This gives the new edge strength sequence for Γ_m and we are finished.

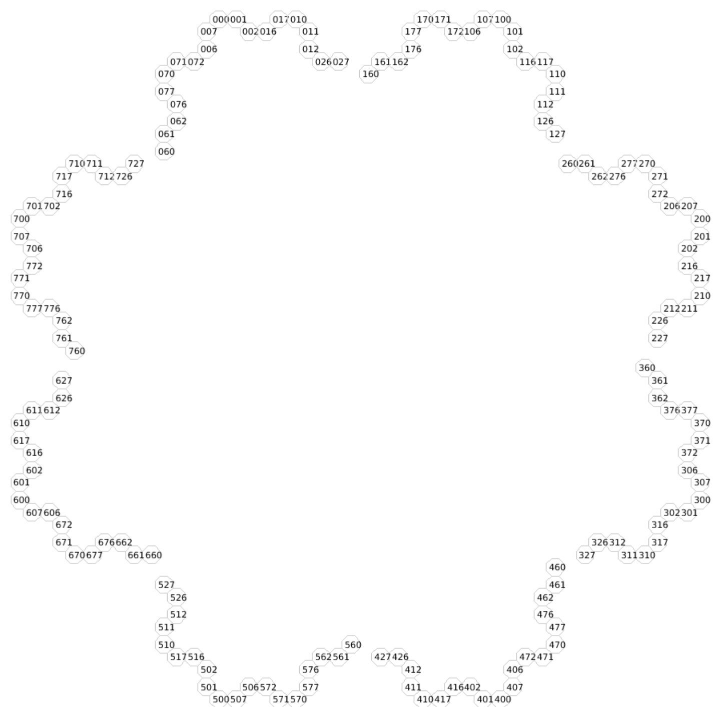
(4) **Inner Boundary identifications**
If an $(m+1)$-cell $w = X_1X_2X_3...X_{m+1} \in \Gamma_{m+1}$ with $X_2 = 3, 4$ or 5 shares n edges with other cells before (4), then w shares $(8-n)$ edges with $(X_1+4)X_2X_3...X_{m+1}$ after (4). The connectivity of other cells remains unchanged.

(5) **Outer Boundary identifications**

If an $(m+1)$-cell $w = X_1X_2X_3...X_{m+1} \in \Gamma_{m+1}$ shares n edges with other cells before (5), then w shares $(8-n)$ edges with $(X_1+4)X_2X_3...X_{m+1}$ after (5).

We show only the outer boundary for level 2 and 3. If you attempt to connect this cells antipodally it is apparent that one is doing something equivalent to the algebraic identification listed above.





5. Eigenvalue and Eigenfunction

Recall that we define the Laplacian $-\triangle u(x) = -\lim_{m\to\infty} r^{-m} \triangle_m u(A_i^m)$., where $x \in A_i^m$ for all m. In order to find the renormalization factor r, we first compute the eigenvalue and eigenfunction of $-\triangle_m$.

The table below indicates a portion of the eigenvalues for levels 1, 2, 3, 4, 5. Observe that the multiplicity of every eigenvalue does not exceed 2.

Level1		*Level2*		*Level3*		*Level4*		*Level5*	
Eigenvalue	*Multiplicity*	*Eigenvalue*	*Multiplicity*	*Eigenvalue*	*Multiplicity*	*Eigenvalue*	*Multiplicity*	*Eigenvalue*	*Multiplicity*
0	1	0	1	0	1	0	1	0	1
2	2	0.4871	2	0.1138	2	0.0259	2	0.0059	2
4	1	0.9783	1	0.2366	1	0.0541	1	0.0122	1
12.5858	2	2.2871	1	0.5609	1	0.1309	1	0.0298	1
15.4142	2	2.3296	2	0.5751	1	0.1340	1	0.0304	1
		2.3820	1	0.5902	2	0.1375	2	0.0313	2
		2.8687	1	0.7301	1	0.1697	1	0.0386	1
		3.0247	2	0.7332	2	0.1711	2	0.0389	2
		3.4427	1	0.8313	1	0.1962	1	0.0446	1
		4.4197	1	1.0996	1	0.2598	1	0.0591	1
		4.4464	2	1.1241	2	0.2638	2	0.0599	2
		4.6180	1	1.1709	1	0.2719	1	0.0616	1
		5.2598	2	1.3120	1	0.3097	1	0.0705	1
		5.3703	1	1.3435	2	0.3116	2	0.0707	2
		5.4460	2	1.3576	1	0.3172	1	0.0720	1
		5.4463	1	1.3945	2	0.3294	2	0.0751	2
		5.4601	1	1.5662	1	0.3731	1	0.0853	1
		6.6192	2	1.6627	2	0.3929	2	0.0895	2
		6.7007	2	1.7328	2	0.4049	2	0.0920	2
		7.8468	1	1.9195	1	0.4505	1	0.1026	1
		7.8481	1	2.0174	1	0.4865	1	0.1114	1
		8.4801	2	2.1509	2	0.5150	2	0.1179	2
		8.5001	2	2.2085	2	0.5213	2	0.1187	2
		8.7486	1	2.2149	1	0.5284	1	0.1204	1
		8.8864	2	2.2517	1	0.5367	1	0.1231	1
		9	1	2.2995	2	0.5469	2	0.1248	2
		9.1163	2	2.3466	2	0.5675	2	0.1296	2
		9.1668	2	2.3763	1	0.5683	2	0.1298	2
		9.2832	1	2.3864	2	0.5712	2	0.1311	2
		9.3041	2	2.3900	2	0.5806	1	0.1353	1
		10.5892	2	2.3992	2	0.5897	2	0.1364	1
		10.8516	2	2.4039	1	0.5905	1	0.1373	2
		11.2427	2	2.4135	1	0.5915	1	0.1375	1
		12.0270	2	2.4259	1	0.5990	1	0.1395	1
		12.5055	2	2.4369	1	0.6014	2	0.1399	2
		12.5145	2	2.4507	1	0.6145	1	0.1430	1
		14.1299	2	2.4604	2	0.6145	1	0.1430	1
		14.1407	2	2.4791	2	0.6183	1	0.1433	1
		15.0161	2	2.5360	1	0.6248	2	0.1450	2
		15.2155	2	2.6906	1	0.6668	1	0.1551	1
				2.7170	2	0.6689	2	0.1558	2
				2.7996	2	0.6815	2	0.1566	2
				2.8159	1	0.6996	2	0.1607	2
				2.8872	2	0.7034	1	0.1635	1

