Circles in the sand

Lionel Levine (Cornell University)
Wesley Pegden (Carnegie Mellon)
Charles Smart (Cornell University)

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The Abelian Sandpile (BTW 1987, Dhar 1990)

- Start with a pile of n chips at the origin in $\mathbb{Z}^d$.
- Each site $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ has $2d$ neighbors

$$x \pm e_i, \qquad i = 1, \ldots, d.$$

- Any site with at least $2d$ chips is unstable, and topples by sending one chip to each neighbor.

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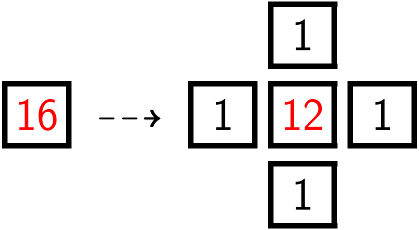
$$x \pm e_i, \qquad i = 1, \ldots, d.$$

- Any site with at least $2d$ chips is unstable, and topples by sending one chip to each neighbor.
- This may create further unstable sites, which also topple.
- Continue until there are no more unstable sites.

Toppling to Stabilize A Sandpile

- Example: n=16 chips in $\mathbb{Z}^2$.
- Sites with 4 or more chips are unstable.



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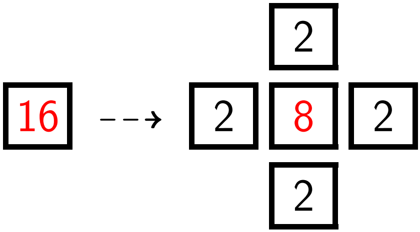

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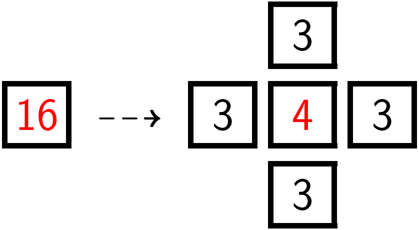

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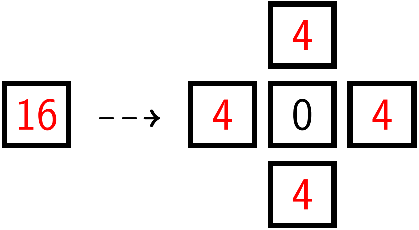

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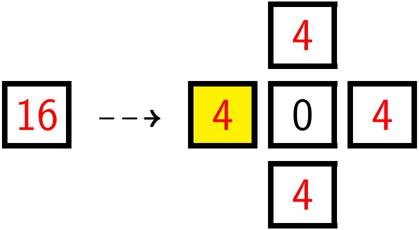

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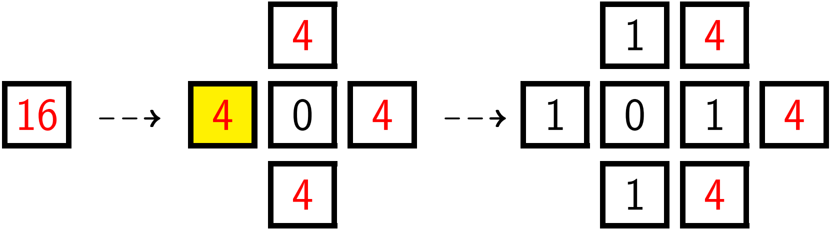

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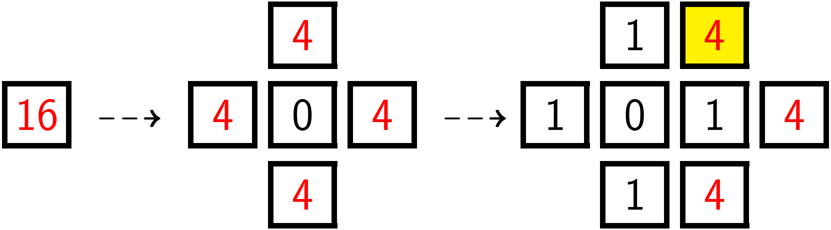

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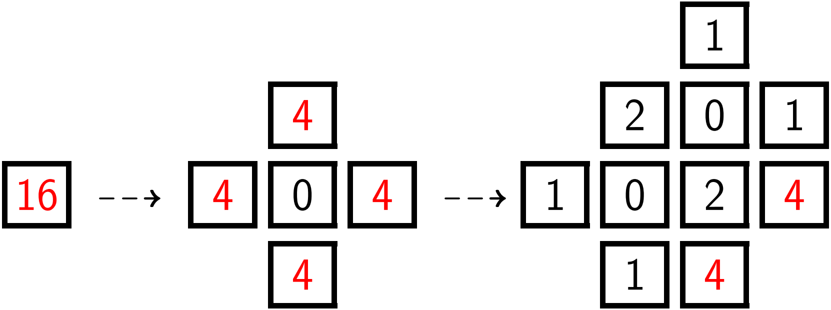

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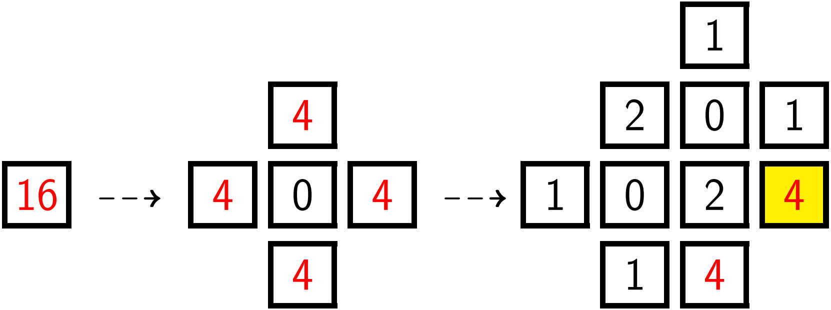

