18.312: Algebraic Combinatorics	Lionel Levine
Lecture 7	
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1 Partially ordered sets

1.1 Definitions

Definition 1 *A* partially ordered set *(*poset *for short) is a set P with a binary relation $R \subseteq P \times P$ satisfying all of the following conditions.*

1. *(reflexivity) $(x, x) \in R$ for all $x \in P$*
2. *(antisymmetry) $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$*
3. *(transitivity) $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$*

In analogy with the order on the integers by size, we will write $(x, y) \in R$ as $x \leq y$ (or equivalently, $y \geq x$). We will use $x < y$ to mean that $x \leq y$ and $x \neq y$. When there are multiple posets in play, we can disambiguate by using the name of the poset as a subscript, e.g. $x \leq_P y$.

Remark 2 *The word "partial" indicates that there's no guarantee that all elements can be compared to each other—i.e. we don't know that for all $x, y \in P$, at least one of $x \leq y$ and $x \geq y$ holds. A poset in which this is guaranteed is called a* totally ordered set.

Partially ordered sets can be visualized via *Hasse diagrams*, which we now proceed to define.

Definition 3 *Given x, y in a poset P, the* interval $[x, y]$ *is the poset $\{z \in P \mid x \leq z \leq y\}$ with the same order as P.*

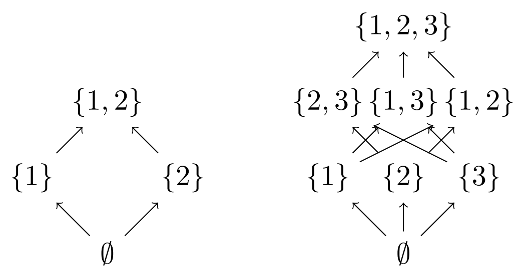
Definition 4 *"y* covers *x" means $[x, y] = \{x, y\}$. That is, no elements of the poset lie strictly between x and y (and $x \neq y$).*

Definition 5 *The* Hasse diagram *of a partially ordered set P is the (directed) graph whose vertices are the elements of P and whose edges are the pairs (x, y) for which y covers x. It is usually drawn so that elements are placed higher than the elements they cover.*

1.2 Examples

1. **n** (handwritten as $\underline{n}$) is the set $[n]$ with the usual order on integers.
2. The *Boolean algebra* B_n is the set of subsets of $[n]$, ordered by inclusion. ($S \leq T$ means $S \subseteq T$).

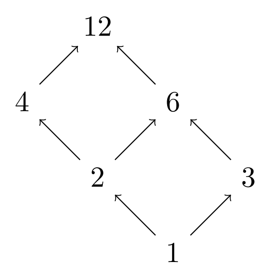
Figure 1: Hasse diagrams of B_2 and B_3



B_2 B_3

3. $D_n = \{\text{all divisors of } n\}$, with $d \leq d' \iff d \mid d'$.

Figure 2: $D_{12} = \{1, 2, 3, 4, 6, 12\}$



4. $\Pi_n = \{\text{partitions of } [n]\}$, ordered by refinement. [1]
5. Generalizing B_n, any collection P of subsets of a fixed set X is a partially ordered set ordered by inclusion. For instance, if X is a vector space then we can take P to be the set of all linear subspaces. If X is a group, we can take P to be the set of all subgroups or the set of all normal subgroups.

[1] A *partition* of a set X is a set of disjoint subsets of X whose union is X. We say that a partition σ *refines* another partition τ (so, in the example, $\sigma \leq \tau$) if every $\sigma_i \in \sigma$ is a subset of some $\tau_{j(i)} \in \tau$.

2 Maps between partially ordered sets

Definition 6 *A function $f : P \to Q$ between partially ordered sets is* order-preserving *if $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$.*

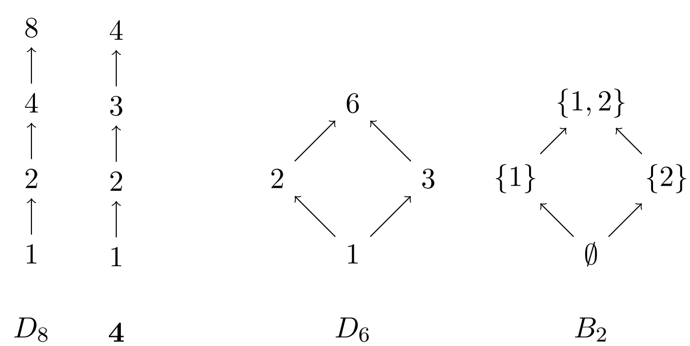
Definition 7 *Two partially ordered sets P and Q are* isomorphic *if there exists a bijective, order-preserving map between them whose inverse is also order-preserving.*