In order to obtain the renormalisation factor, we want to compare the ratio between the eigenvalues in different levels. The difficulty is that some eigenvalues are not born from the previous level. Nevertheless, since the eigenfucntions corresponding to the eigenvalue of multiplicity 1 have good symmetry, as stated below, we can compare them easily.

Proposition 5.1. *Given an eigenfunction u and the corresponding eigenvalue with multiplicity* 1*, one the the following is true about u: invariant under D_8 (O), symmetric about a line through the mid-points of the opposite edges of an octagon (E), invariant under a $\frac{\pi}{4}$ rotation (R), or symmetric about a line through two opposite corners of an octagon (C).*

Proof. Let E_λ be an λ-eigenspace of dimension 1 spanned by u. Then, by the construction of laplacian, one can easily see that for all $g \in D_8$, $\Delta(u \circ g) = \lambda u \circ g$. Since ${}_\lambda = 1$, we have $u \circ g = \rho_g u$ for some $\rho_g \in \mathbb{C}$. Note that $u = u \circ g^8 = \rho_g^8 u$, and

u is not zero everywhere since it spans E_λ. Hence, $\rho_g^8 = 1$. In particular, $\rho_g \neq 0$. Since the matrix used to compute eigenfunction is symmetric, by spectral theorem, we can choose u to be real-valued. Now pick $x \in POG$ such that $u(x) \neq 0$, we have $u \circ g(x) = \rho_g u(x) \neq 0$, which forces $\rho_g \in \mathbb{R}$ since u is real-valued. Therefore $\rho_g = +1$ or -1. As D_8 is generated by reflection along the opposite edges of an octagon e and rotation r, the symmetry of u is completely determined by ρ_e and ρ_r. One can easily show that the four possible cases $\rho_e = \rho_r = 1$, $\rho_e = 1$ and $\rho_r = -1$, $\rho_e = -1$ and $\rho_r = 1$, $\rho_e = \rho_r = -1$ correspond to the above cases respectively. □

Remark 5.2. Let e, r and $c \in D_8$ be the reflection about a line through the midpoints of the opposite edges of an octagon, rotation anticlockwise by $\frac{\pi}{4}$ and reflection about a line through two opposite corners of an octagon respectively. If u is of type (E), then $u \circ r = u \circ c = -u$. If u is of type (R), then $u \circ e = u \circ c = -u$. If u is of type (C), then $u \circ r = u \circ e = -u$.

Remark 5.3. There is an interesting fact related to the number of eigenvalues with multiplicity 1. Let O_m, E_m, R_m and C_m be the number of eigenvalues of type O, E, R and C respectively. We find that $O_m = C_m = 4 \times 8^{m-2} + 2^{m-2}$ and $E_m = R_m = 4 \times 8^{m-2} - 2^{m-2}$ for $m = 1, 2, 3$ and 4.

The below table shows a part of the comparison between the eigenvalues of multiplicity 1 corresponding to $-\triangle_1$, $-\triangle_2$, $-\triangle_3$, $-\triangle_4$ and $-\triangle_5$, which gives us the renormalisation factor $r \approx 4.2$.

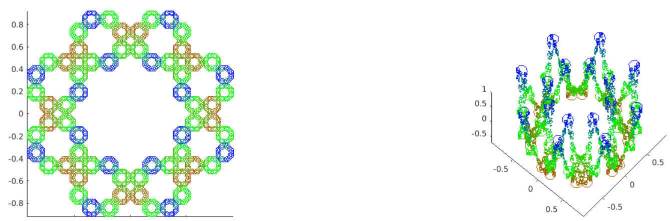
*Level*1	*Level*2	*Level*3	*Level*4	*Level*5	*Type*	R_1/R_2	R_2/R_3	R_3/R_4	R_4/R_5
0	0	0	0	0	O				
4	0.9783	0.2366	0.0541	0.0122	C	4.0887	4.1348	4.3734	4.4242
	2.2871	0.5609	0.1309	0.0298	O		4.0776	4.2850	4.4000
	2.3820	0.5751	0.1340	0.0304	E		4.1419	4.2918	4.4011
	2.8687	0.7301	0.1697	0.0386	R		3.9292	4.3023	4.4013
	3.4427	0.8313	0.1962	0.0446	C		4.1413	4.2370	4.3958
	4.4197	1.0996	0.2598	0.0591	O		4.0194	4.2325	4.3956
	4.6180	1.1709	0.2719	0.0616	E		3.9440	4.3064	4.4119
	5.3703				C				
	5.4463	1.3120	0.3097	0.0705	O		4.1511	4.2364	4.3929
	5.4601	1.3576	0.3172	0.0720	C		4.0219	4.2799	4.4036
		1.5662	0.3731	0.0853	C			4.1978	4.3726
	7.8468	1.9195	0.4505	0.1026	O		4.0879	4.2608	4.3926
	7.8481	2.0174	0.4865	0.1114	R		3.8902	4.1468	4.3691

