18.312: Algebraic Combinatorics — Lionel Levine

Problem Set 1

Due in class on Feb 10, 2011

You are allowed (in fact, encouraged) to work in groups, but the solutions you hand in should be written by you only. If you work in a group, you must write the names of your collaborators at the top of your assignment. Explain your reasoning to receive full credit, even for computational questions.

P1 Let p be prime and let n and k be positive integers. Prove that $n^{p^k} - n^{p^{k-1}}$ is divisible by p^k.

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P2 Let P_n be the product of all primes $\leq n$. Show that $\begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix}$ is divisible by every prime number strictly between n and $2n$, and use this to prove that

$$P_n \leq 4^n.$$

(According to the Prime Number Theorem, which we will not prove in this class, the actual size of P_n is about e^n. More precisely, $(\ln P_n)/n \to 1$ as $n \to \infty$.)

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P3 Write down the first 16 rows of Pascal's triangle (mod 2). What's going on? Formulate a conjecture and try to prove it. For example, your conjecture could take the form

$$\begin{pmatrix} n \\ k \end{pmatrix} \equiv \begin{pmatrix} f(n,k) \\ g(n,k) \end{pmatrix} \pmod 2$$

where f and g are functions you define. Or it could take the form

"$\begin{pmatrix} n \\ k \end{pmatrix}$ is even if and only if ..."