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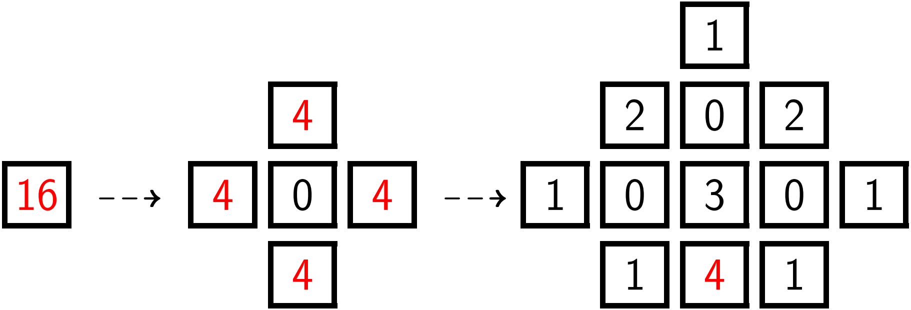

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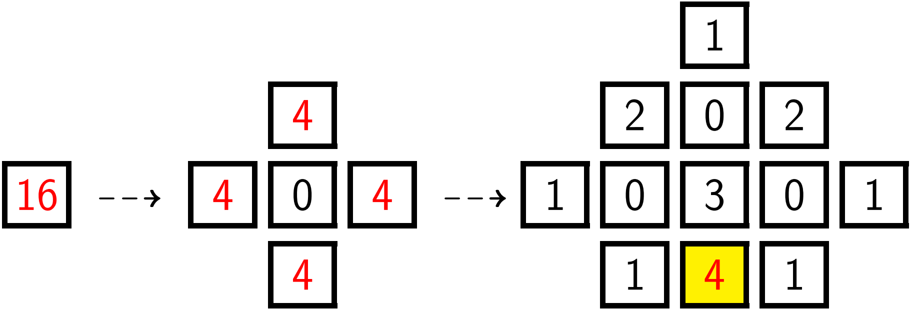

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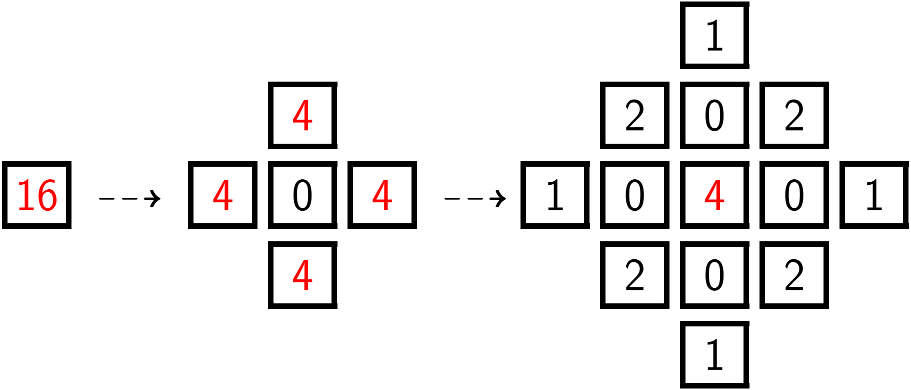

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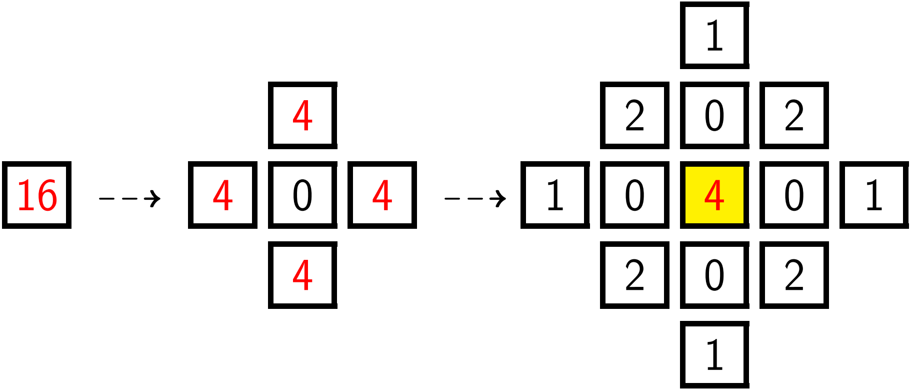

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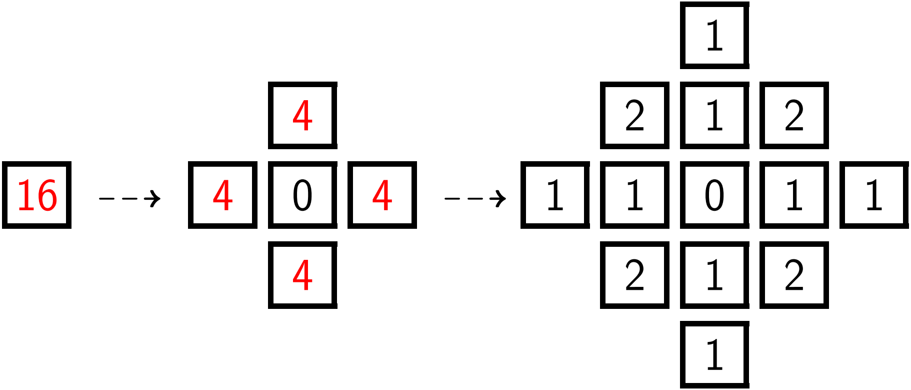
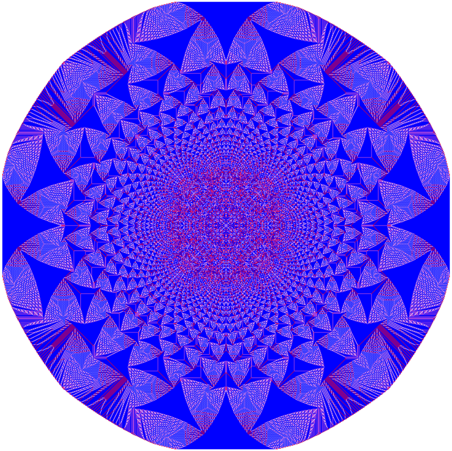


Stable.