The followings are the picture for the eigenfunctions corresponding to some of the eigenvalues listed above. It has an overlay of the eigenfunctions lined up on consecutive levels.

Multiplicity 1 Eigenfunctions

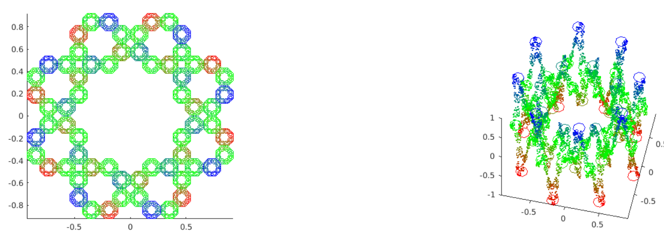
4th Eigenfunction $\lambda = 4, 4.1088, 4.07425, 3.8195$

5th Eigenfunction $\lambda = 9.60582, 9.65898, 9.2418$

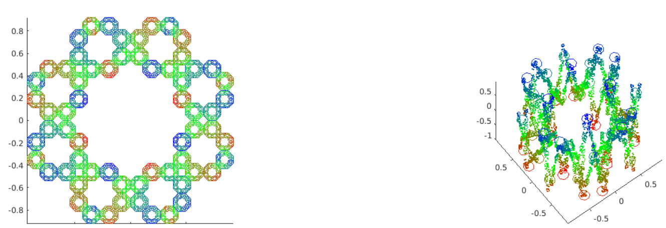


6th Eigenfunction $\lambda = 10.0044, 9.903222, 9.460668$

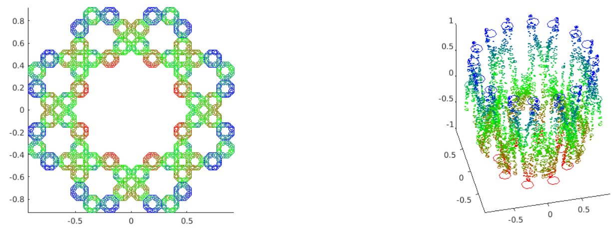
9th Eigenfunction $\lambda = 12.04854, 12.5723, 11.9811$



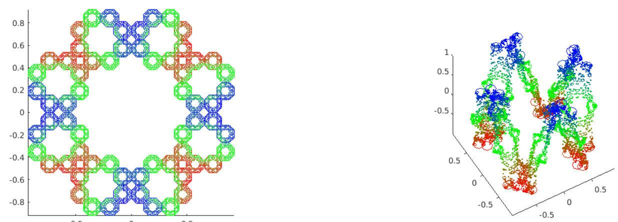
12th Eigenfunction $\lambda = 14.45934, 14.31498, 13.852112$



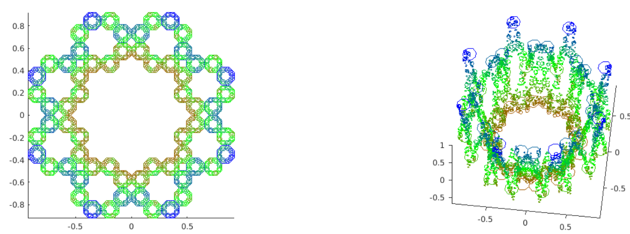
13th Eigenfunction $\lambda = 18.56274, 22.59264, 21.865439$



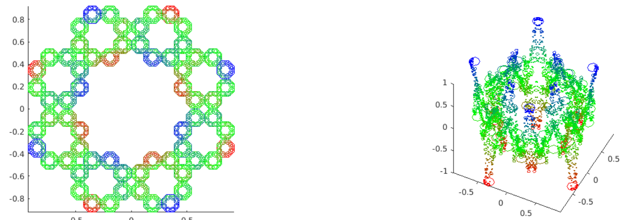
16th Eigenfunction $\lambda = 19.3956, 20.162898, 19.1966838$



