18.095: Calculus of Finite Differences

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Studying Sequences As If They Were Functions

- Why do this?
- For fun (sequences are cool!)
- To count things (combinatorics, computer science).
- To model reality (numerical solution of differential equations).
- Some problems about functions are most easily solved by translating into a problem about sequences (power series, Fourier series) and vice versa (generating functions).
- Shows the power of reasoning by analogy.

Three Levels of Structure

1.	Real numbers: $-2, \frac{7}{3}, \sqrt{5}, e, \pi,$	$\ldots$
2.	Functions $f : \mathbb{R} \to \mathbb{R}$	Sequences $s : \mathbb{N} \to \mathbb{R}$
3.	Differential Operators: $\frac{d}{dt}$	Difference Operators: $D, E, \ldots$

- Differential operators map functions to functions.
- Difference operators map sequences to sequences.

Adding and Multiplying Operators

- Addition is term by term: if

$$A(s_0, s_1, s_2, \dots) = (a_0, a_1, a_2, \dots)$$

and

$$B(s_0, s_1, s_2, \dots) = (b_0, b_1, b_2, \dots)$$

then we define

$$(A + B)(s_0, s_1, s_2, \dots) = (a_0 + b_0, a_1 + b_1, a_2 + b_2, \dots)$$

- Multiplication is composition: $(AB)(s) = A(B(s))$

The Shift Operator and the Difference Operator

- Shift operator: $E(s_0, s_1, s_2, \ldots) = (s_1, s_2, s_3, \ldots)$.
- Identity Operator: $I(s_0, s_1, s_2, \ldots) = (s_0, s_1, s_2, \ldots)$.
- Difference Operator: $D = E - I$.

$$D(s_0, s_1, s_2, \ldots) = (s_1 - s_0, s_2 - s_1, s_3 - s_2, \ldots).$$

The Fibonacci Sequence

- Let $F_0 = 0$, $F_1 = 1$, and

$$F_{n+2} = F_{n+1} + F_n \qquad \text{for } n = 0, 1, 2, \ldots.$$

- This recurrence relates the sequence F to its shifts:

$$\begin{array}{rcl} F & = & (0, \; 1, \; 1, \; 2, \; 3, \; 5, \; 8, \; 13, \; 21, \; \ldots) \\ EF & = & (1, \; 1, \; 2, \; 3, \; 5, \; 8, \; 13, \; 21, \; 34, \; \ldots) \\ E^2F & = & (1, \; 2, \; 3, \; 5, \; 8, \; 13, \; 21, \; 34, \; 55, \; \ldots) \end{array}$$

- Infinitely many equations encoded in one:

$$E^2F = EF + F.$$

Factoring An Operator

- The Fibonacci sequence F satisfies

$$(E^2 - E - \mathbf{1})F = 0.$$

- Suppose we factor this quadratic:

$$(E - \phi)(E - \bar{\phi})F = 0,$$

where

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \bar{\phi} = \frac{1-\sqrt{5}}{2}.$$

Eigenvectors of the Shift Operator

- How would we solve the simpler equation

$$(E - \phi)s = 0?$$

- This says $Es = \phi s$, or

$$(s_1, s_2, s_3, \dots) = (\phi s_0, \phi s_1, \phi s_2, \dots).$$

- So

$$\begin{aligned} s_1 &= \phi s_0 \\ s_2 &= \phi s_1 = \phi^2 s_0 \\ s_3 &= \phi s_2 = \phi^3 s_0 \\ &\dots \\ s_n &= \phi s_{n-1} = \phi^n s_0. \end{aligned}$$

The Charm of Commutativity

- Likewise, the general solution to

$$(E - \bar{\phi})t = 0$$

is $t_n = t_0 \bar{\phi}^n$.

- The sequences s and t satisfy

$$(E^2 - E - \mathbf{1})s = (E - \bar{\phi})(E - \phi)s = 0.$$

$$(E^2 - E - \mathbf{1})t = (E - \phi)(E - \bar{\phi})t = 0.$$

A Basis For The Solution Space

- The set of all sequences u satisfying the Fibonacci recurrence

$$(E^2 - E - \mathbf{1})u = 0$$

  is a 2-dimensional vector space.
- The sequences $s_n = \phi^n$ and $t_n = \bar{\phi}^n$ form a basis for this space.
- Let's write the Fibonacci sequence in this basis:

$$F = as + bt.$$

- Solving for a and b gives the famous formula

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \bar{\phi}^n \right)$$

- Since $\bar{\phi} = \frac{1-\sqrt{5}}{2} \approx -0.618$, the second term is extremely tiny.
- So F_n is the closest integer to $\phi^n/\sqrt{5}$.

$$F_{10} = 55, \quad \phi^{10}/\sqrt{5} = 55.0036.$$

$$F_{11} = 89, \quad \phi^{11}/\sqrt{5} = 88.9978.$$

How Fast Do Rabbits Multiply?