Abelian Property

- The final stable configuration does not depend on the order of topplings.
- Neither does the number of times a given vertex topples.

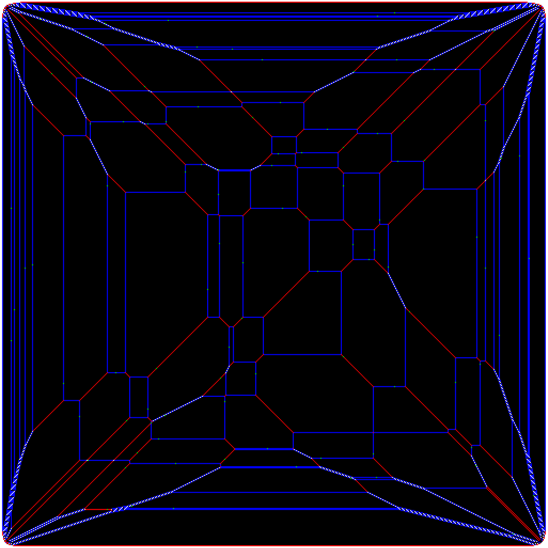
Sandpile of $1,000,000$ chips in $\mathbb{Z}^2$



- Ostojic 2002, Fey-Redig 2008, Dhar-Sadhu-Chandra 2009, L.-Peres 2009, Fey-L.-Peres 2010, Pegden-Smart 2011
- Open problem: Determine the limit shape! (It exists.)

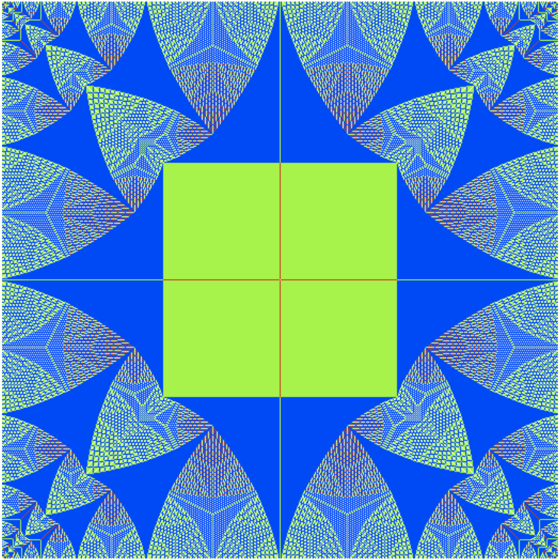
Limit shape 1: The sandpile computes an area-minimizing tropical curve through n given points

Caracciolo-Paoletti-Sportiello 2010, Kalinin-Shkolnikov 2015

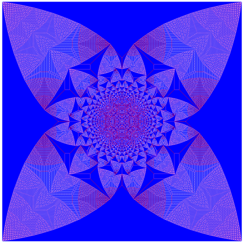


Limit shape 2: Identity element of the sandpile group of an $n \times n$ square grid

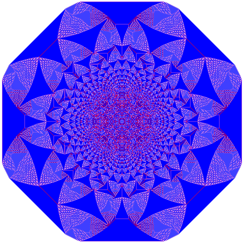
Le Borgne-Rossin 2002. Sportiello 2015+



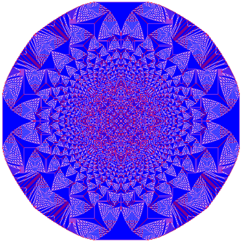
Sandpiles of the form $h + n\delta_0$



$h = 2$



$h = 1$



$h = 0$

What about $h = 3$?

3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3

3	3	3	3	3	3	3
3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	4	0	4	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3
3	3	3	3	3	3	3

3	3	3	3	3	3	3
3	3	3	4	3	3	3
3	3	5	0	5	3	3
3	4	0	4	0	4	3
3	3	5	0	5	3	3
3	3	3	4	3	3	3
3	3	3	3	3	3	3

3	3	3	4	3	3	3
3	3	5	0	5	3	3
3	5	1	4	1	5	3
4	0	4	0	4	0	4
3	5	1	4	1	5	3
3	3	5	0	5	3	3
3	3	3	4	3	3	3

3	3	5	0	5	3	3
3	5	1	4	1	5	3
5	1	5	0	5	1	5
0	4	0	4	0	4	0
5	1	5	0	5	1	5
3	5	1	4	1	5	3
3	3	5	0	5	3	3

. . . Never stops toppling!

3	5	1	4	1	5	3
5	1	5	0	5	1	5
1	5	1	4	1	5	1
4	0	4	0	4	0	4
1	5	1	4	1	5	1
5	1	5	0	5	1	5
3	5	1	4	1	5	3

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1	5	1	4	1	5	1
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. . . Never stops toppling!