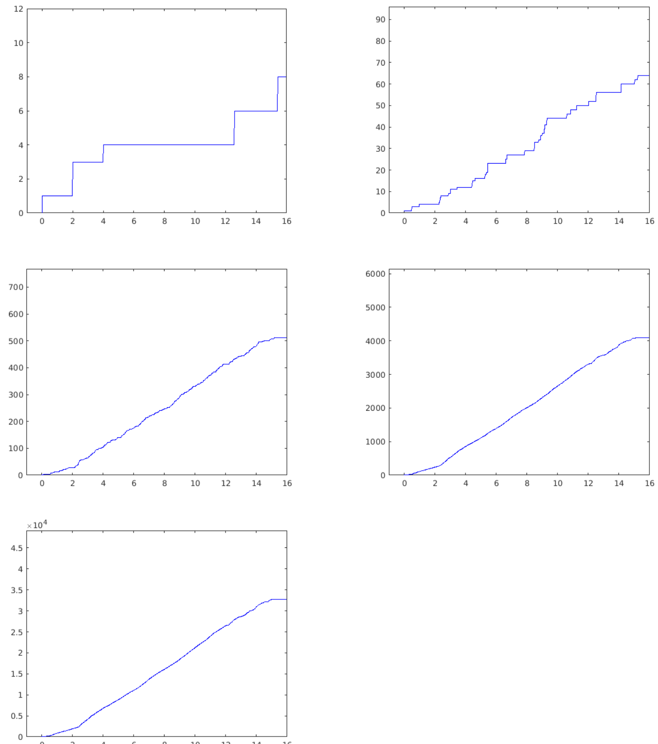
17th Eigenfunction $\lambda = 22.87446, 22.59264, 21.8654394$



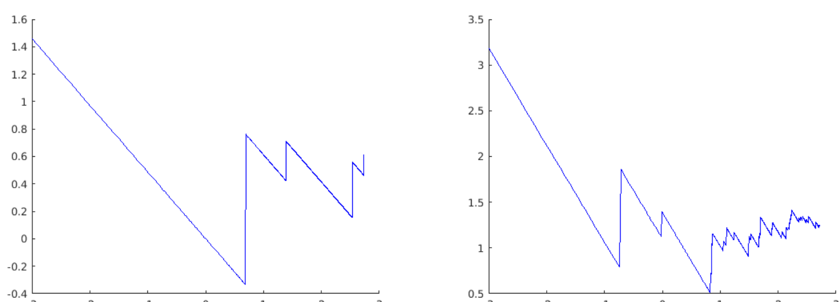
20th Eigenfunction $\lambda = 22.55526, 23.377872, 22.3949544$

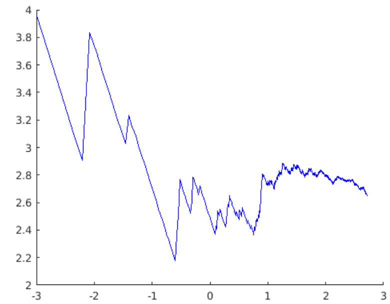


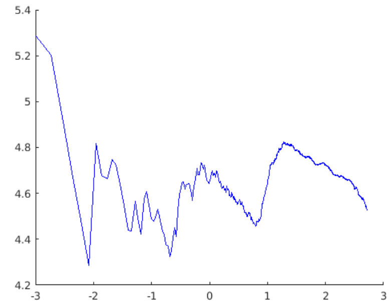
Let $N(x)$ be the eigenvalue counting function, which denotes the number of eigenvalues less than or equal to x. It is of interest to study the growth rate of $N(x)$ and the gaps between the eigenvalues. The followings are the graph of x against $N(x)$ for level 1, 2 ,3 ,4 and 5 respectively.

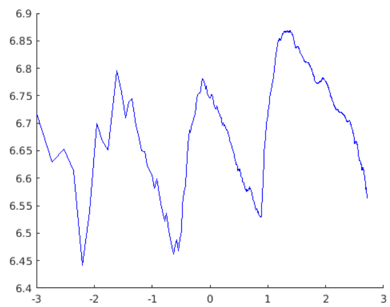


In order to study the spectral asymptotics of the projective octagasket, we try to use a finite graph approach. We first take the log-log scale of the x against $N(x)$ and then take the segment that is most suitable for a linear fit. Let α be the slope of this line and the Weyl ratio be $N(t)/t^{\alpha}$. Then, plotting $log(t)$ against $log(N(t)/t^{\alpha})$ gives us the Weyl plot. The following are the Weyl plots of level 1 through level 4.









The α's for level 1, 2, 3, 4, 5 are 0.4863, 1.0615, 1.3181, 1.3957, 1.4105 respectively. Therefore, we would expect that the $N(x) \geq O(x^{1.4})$ for the eigenvalue counting function in the projective octagasket..

6. Heat Equation

The heat equation

$$\frac{\partial(u(x,t))}{\partial t} = \triangle u(x,t)$$

on our graph approximations given initial conditions

$$u(x,0) = f(x)$$

Can be solved by using the heat kernel. A construction which utilizes an orthonormal basis of the eigenspace for the Laplacian. The eigenfunction we found at each level can be renormalized to construct such a basis.