- Now we can answer Fibonacci's original question:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\text{closest integer to } \phi^{n+1}/\sqrt{5}}{\text{closest integer to } \phi^n/\sqrt{5}} = \phi \approx 1.618.$$

The Fibonacci Recurrence Is Just The Tip of The Iceberg

- How do we solve recurrences like

$$s_{n+2} = 2s_{n+1} + s_n$$

and

$$s_{n+3} = s_{n+2} + s_{n+1} + s_n \ ?$$

Linear Recurrences

- **Definition**: A sequence of complex numbers $s = (s_0, s_1, s_2, \ldots)$ obeys a linear recurrence of order k if there exist constants $a_0, \ldots, a_{k-1} \in \mathbb{C}$, with $a_0 \neq 0$, such that

$$s_{n+k} = \sum_{i=0}^{k-1} a_i s_{n+i}, \qquad n = 0, 1, 2, \ldots.$$

The General Method

- Write the recurrence in the form

$$(p(E))s = 0$$

for some polynomial p.

- Factor the polynomial

$$p(E) = (E - \phi_1)\dots(E - \phi_k).$$

- If the complex numbers $\phi_1, \dots, \phi_k$ are distinct, we say that s obeys a *simple linear recurrence*.

Main Theorem

- A sequence of complex numbers $s = (s_0, s_1, s_2, \ldots)$ obeys a simple linear recurrence of order k if and only if it can be written in the form

$$s_n = c_1 \phi_1^n + \ldots + c_k \phi_k^n$$

for some complex numbers $\phi_1, \ldots, \phi_k$ and $c_1, \ldots, c_k$.

The Converse Direction

- According to the Main Theorem, the sequence $s = (0, 1, 5, 19, 65, 211, \ldots)$, whose n-th term is

$$s_n = 3^n - 2^n$$

obeys a linear recurrence of order 2.
- How do we find this recurrence?

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- How do we find this recurrence?
- We have $\phi_1 = 3$ and $\phi_2 = 2$, so the recurrence is $(p(E))s = 0$, where

$$p(E) = (E - 3)(E - 2) = E^2 - 5E + 6.$$

- In other words,

$$s_{n+2} - 5s_{n+1} + 6s_n = 0.$$

What If The Polynomial $p(E)$ Has Multiple Roots?

- Say we want to solve the recurrence

$$s_{n+3} = 3s_{n+2} - 3s_{n+1} + s_n.$$

- We can write this as $p(E)s = 0$, where

$$p(E) = E^3 - 3E^2 + 3E - 1 = (E - 1)^3.$$

- Our method so far only gives one solution

$$s_n = 1.$$

There should be two more...

The Difference Operator

- Remember that $D = E - 1$, so we can write our recurrence as

$$D^3 s = 0.$$

- Reasoning by analogy, the differential equation

$$\frac{d^3}{dx^3}[f] = 0$$

has the three solutions $f(x) = 1, x, x^2$.

- So let's try out the three sequences

$$s_n = 1, \quad s_n = n, \quad s_n = n^2.$$

Polynomial Solutions

- The general solution to $D^3 s = 0$ has the form

$$s_n = an^2 + bn + c.$$

- More generally, we have

$$D^m s = 0$$

  if and only if

$$s_n = q(n)$$

  for some polynomial q of degree $\leq m - 1$.
- This suggests how to deal with the issue of multiple roots...

Main Theorem, General Version

- A sequence of complex numbers $s = (s_0, s_1, s_2, \dots)$ obeys the linear recurrence

$$\prod_{i=1}^{k} (E - \phi_i)^{m_i} s = 0$$

if and only if it can be written in the form

$$s_n = q_1(n)\phi_1^n + \dots + q_k(n)\phi_k^n$$

where each q_i is a polynomial of degree $\leq m_i - 1$.