A dichotomy

Any sandpile $\tau : \mathbb{Z}^d \to \mathbb{N}$ is either

- *stabilizing*: every site topples finitely often
- or *exploding*: every site topples infinitely often

An open problem

- Given a probability distribution μ on $\mathbb{N}$, decide whether the i.i.d. sandpile $\tau \sim \prod_{x \in \mathbb{Z}^2} \mu$ is stabilizing or exploding.
- For example, find the smallest λ such that i.i.d. Poisson(λ) is exploding.

How to prove an explosion

- **Claim**: If every site in $\mathbb{Z}^d$ topples at least once, then every site topples infinitely often.

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- **Claim**: If every site in $\mathbb{Z}^d$ topples at least once, then every site topples infinitely often.
- Otherwise, let x be the first site to finish toppling.
- Each neighbor of x topples at least one more time, so x receives at least $2d$ additional chips.
- So x must topple again. $\Rightarrow\Leftarrow$

The Odometer Function

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The Odometer Function

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- Discrete Laplacian:

$$\Delta u(x) = \sum_{y \sim x} u(y) - 2d\, u(x)$$
$$= \text{chips received} - \text{chips emitted}$$
$$= \tau_\infty(x) - \tau(x)$$

where τ is the initial unstable chip configuration
and τ_∞ is the final stable configuration.

Stabilizing Functions

- Given a chip configuration τ on $\mathbb{Z}^d$ and a function $u_1 : \mathbb{Z}^d \to \mathbb{Z}$, call u_1 stabilizing for τ if

$$\tau + \Delta u_1 \leq 2d - 1.$$

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- If u_1 and u_2 are stabilizing for τ, then

$$\begin{aligned} \tau + \Delta \min(u_1, u_2) &\leq \tau + \max(\Delta u_1, \Delta u_2) \\ &\leq 2d - 1 \end{aligned}$$

so $\min(u_1, u_2)$ is also stabilizing for τ.

Least Action Principle

- Let τ be a sandpile on $\mathbb{Z}^d$ with odometer function u.
- Least Action Principle:

  If $v : \mathbb{Z}^d \to \mathbb{Z}_{\geq 0}$ is stabilizing for τ, then $u \leq v$.

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- So the odometer is minimal among all nonnegative stabilizing functions:

$$u(x) = \min\{v(x) \mid v \geq 0 \text{ is stabilizing for } \tau\}.$$

- Interpretation: "Sandpiles are lazy."

The Green function of $\mathbb{Z}^d$

- $G : \mathbb{Z}^d \to \mathbb{R}$ and $\Delta G = -\delta_0$.
- In dimensions $d \geq 3$,

$$G(x) = \mathbb{E}_0 \#\{k | X_k = x\}$$

  is the expected number of visits to x by simple random walk started at 0.
- As $|x| \to \infty$,

$$G(x) \sim g(x) = \begin{cases} c_d |x|^{2-d} & d \geq 3 \\ c_2 \log |x| & d = 2. \end{cases}$$

An integer obstacle problem

- The odometer function for n chips at the origin is given by

$$u = nG + w$$

where G is the Green function of $\mathbb{Z}^d$, and w is the pointwise smallest function on $\mathbb{Z}^d$ satisfying

$$w \geq -nG$$

$$\Delta w \leq 2d - 1$$

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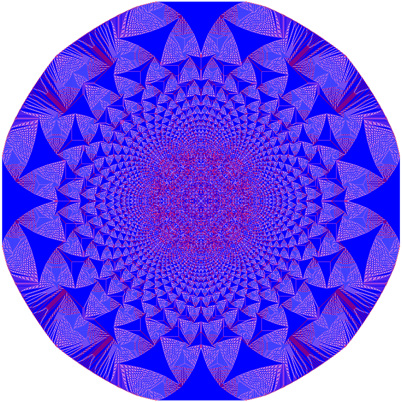
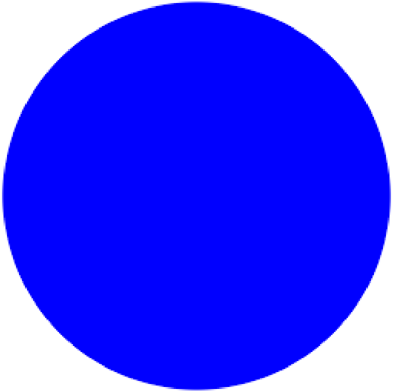
- What happens if we replace $\mathbb{Z}$ by $\mathbb{R}$?



Abelian sandpile
(Integrality constraint)



Divisible sandpile
(No integrality constraint)

Scaling limit of the abelian sandpile in $\mathbb{Z}^d$

- Consider $s_n = n\delta_0 + \Delta u_n$, the sandpile formed from n chips at the origin.
- Let $r = n^{1/d}$ and

$$\bar{s}_n(x) = s_n(rx) \qquad \text{(rescaled sandpile)}$$
$$\bar{w}_n(x) = r^{-2}u_n(rx) - nG(rx) \qquad \text{(rescaled odometer)}$$

Theorem (Pegden-Smart, 2011)

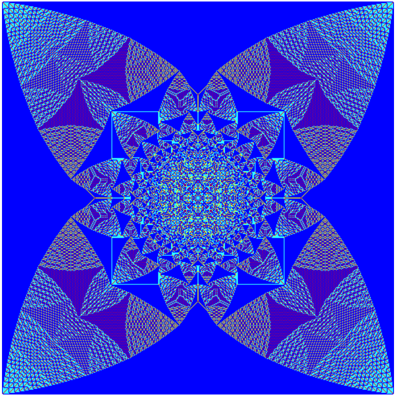
- There are functions $w, s : \mathbb{R}^d \to \mathbb{R}$ such that as $n \to \infty$,

$$\bar{w}_n \to w \qquad \text{locally uniformly in } C(\mathbb{R}^d)$$
$$\bar{s}_n \to s \qquad \text{weakly-}* \text{ in } L^\infty(\mathbb{R}^d).$$