Given an orthonormal eigenbasis $\{\phi_i\}$ the heat kernel is defined as

$$h_t(x,y) = \sum_{\lambda} e^{-\lambda t} P_\lambda(x,y)$$

where $P_\lambda(x,y) = \sum_i \phi_i(x)\phi_i(y)$ Now, given our initial heat distribution $f(x)$ we can solve for $u(x,t)$ for any time t by simply taking the integral of the product of the heat kernel and the initial $f(x)$ with respect to our measure. Thereby a solution to the heat equation is given as

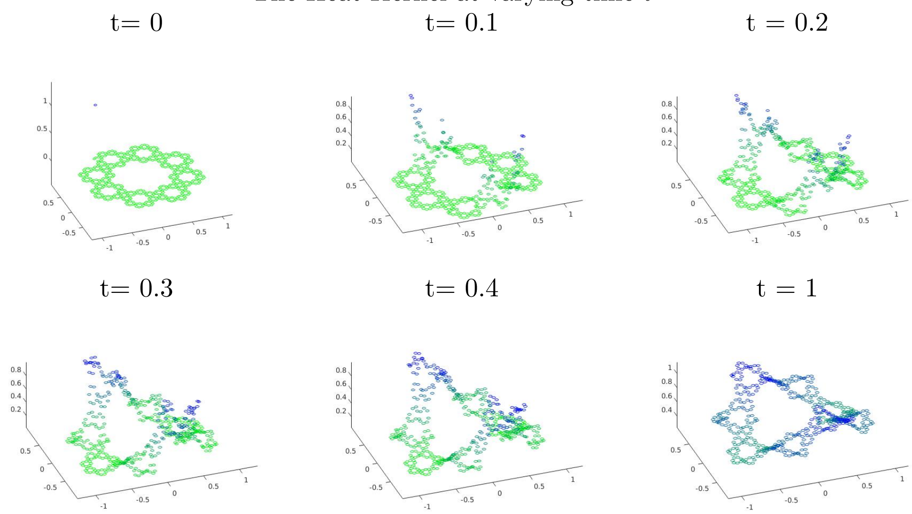
$$u(x,t) = \int_K h_t(x,y) f(y) d\mu(y)$$

This technique was demonstrated by Strichartz in [3].

Of course since we only have approximations to work with our integral turns into a finite sum. We were able to find solutions to heat equation especially when setting $f(x_0) = 1$ and $f(x) = 0$ for all $x \neq x_0$. This allowed us to simplify the finite sum to $u(x,t) = h_t(x, x_0)$.

The following is the solution to the heat equation when setting $f(00...0) = 1$ and $f(x) = 0$ if $x \neq 00...0$.

The Heat Kernel at varying time t



7. Wave Equation

Similar to the heat equation, we can solve the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \triangle u(x,t) = 0$$

on our graph approximations given the initial conditions

$$\begin{cases} u(x,0) = f_0(x) \\ \frac{\partial u}{\partial t}(x,0) = f_1(x) \end{cases}$$

If $f_0 = 0$, then the solution is given by

$$u(x,t) = \sum_{\lambda} \frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}} \int_K P_\lambda(x,y) f_1(y) d\mu(y)$$

, where $P_\lambda(x,y)$ is defined as the same as before [2].

Since the solution is not obvious when putting on the paper, we refer the interested reader to our website.

8. Geometry

8.1. **Metric in the Finite Graph.**

Given Γ_m, there exists a natural metric D_m on it. We call l a path in Γ_m if l is a sequence of m-cells $c_0c_1c_2...c_n$ with each pair of consecutive cells (c_i, c_{i+1}) is adjacent but not equal. In this case, we call l is of length n. Let w, v be m-cells in Γ_m, then we define $D_m(w, v)$ to be the minimum of the length of all paths with end points w and v. Let $d_m = diam(\Gamma_m)$. Our experimental results show that $d_1 = 2, d_2 = 7, d_3 = 15$, $d_4 = 29$ and $d_5 = 57$, and d_m is attained if $w = X00...0$, $v = (X \pm 2)00...0$ for some $X \in \mathbb{Z}_8$, which gives us the following conjecture:

Conjecture 8.1. *$d_m = 2^{m-2} \times 7 + 1$ for $m > 2$*

Though we cannot prove it at this stage, we have the following propositions instead.

Proposition 8.2. *$d_{m+1} \leq 2d_m + 2^m + 1$ for all $m > 0$*

Proof. Let $u = u_1u_2...u_{m+1}, v = v_1v_2...v_{m+1}$ be $(m+1)$-cells in Γ_m. Let U and V be the 1-cell containing u and v respectively.

If $U = V$ (i.e $u_1 = v_1$), then obviously , by the definition of d_m, $D_{m+1}(u, v) \leq d_m$.

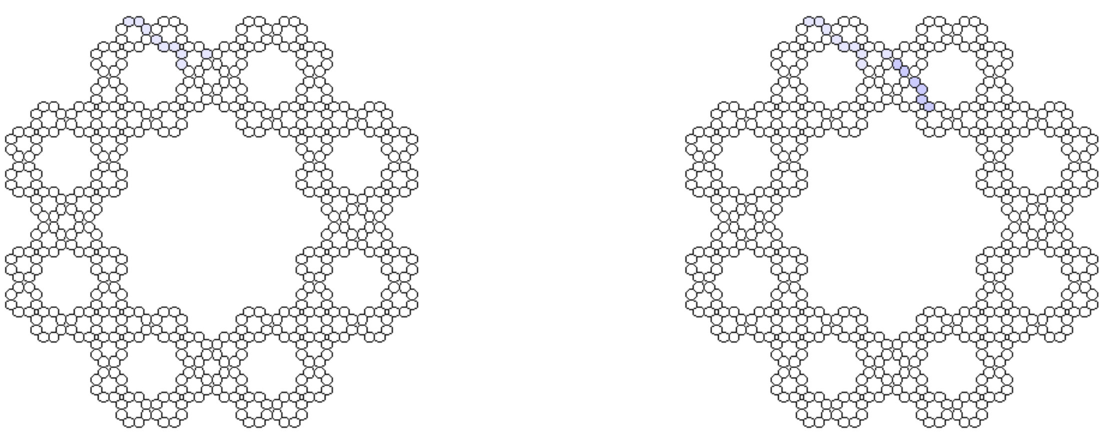
If the 1-cell containing u and the 1-cell containing v are adjacent (i.e. $u_1 = v_1 \pm 1$ or $v_1 \pm 4$), one can easily construct a path l starting from u, passing through the boundary between U and V, and then ending at v with $length(l) \leq 2d_m + 1$. So, $D_{m+1}(u, v) \leq 2d_m + 1$