Remark 8 *For those familiar with topology, this should look like the definition of homeomorphic spaces—spaces linked by a continuous bijection whose inverse is also continuous. A continuous bijection can fail to have a continuous inverse if the topology of the domain has extra open sets; and an order-preserving bijection between posets can fail to have a continuous inverse if the codomain has extra order information.*

2.1 Examples

1. $D_8 \simeq \mathbf{4}$
2. $D_6 \simeq B_2$

Figure 3: Hasse diagrams of isomorphic posets



3 Operations on partially ordered sets

Given two partially ordered sets P and Q, we can define new partially ordered sets in the following ways.

1. (Disjoint union) $P + Q$ is the disjoint union set $P \sqcup Q$, where $x \leq_{P+Q} y$ if and only if one of the following conditions holds.

   - $x, y \in P$ and $x \leq_P y$
   - $x, y \in Q$ and $x \leq_Q y$

   The Hasse diagram of $P + Q$ consists of the Hasse diagrams of P and Q, drawn together.

2. (Ordinal sum) $P \oplus Q$ is the set $P \sqcup Q$, where $x \leq_{P \oplus Q} y$ if and only if one of the following conditions holds.

   - $x \leq_{P+Q} y$
   - $x \in P$ and $y \in Q$

   Note that the ordinal sum operation is not commutative. In $P \oplus Q$, everything in P is less than everything in Q.

3. (Cartesian product) $P \times Q$ is the Cartesian product set, $\{(x, y) \mid x \in P, y \in Q\}$, where $(x, y) \leq_{P \times Q} (x', y')$ if and only if both $x \leq_P x'$ and $y \leq_Q y'$.

   The Hasse diagram of $P \times Q$ is the Cartesian product of the Hasse diagrams of P and Q.

   Example 9 $B_n \simeq \underbrace{\mathbf{2} \times \cdots \times \mathbf{2}}_{n \text{ times}}$

   Proof: Define a candidate isomorphism

   $$\begin{aligned} f : \mathbf{2} \times \cdots \times \mathbf{2} &\to B_n \\ (b_1, \cdots, b_n) &\mapsto \{i \in [n] \mid b_i = 2\}. \end{aligned}$$

   It's easy to show that f is bijective. To check that f and f^{-1} are order-preserving, just observe that each of the following conditions is equivalent to the ones that come before and after it.

   - $(b_1, \cdots, b_n) \leq (b'_1, \cdots, b'_n)$
   - $b_i \leq b'_i$ for all i
   - $\{i \mid b_i = 2\} \subseteq \{i \mid b'_i = 2\}$
   - $f((b_1, \cdots, b_n)) \leq f((b'_1, \cdots, b'_n))$

   □

   Example 10 *If $k = p_1 \cdots p_n$ is a product of n distinct primes, then $D_k \simeq B_n$.*

The proof of Example 10 is similarly easy, using the isomorphism $f : D_k \to B_n$ defined by $\prod_{i\in S} p_i \mapsto S$.

4. P^Q is the set of order-preserving maps from Q to P, where $f \leq_{P^Q} g$ means that $f(x) \leq_P g(x)$ for all $x \in Q$.

   The notation P^Q can be motivated by a basic example.

   Example 11

$$P = \overbrace{\mathbf{1} + \cdots + \mathbf{1}}^{n}$$
$$Q = \overbrace{\mathbf{1} + \cdots + \mathbf{1}}^{k}$$
$$P^Q \simeq \overbrace{\mathbf{1} + \cdots + \mathbf{1}}^{n^k}$$

   Perhaps more importantly, the following properties hold (the proof is the 15th homework problem).

$$P^{Q+R} \simeq P^Q \times P^R$$
$$\left(P^Q\right)^R \simeq P^{Q\times R}$$

   Example 12 *The partially ordered set* $\mathbf{2}^{\mathbf{2}}$ *is isomorphic to* $\mathbf{3}$.