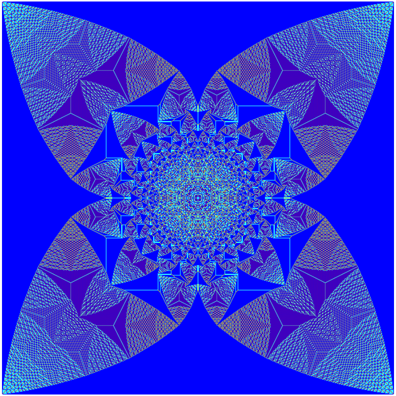
  Moreover s is a weak solution to $\Delta w = s$.

Two Sandpiles of Different Sizes

$n = 100,000$

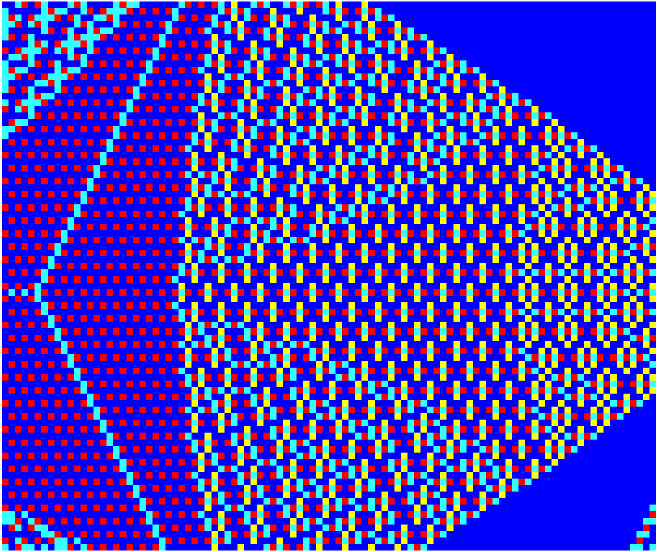


$n = 200,000$



(scaled down by $\sqrt{2}$)

Locally constant “steps” of s correspond to periodic patterns:



Limit of the least action princpile

$$w = \min\{v \in C(\mathbb{R}^d) \mid v \geq -g \text{ and } D^2(v+g) \in \Gamma\}.$$

- g encodes the initial condition (rotationally symmetric!)
- Γ is a set of symmetric $d \times d$ matrices, to be described. It encodes the sandpile “dynamics.”

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- g encodes the initial condition (rotationally symmetric!)
- Γ is a set of symmetric $d \times d$ matrices, to be described. It encodes the sandpile "dynamics."
- $D^2 u \in \Gamma$ is interpreted in the sense of viscosity:

$$D^2\phi(x) \in \Gamma$$

whenever ϕ is a C^∞ function touching u from below at x (that is, $\phi(x) = u(x)$ and $\phi - u$ has a local maximum at x).

The set Γ of stabilizable matrices

- $\Gamma = \Gamma(\mathbb{Z}^d)$ is the set of $d \times d$ real symmetric matrices A for which there exists a slope $b \in \mathbb{R}^d$ and a function $v : \mathbb{Z}^d \to \mathbb{Z}$ such that

$$\Delta v(x) \leq 2d - 1 \quad \text{and} \quad v(x) \geq \tfrac{1}{2} x \cdot Ax + b \cdot x$$

for all $x \in \mathbb{Z}^d$.

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  for all $x \in \mathbb{Z}^d$.

- How to test for membership in Γ?
  - Start with $v(x) = \left\lceil \frac{1}{2} x \cdot Ax + b \cdot x \right\rceil$.
  - For each $x \in \mathbb{Z}^d$ such that $\Delta v(x) \geq 2d$, increase $v(x)$ by 1. Repeat.

Testing for membership in Γ

- $A \in \Gamma$ if and only if there exists b such that the sandpile

$$s_{A,b} = \Delta \lceil q_{A,b} \rceil$$

stabilizes, where $q_{A,b}(x) = \frac{1}{2} x \cdot Ax + b \cdot x$.

- if A and b have rational entries, then $s_{A,b}$ is periodic.
- Topple until stable, or until every site has toppled at least once.

The structure of $\Gamma(\mathbb{Z}^2)$

Parameterize 2×2 real symmetric matrices by

$$M(a, b, c) = \frac{1}{2} \begin{bmatrix} c + a & b \\ b & c - a \end{bmatrix}.$$

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Parameterize 2×2 real symmetric matrices by