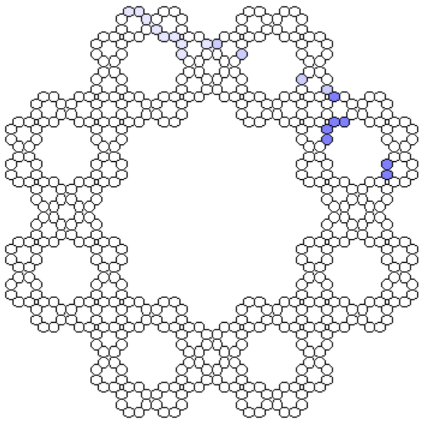
If $u_1 = v_1 \pm 2$, with loss of generality, say $u_1 = v_1 - 2$, then one can construct a path l_1 from u to $u_1200...0$, a path l_2 from $(u_1+1)600...0$ to $(u_1+1)200...0$, and a path l_3 from $(u_1+2)600...0$ to v with $length(l_1), length(l_3) \leq d_m$ and $length(l_2) \leq 2^m - 1$ (in fact it suffices to take l_2 to be the straight line from $(u_1+1)600...0$ to $(u_1+1)200...0$ in the Euclidean sense). Connecting them, we obtain a path from u to v with length $\leq 2d_m + 2^m + 1$. Hence, $D_{m+1}(u, v) \leq 2d_m + 2^m + 1$.

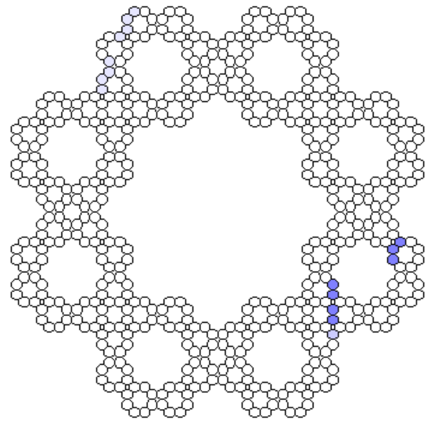
If $u_1 = v_1 \pm 3$, with loss of generality, say $u_1 = v_1 - 3$, then one can construct a path l_1 from u to $u_1600...0$, and a path l_2 from $(u_1-3)600...0$ with $length(l_1), length(l_2) \leq d_m$. Note that $(u_1+4)6000...0$ is adjacent to both $u_1600...0$ and $(u_1+3)200...0$. Connecting them, we obtain a path from u to v with length $\leq 2d_m + 2$. Hence, $D_{m+1}(u, v) \leq 2d_m + 2$

The following figure shows how we construct path from $u_1u_2...u_{m+1}$ to $u_1x_2...x_{m+1}$, $(u_1+1)x_2...x_{m+1}$, $(u_1+2)x_2...x_{m+1}$ and $(u_1+3)x_2...x_{m+1}$ respectively.

It follows that $d_{m+1} \leq 2d_m + 2^m + 1$. □







Corollary 8.3. *$d_m \leq 2^{m-1}(m+2) - 1$ for all $m > 0$*

Proof. Directly follows from Proposition 8.1 and the fact that $d_1 = 2$. □

Proposition 8.4. *$d_m \geq 2^{m-1} - 1$ for all $m > 0$*

Proof. It is trivial for $m = 1$.

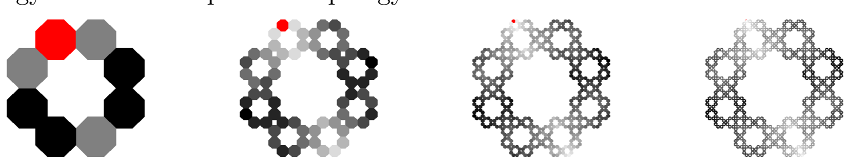
For $m > 1$, first observe that $d_m \geq D_m(\text{outer boundary}, \text{inner boundary})$. What we need is in fact $D_m(\text{outer boundary}, \text{inner boundary}) \geq 2^{m-1} - 1$.

□

The following pictures display the distance between 00...0 and other m-cells in Γ_m. The darkness of the cell represents the distance between that cell and 00...0, and the red cell is 00...0. The similarity of these pictures gives us an evidence to the existence of the metric $D : POG \times POG \to [0, 1]$, where D is determined by

$$D(v, w) = \lim_{m\to\infty} \frac{D_m(V_m, W_m)}{d_m}$$

for all $v, w \in V_*$, where V_m and W_m are m-cells in Γ_m with $\bigcap_{n=1}^{\infty} V_n = \{v\}$ and $\bigcap_{n=1}^{\infty} W_n = \{w\}$. In addition, it is believed that the metric induces the same topology as the usual quotient topology.