Exponential Generating Functions

- The exponential generating function of a sequence $s = (s_0, s_1, s_2, \ldots)$ is

$$\mathcal{F}_s(x) = s_0 + s_1 x + s_2 \frac{x^2}{2} + s_3 \frac{x^3}{6} + \ldots + s_n \frac{x^n}{n!} + \ldots.$$

- For example, if $s = (1, 1, 1, 1, \ldots)$, then

$$\mathcal{F}_s(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!} + \ldots = e^x.$$

Relating E and $\frac{d}{dx}$

- Since

$$\frac{d}{dx}\left[\frac{x^n}{n!}\right] = \frac{nx^{n-1}}{n!} = \frac{x^{n-1}}{(n-1)!}$$

we have

$$\begin{aligned}\frac{d}{dx}\left[\mathcal{F}_s(x)\right] &= \frac{d}{dx}\left[s_0 + s_1 x + s_2\frac{x^2}{2} + s_3\frac{x^3}{6} + \ldots + s_n\frac{x^n}{n!} + \ldots\right] \\ &= s_1 + s_2 x + s_3\frac{x^2}{2} + \ldots + s_{n+1}\frac{x^n}{n!} + \ldots \\ &= \mathcal{F}_{Es}(x).\end{aligned}$$

Recurrences Become Differential Equations

- If the sequence s obeys a linear recurrence $(p(E))s = 0$, then its exponential generating function obeys the differential equation

$$p\left(\frac{d}{dx}\right)[\mathcal{F}_s(x)] = F_{(p(E))s}(x) = F_0(x) = 0.$$

Example: Solving the Differential Equation $f'' = f' + f$

- We can write this equation as

$$\left(\frac{d^2}{dx^2} - \frac{d}{dx} - \mathbf{1}\right) f = 0.$$

- Since the Fibonacci sequence obeys the corresponding recurrence

$$(E^2 - E - \mathbf{1})F = 0$$

its exponential generating function

$$f(x) = \sum_{n=1}^{\infty} F_n \frac{x^n}{n!}$$

is a solution to $f'' = f' + f$.

Example: Deriving The Power Series for $\sin x$

- Since

$$\frac{d^2}{dx^2}[\sin x] = -\sin x$$

we have

$$\sin x = \mathcal{F}_s(x)$$

for a sequence s obeying the recurrence

$$E^2 s = -s.$$

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- Since $s_0 = \sin(0) = 0$ and $s_1 = \frac{d}{dx}[\sin x]|_{x=0} = \cos(0) = 1$, we get
$$s = (0, 1, 0, -1, 0, 1, 0, -1, \dots)$$
so
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots.$$

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Example: Taylor Series

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2}t^2 + \frac{f'''(0)}{6}t^3 + \ldots + \frac{f^{(n)}(0)}{n!}t^n + \ldots$$

- How would anyone discover this formula?
- If we allow ourselves a few "non-rigorous" steps, we can derive it from finite differences...

We Can Shift Functions Too

- $(Ef)(x) = f(x+1)$
- $(E^2 f)(x) = E(E(f))(x) = (E(f))(x+1) = f(x+2)$
- $(E^3 f)(x) = f(x+3)$
- $(E^h f)(x) = f(x+h)$.

Relating $\frac{d}{dx}$ And E As Operators

- From the definition of the derivative,

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \left(\frac{E^h - \mathbf{1}}{h} f \right)(x).$$

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- $\lim_{h\to 0} \frac{t^h-1}{h} = \lim_{h\to 0} \frac{e^{h\ln t}-1}{h} = \lim_{h\to 0} \frac{(\ln t)e^h}{1} = \ln t.$
- So $\boxed{\frac{d}{dx} = \ln E}$.
- How to interpret this??

Expanding $e^{\frac{d}{dx}}$ As A Series

- If $\frac{d}{dx} = \ln E$, then $E = e^{\frac{d}{dx}}$. Let's try expanding this:

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \ldots + \frac{t^n}{n!} + \ldots.$$

$$e^{t\frac{d}{dx}} = \mathbf{1} + t\frac{d}{dx} + \frac{t^2}{2}\frac{d^2}{dx^2} + \frac{t^3}{6}\frac{d^3}{dx^3} + \ldots + \frac{t^n}{n!}\frac{d^n}{dx^n} + \ldots$$

- This is an equation of operators, so we can plug in a function:

$$e^{t\frac{d}{dx}}[f] = f + tf' + \frac{t^2}{2}f'' + \frac{t^3}{6}f''' + \ldots + \frac{t^n}{n!}f^{(n)} + \ldots$$

Taylor Series

- But $e^{t\frac{d}{dx}}$ is just E^t, so:

$$f(t) = (E^t f)(0) = e^{t\frac{d}{dx}}[f](0)$$
$$= f(0) + f'(0)t + \frac{f''(0)}{2}t^2 + \frac{f'''(0)}{6}t^3 + \ldots + \frac{f^{(n)}(0)}{n!}t^n + \ldots$$

which is what we wanted!

Further Reading

- Kelley and Peterson, *Difference Equations*, 1991.
- Jordan, *Calculus of Finite Differences*, 1965.
- Stanley, *Enumerative Combinatorics* vol. 1, chapter 4.

- These slides & the homework problems will be posted at http://math.mit.edu/∼levine/18.095