   Proof: The order-preserving maps are specified by $f_1(1) = f_1(2) = 1$, $f_2 = id$, and $f_3(1) = f_3(2) = 2$; so $f_1 \leq f_2 \leq f_3$. □

4 Graded posets

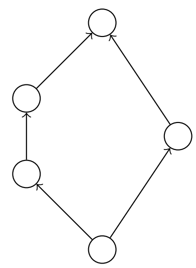
Definition 13 *A* chain *of a partially ordered set P is a totally ordered subset $C \subseteq P$—i.e. $C = \{x_0, \cdots, x_\ell\}$ with $x_0 \leq \cdots \leq x_\ell$. The quantity $\ell = |C| - 1$ is its* length *and is equal to the number of edges in its Hasse diagram.*

Definition 14 *A chain is* maximal *if no other chain strictly contains it.*

Definition 15 *The* rank *of P is the length of the longest chain in P.*

Definition 16 *P is* graded *if all maximal chains have the same length.*

Figure 4: Hasse diagram of a poset that is not graded



Definition 17 *A* rank function *on a poset P is a map $r : P \to \{0, \cdots, n\}$ for some n, satisfying the following properties.*

1. *$r(x) = 0$ for all minimal x (i.e. there is no $y < x$).*
2. *$r(x) = n$ for all maximal x.*
3. *$r(y) = r(x) + 1$ whenever y covers x.*

Lemma 18 *P is graded of rank $n \iff$ there exists a rank function $r : P \to \{0, \cdots, n\}$.*

Example 19 *B_n is graded, and cardinality is a rank function on B_n.*

Proof:

$\Rightarrow$: If P is graded of rank n, define $r(x) = \#\{y \in C \mid y < x\}$ where C is a maximal chain containing x. To check that this is well-defined, we need to show that it is independent of C.

So suppose C and C' are maximal chains containing x. Write

$$C = C_0 \sqcup \{x\} \sqcup C_1$$
$$C' = C'_0 \sqcup \{x\} \sqcup C'_1$$

where $C_0 = \{y \in C \mid y < x\}$ and $C'_0 = \{y \in C' \mid y < x\}$. If $|C_0| \neq |C'_0|$, then assuming without loss of generality that $|C_0| > |C'_0|$, the chain $C_0 \cup x \cup C'_1$ would have length greater than n. P being graded of rank n disallows this, so $|C_0| = |C'_0| = r(x)$.

This establishes that $r(x)$ is well-defined. It is easy to see by maximality of the chains involved that r is indeed a rank function.

$\Leftarrow$: Given a rank function $r : P \to \{0, \cdots, n\}$ and a maximal chain $C = \{x_0, \cdots, x_\ell\}$, we observe that

- x_0 is minimal (otherwise C could be extended by anything less than x_0),
- x_ℓ is maximal (otherwise C could be extended by anything greater than x_ℓ), and
- x_{i+1} covers x_i (otherwise the element between them could be inserted into C).

Then $r(x_0) = 0$, $r(x_\ell) = n$, and $r(x_{i+1}) = r(x_i) + 1$ for $i = 0, 1, \ldots, \ell - 1$, so we see that $\ell = n$.

□

Remark 20 *If a rank function exists, it is in fact uniquely defined.*

Corollary 21 *Any interval in a graded poset is graded.*

Proof: For $[x, y] \subset P$, use the rank function $r_{[x,y]}(z) = r_P(z) - r_P(x)$. □

5 Lattices

Definition 22 *A poset L is a* lattice *if every pair of elements x, y has*

- *a* least upper bound $x \vee y$ *(a.k.a.* join*), and*
- *a* greatest lower bound $x \wedge y$ *(a.k.a.* meet*);*

i.e.

$$z \geq x \vee y \iff z \geq x \text{ and } z \geq y$$
$$z \leq x \wedge y \iff z \leq x \text{ and } z \leq y.$$

Example 23 *B_n is a lattice. The meet and join can be explicitly specified as*

$$S \cap T = S \wedge T \qquad\qquad S \cup T = S \vee T,$$

and this can serve as a mnemonic for the symbols.

Figure 5: Hasse diagram of part of a lattice