CoEulerian Graphs

LIONEL LEVINE

joint work with MATT FARRELL

CMU ACO Seminar
4/16/2020
$G = (V,E)$ finite directed graph

Assume strongly connected: $\forall x, y \in V$

$\exists$ paths $x \rightarrow y, y \rightarrow x.$

**Def** $G$ is Eulerian if $\exists$ tour $e_0, e_1, \ldots, e_m$

such that each directed edge of $G$ appears exactly once.

Then $G$ Eulerian ($\implies$) $\forall x \in V$ $\text{indeg}(x) = \text{outdeg}(x)$.
A **directed graph Laplacian** $\Delta : \mathbb{Z}^v \rightarrow \mathbb{Z}^v$

$$\Delta f(v) = \partial_v f(v) - \sum_{e : e^+ = v} f(e^-)$$

and

$$\partial_v = \text{outdeg}(v) = \# \{ e \in E \mid e^- = v \}$$

and

$e$ is oriented from $e^-$ to $e^+$

$$\Delta = \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix} - \begin{pmatrix} a_{ij} \end{pmatrix}$$
DEF: An oriented spanning tree of $G$ oriented toward $\text{rev}(V)$ is an acyclic subgraph $T = (V, A)$ satisfying

$$\text{outdeg}_T(r) = 0$$
$$\text{outdeg}_T(v) = 1 \quad \forall v \in V \setminus r$$

MTT, MCTT, BEST: 3 cornerstones of "Alg. Dir. Graph Thy"
**Matrix Tree Theorem:** Let $K(r)$ be the number of oriented SP. trees of $G$ oriented toward $r$. Then

$$K(r) = \det(\Delta_r)$$

Cross off row $r$ & col $r$ from $\Delta$.

Pham 2016: $\phi = \gcd \{ K(r) \} \subseteq \mathbb{V}$.

Measures "Eulerianiness" of $G$.

$\phi = K(r) \quad \forall r \iff G \text{ is Eulerian}$.

**Def:** $G$ is coEulerian if $\phi = 1$. 
\[ x(r) = 3^{d-k} \]
\[ \phi = \gcd(3^n, 2^n) = 1. \]

**DEF:** CHIP CONFIG

**FIRING** VERTEX \( V \): SEND 1 CHIP FROM \( V \) ALONG EACH EDGE WITH \( e^{-} = V \).

**LEGAL** IF \( \sigma(V) \geq d_v \)

**DEF:** STABILIZES IF \( \exists \) LEGAL FIRING SEQ. \( \sigma_1, \sigma_2, \ldots, \sigma_k \) WITH \( \sigma_k \) STABLE: \( \sigma_k(V) < d_v \) \( \forall \varepsilon \).
HALTING PROBLEM FOR CHIP-FIRING:

Given (the adjacency matrix of) $G$, chip config $\sigma \in \mathbb{Z}^V$

Decide whether $\sigma$ stabilizes.

Sometimes easy: if $|\sigma_1| > |\sigma_{\text{max}}| = \sum_{v \in V} (d_v - 1)$,

by pigeonhole there is always an unstable vertex

$0 \ldots 0$

$|\sigma_1| = |\sigma_1| \Rightarrow \exists v \ s.t. \ \sigma_v(v) \geq d_v.$

Converse? Usually fails, but...
(CLASSICAL) EULERIAN

**THM:**

\[
\ker(\Delta) = \mathbb{Z}_1
\]

\[\Downarrow\]

\[\phi = \chi(s) \quad \forall s \in V\]

\[\Downarrow\]

\[G \text{ has Eul. Tour}\]

\[\mathbf{CoEULERIAN}\]

\[\text{Im}(\Delta) = \mathbb{Z}_0^V\]

\[\Downarrow\]

\[\phi = 1\]

\[\Downarrow\]

\[\forall s \in V \text{ with } |s| \leq |s_{\text{max}}|, \quad \sigma \text{ stabilizes!}\]
G. Tardos 1988: \( G \) bidirected, if \( \exists \) legal firing seq.

For \( \Gamma \) where every \( uv \) fires at least once, then \( \Gamma \) does not stabilize.

\[ \rightarrow \text{poly. time algo to check for stabilization.} \]

B. Berger-Lovasz 1992: \( \exists! \) primitive \( \pi \in \mathbb{N}^V \)

such that \( \Delta \pi = 0 \). If each \( uv \) fires \( \geq \pi(v) \) times,
then \( \Gamma \) does not stabilize.

But \( \lvert \pi \rvert \) can be exponentially large!

**Markov Chain Tree Thm:**

\[
\pi = \frac{1}{\phi} \mathbf{e}
\]
The Halting Problem for CHIP-FRNC is NP-complete for S.C. Multicographs.

Open: Is it still NP-complete for Simple Directed Graphs?

Thm (Farrell-L.): P(\(G(n,p)\) is coEulerian) \(\Rightarrow\) \(\prod_{k=1}^{\infty} \frac{1}{\Phi(k)} \approx 0.436\ldots\) as \(n\to\infty\)

Fix \(q\) prime

P(\(\Delta: \mathbb{Z}_q \to \mathbb{Z}_q^V\) onto) 

Limit doesn't depend on \(p\).