8.2. Metric in the Projective Octagasket.

8.3. Cardinality of Metric Balls in Finite Graph.

The existence of metric allows us to discuss balls of radius n of an m-cell v in Γ_m. The experimental results show that the cardinality of $B_m(v, n)$ for $m = 2, 3, 4$ are $C_2 n^{2.5377}, C_3 n^{2.7967}, C_4 n^{2.9483}, C_5 n^{3.0203}$ approximately for some constants C_2, C_3, C_4, C_5 when r is not too large. In fact, using the similar technique developed by [4], we have the following bound: $\#B_{m+1}(v, n) \leq \sqrt[3]{32}n^3$ if $n < d_m$.

Before getting into the bound, we first give some definitions and develop some lemmas. We first define an m-edge in Γ_{m+k}. Basically, an m-edge is a path travelling from a corner point of an octagon to the adjacent corner point with its locus is totally contained in the outer boundary of the m-cell, where the octagon is the m-cell in Γ_m with the same address as that m-cell in Γ_{m+k}.

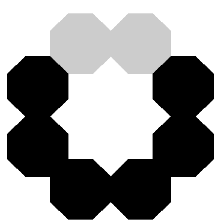
Definition 8.5. In Γ_{m+k} with $m > 0$, an m-edges E is a piecewise straight line in the outer boundary of the m-cell $X_1X_2...X_m$ travelling from the point $X_1X_2...X_m00...$ to $X_1X_2...(X_m+1)00....$ We write $E \in E_{vw}$ if $E \subseteq v \cap w$. If $E = v \cap w$, we write $E = E_{vw}$. A cell u is on an m-edge E if $u \cap E \neq \emptyset$. Two edges E, E' are adjacent if $E \cap E' \neq \emptyset$.

Remark 8.6. Each m-cell is on 8 distinct m-edge. Two m-cells may intersect on more than one m-edges.

Lemma 8.7. *In Γ_{m+k} with $m, k > 0$, if a path, with length $< 2^k - 1$, lies in an m-cell and begins at an $(m+k)$-cell on the m-edge, then it cannot pass through nor end at any $(m+k)$-cell on the non-adjacent m-edge.*

Proof. The proof is similar to Proposition 8.4, which is basically by induction on k. Without loss of generality, we first assume the m-edge is from the top left corner to the top right corner. Let $\mathcal{C}_{k,1}$ be the collection of the $(m+k)$-cell on that m-edge. We will construct layers of cells $\{\mathcal{C}_{k,i}\}_{i=1}^{2^k}$ in the m-cell such that the $\cup_{U \in C_{k,i}} U$ only intersect $\cup_{U \in C_{k,i+1}} U$ and $\cup_{U \in C_{k,i-1}} U$

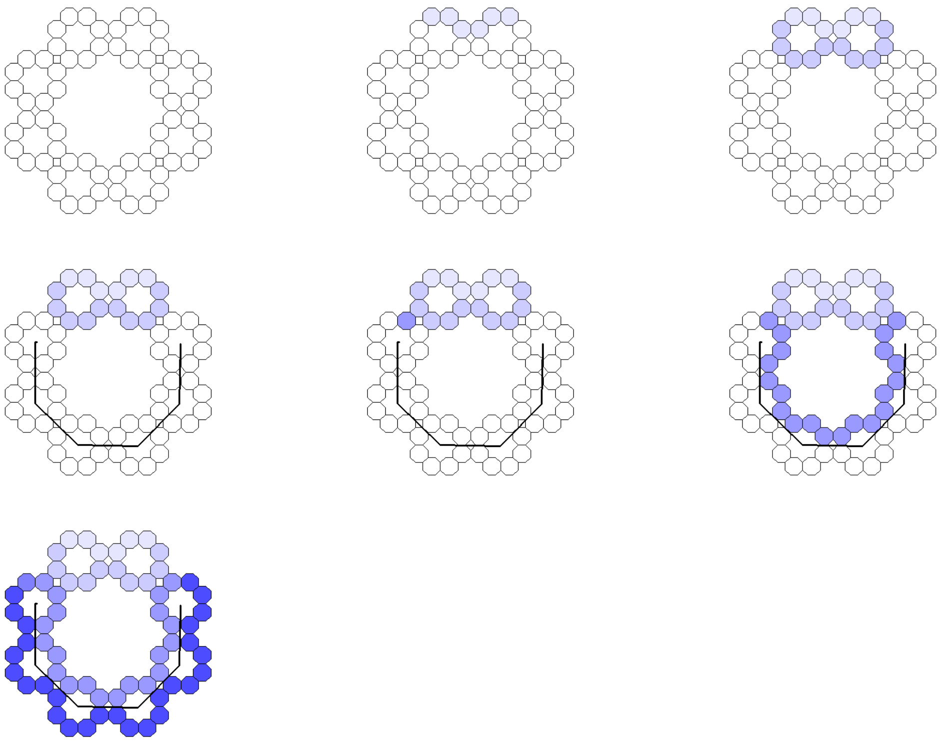
For $k = 1$, it is suffices to take the following layers, where grey denotes the cells in $\mathcal{C}_1$, black denotes the cells in $\mathcal{C}_2$.