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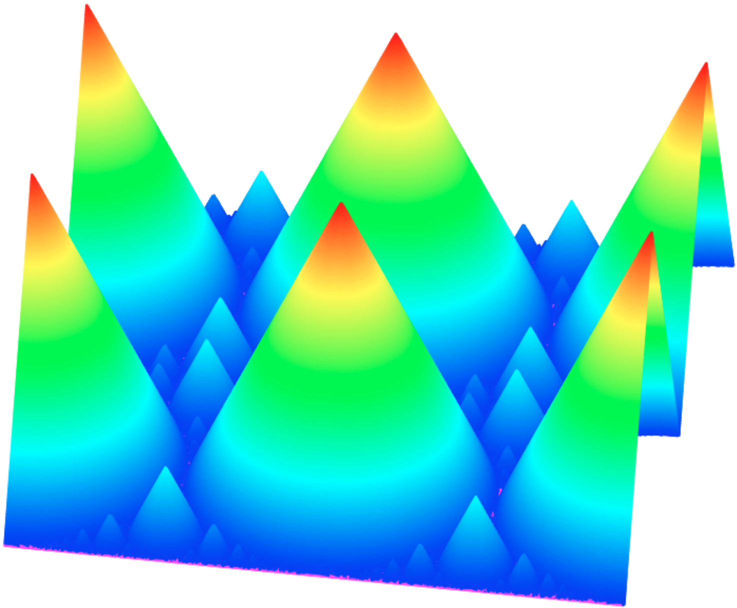
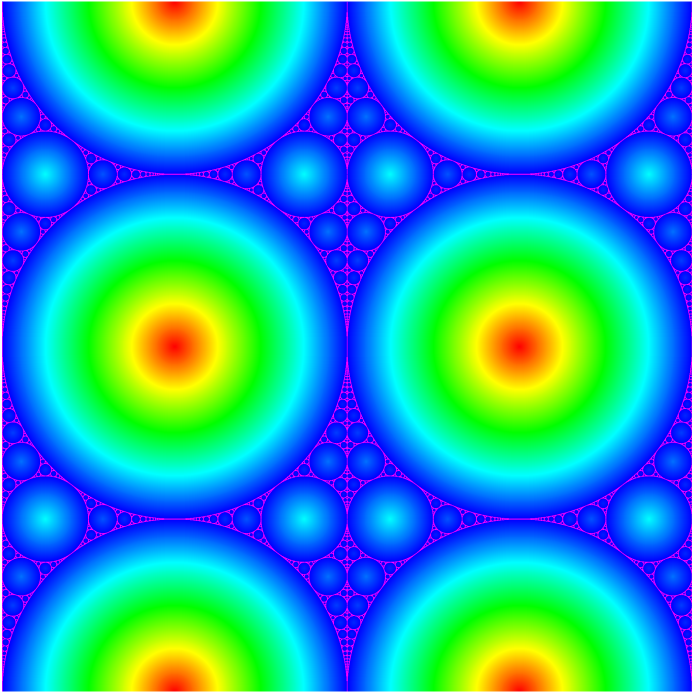
Note that if $A \leq B$ (that is, $B - A$ is positive semidefinite) and $B \in \Gamma$ then $A \in \Gamma$. In particular,

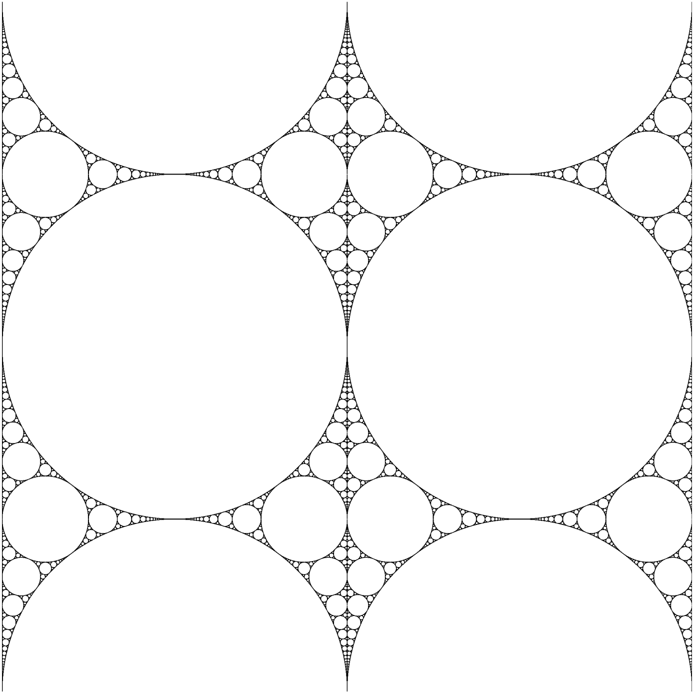
$$\Gamma = \{M(a, b, c) \mid c \leq \gamma(a, b)\}$$

for some function $\gamma : \mathbb{R}^2 \to \mathbb{R}$.

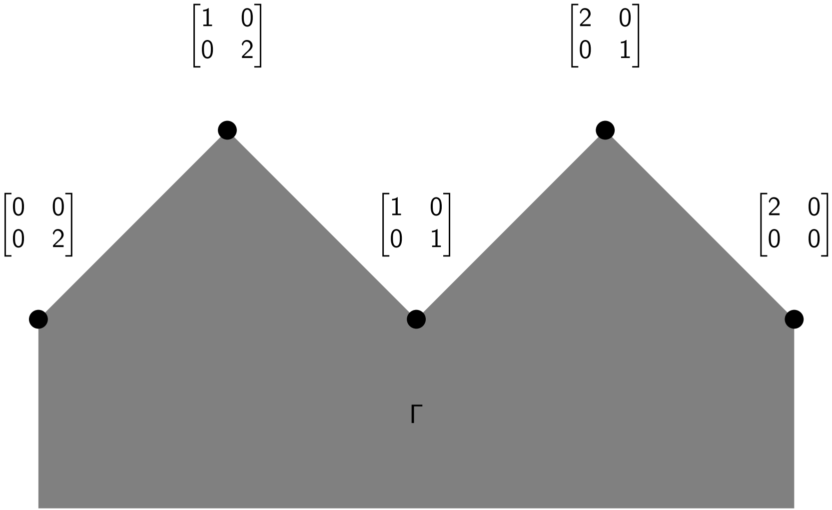
Graph of $\gamma(a, b)$







Cross section



Cross section

The Laplacian of

$$v(x) = \frac{1}{2}x_1(x_1+1) + \frac{1}{2}x_2(x_2+1)$$

is

2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2

Cross section

The Laplacian of

$$v(x) = \tfrac{1}{2}x_1(x_1+1) + \tfrac{1}{2}x_2(x_2+1) + \lceil \varepsilon x_2^2 \rceil$$

is

3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2

Cross section

The Laplacian of

$$v(x) = \tfrac{1}{2}x_1(x_1+1) + x_2(x_2+1)$$

is

3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3

Cross section

The Laplacian of

$$v(x) = \frac{1}{2}x_1(x_1+1) + \frac{1}{2}x_2(x_2+1) + \lceil \varepsilon x_1^2 \rceil$$

is

2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
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2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2
2	3	2	2	2	3	2	2	2	3	2	2