Suppose the statement is true for $k = n$. We split the $\mathcal{C}_{n,i}$ into two layers. $\mathcal{C}_{n+1,1}$ is clear by the identification algorithm in section 4. $\mathcal{C}_{n+1,2}$ will be the collection of the remaining $m+n+1$-cells with the same prefix as the cells in $\mathcal{C}_{n,1}$.

To construct other layer $\mathcal{C}_{n+1,2i-1}$, one first draw a piecewise straight line l to connecting the centres of the every pair of adjacent cells in the $\mathcal{C}_{n,i}$. Now replace the octagons in $\mathcal{C}_{n,i}$ by Γ_1. One can inductively define $\mathcal{C}_{n+1,2i-1}$ by including the most top-left cell adjacent to some cells in $\mathcal{C}_{n+1,2i-2}$ and the cell clockwisely adjacent to the previous cell, unless the previous cell touch the line l. If this is the case, one will move to the next copy of Γ_1, and the cell adjacent (in the sense of octagasket) to the previous cell and repeat the above procedures, until reach the most top-right cell which is adjacent to some cells in $\mathcal{C}_{n+1,2i-2}$.

Then, $\mathcal{C}_{n+1,2i}$ will be the collection of the remaining $m+n+1$-cells with the same prefix as the cells in $\mathcal{C}_{n,1}$. The following figure shows how to construct the layers from $k = 1$ to $k = 2$.



One can see that the layers constructed satisfy our requirement, only the cells on $C_{n+1,2^{n+1}}$ are on the non-adjacent m-edge. Now given any path from the original m-edge to any non-adjacent m-edge, it must across every layer if it does not go outside the m-cell, which require the path of at least length $2^k - 1$, so our statement follows. □

Definition 8.8. Consider Γ_{m+k} with $m, k > 0$. Write a path l in Γ_{m+k} as a sequence of $m+k$-cells $c_0, c_1, ..., c_n$, with each c_i has address $c_{i,1}c_{i,2}...c_{i,m+k}$. We can form a sequence of m-cells $\{V_i\}$ associate with l as follows:

(1) For each c_i, let $V_i = c_{i,1}c_{i,2}...c_{i,m}$.
(2) While consecutive elements in the new sequence are equal, delete all but one of them, and relabel the index if necessary.

We call the set $\{V_i\}$ the m-sequence of l.

Definition 8.9. Given an m-sequence $\{V_i\}$ for a path l in Γ_{m+k}, we call a tuple (i, X, Y) an m-segment-of-two if $X = V_i, Y = V_{i+1}$ and $V_{i-1} \neq X$ and Y (if it exists).

A path l is said to enter an m-segment-of-two (i, X, Y) through an m-edge E if there exists some consecutive cells $c_n, c_{[}n+1$ such that both of them are on E and only c_{n+1} is within (i, X, Y). The notion of a path exiting is analogous, with c_n in (i, X, Y) instead. The collection of the m-edge along which the path enters is denoted by $Ent(i)$, and the m-edge along which it exists is denoted by $Exit(i)$.

Lemma 8.10. *Let (i, X, Y) be an m-segment-of-two associated to a path of length at most $2^m - 1$.*

(1) Let $E_1 \in Ent(i)$ (if exists). $E_1 \notin E_{XY}$, but there exists some $E_2 \in E_{XY}$ such that E_1 intersects E_2. Similar results also hold for $Exit(i)$.
(2) If $Ent(i)$ and $Exit(i)$ exists, then there exists unique $E_1 \in Ent(i), E_2 \in Exit(i), E_3 \in E_{XY}$ such that $E_1 \cap E_2 \cap E_3$ is a singleton, which is called the centre of the (i, X, Y).

Lemma 8.11. *If l is a path of length at most $2^m - 1$ and at least two m-segments-of-two, then the centres of the m-segments-of-two are the same.*

Proposition 8.12. *For a path l of length at most $2^m - 1$. Let U be the first k-cell in the m-sequence associated with l. Then there exists a point x inside V_i for all i.*

Corollary 8.13. *Let $x\Gamma_{m+k}$, V_x be a k-cell containing x, $\mathcal{V}$ be the collection of all k-cells adjacent to V_x. Any path starts at V_x with length at most $2^m - 2$ cannot leave $\bigcup_{V \in \mathcal{V}} V$; i.e. $B_{m+k}(V_x, 2^m - 2) \subseteq \bigcup_{V \in \mathcal{V}} V$.*

Lemma 8.14. $\#\mathcal{V} \leq 6$

Proposition 8.15. $\#B_{m+1}(v, n) \leq^3$ *if* $n < d_m$.

9. Conclusion

We have worked out a rigorous recursive algorithm for the construction of a Laplacian on the cell graph approximation of the Projective Octagasket. The computation of higher levels is not feasible due to the rapid growth rate of Γ_m. The asymptotics of the eigenvalue counting function convey important information and our estimates could be improved by constructing another iteration of Γ_m. Our cell graph approximation was used to ease the computational complexity of the identification algorithm. A more thorough analysis using the vertex identification would perhaps yield better numeric results, including a renormalization factor that stays consist from level to level. Overall our experimental data suggests that the Projective Octagasket is a difficult yet worthwhile example in the class of non-finitely ramified fractals.

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[3] M. Begue, T. Kalloniatis, and R. S. Strichartz, Harmonic functions and the spectrum of the Laplacian on the Sierpinski carpet, Fractals 21 (2013), no. 1.

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