Cross section

The Laplacian of

$$v(x) = \tfrac{1}{2}x_1(x_1+1) + \tfrac{1}{2}x_2(x_2+1) + \lceil \varepsilon x_1^2 + \varepsilon x_2^2 \rceil$$

is

2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
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2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2
2	2	2	2	2	2	2	2	3	2	2	2

Another example

We have $\frac{1}{4}\left[\begin{smallmatrix}5 & 2\\ 2 & 4\end{smallmatrix}\right] \in \partial\Gamma$ because

$$v(x) = \left\lceil \frac{1}{8}(5x_1^2 + 4x_1x_2 + 4x_2^2 + 2x_1 + 4x_2) \right\rceil$$

has Laplacian

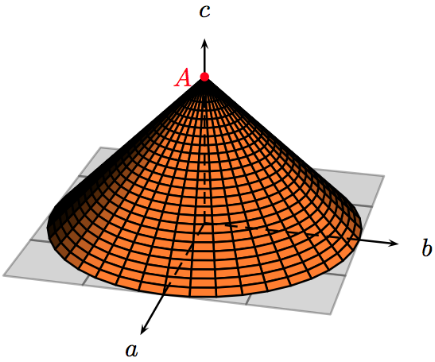
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3
3	3	3	0	3	3	3	0	3	3	3	0
3	0	3	3	3	0	3	3	3	0	3	3

Rank-1 cones

The set $\Gamma(\mathbb{Z}^2)$ is a union of downward cones

$$\{B \mid B \leq A\},$$

for a set of *peaks* $A \in \mathcal{P}$.



Periodicity

Since the matrices

$$M(2,0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad M(0,2,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

have integer valued discrete harmonic quadratic forms

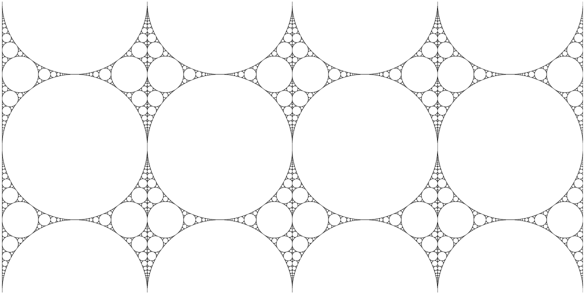
$$u(x) = \tfrac{1}{2}x_1(x_1+1) - \tfrac{1}{2}x_2(x_2+1) \quad \text{and} \quad u(x) = x_1 x_2,$$

we see that γ is $2\mathbb{Z}^2$-periodic.

Associating a matrix to each circle

If C is a circle of radius r centered at $a + b\mathbf{i}$, define

$$A_C := \frac{1}{2}\begin{bmatrix} a+2+r & b \\ b & -a+2+r \end{bmatrix}.$$

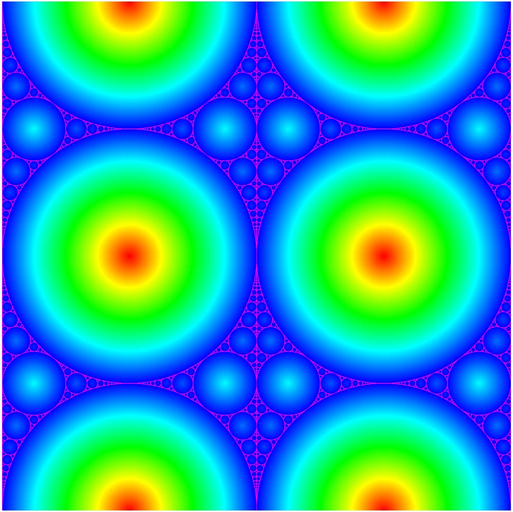


Let $\mathcal{A}$ be the circle packing in the (a,b)-plane generated by the vertical lines $a=0$, $a=2$ and the circle $(a-1)^2+b^2=1$, repeated horizontally so it is $2\mathbb{Z}^2$-periodic.

The Apollonian structure of Γ

Theorem (L-Pegden-Smart 2013)

$B \in \Gamma$ if and only if $B \leq A_C$ for some $C \in \mathcal{A}$.



Analysis of the peaks

Theorem (L-Pegden-Smart 2013)

$B \in \Gamma$ if and only if $B \leq A_C$ for some $C \in \mathcal{A}$.

Proof idea: It is enough to show that each peak matrix A_C lies on the boundary of Γ.

Analysis of the peaks

Theorem (L-Pegden-Smart 2013)

$B \in \Gamma$ if and only if $B \leq A_C$ for some $C \in \mathcal{A}$.

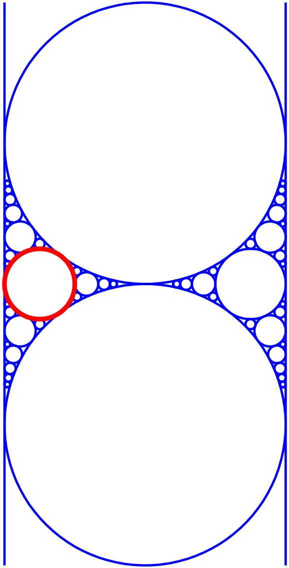
Proof idea: It is enough to show that each peak matrix A_C lies on the boundary of Γ.

For each A_C we must find $v_C : \mathbb{Z}^2 \to \mathbb{Z}$ and $b_C \in \mathbb{R}^2$ such that

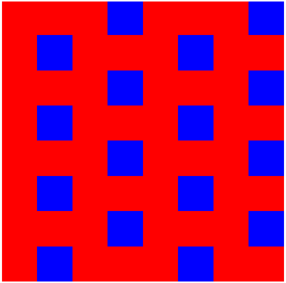
$$\Delta v_C(x) \leq 3 \quad \text{and} \quad v_C(x) \geq \tfrac{1}{2} x \cdot A_C x + b_C \cdot x$$

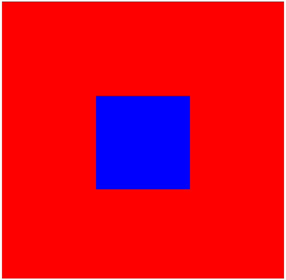
for all $x \in \mathbb{Z}^2$.

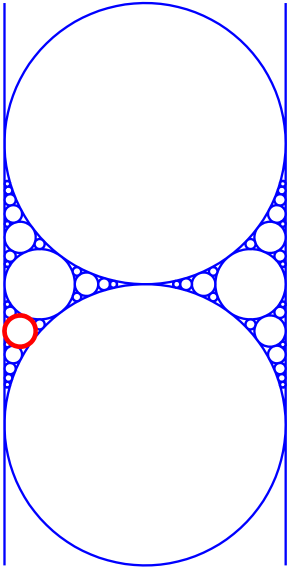
We use the recursive structure of the circle packing to construct v_C and b_C.



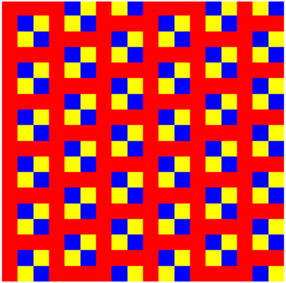
(4,1,4)

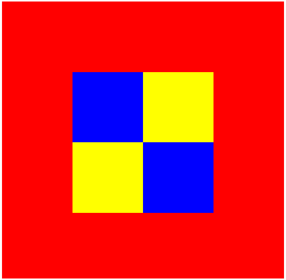


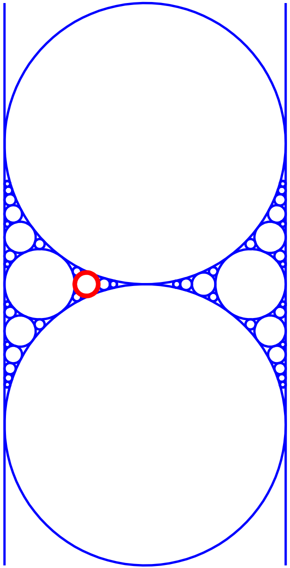




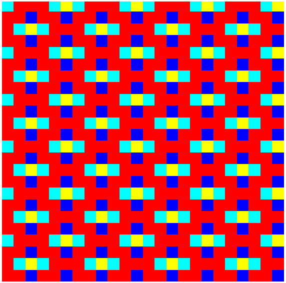
(9,1,6)

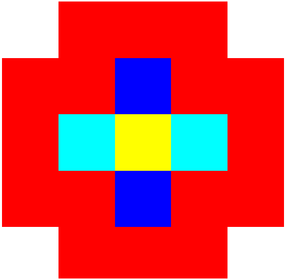


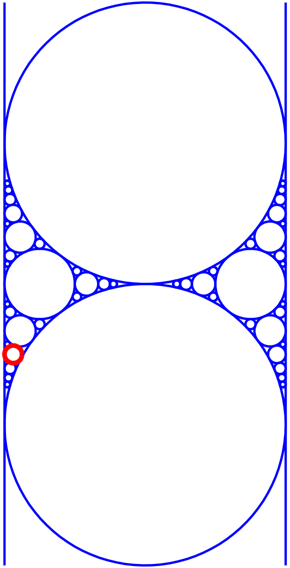




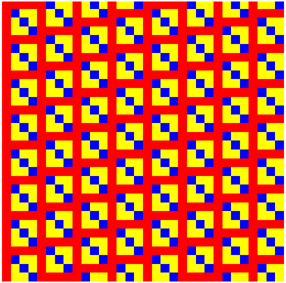
(12,7,12)

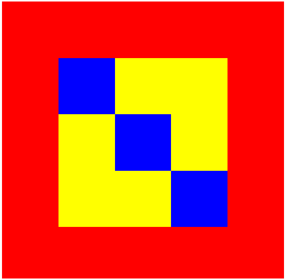


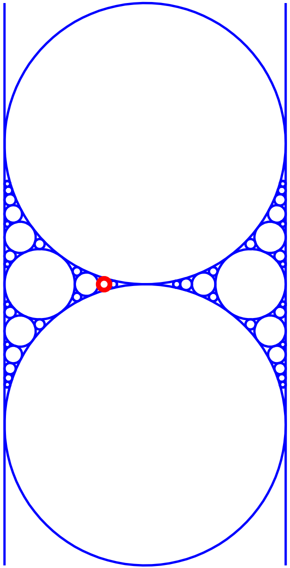




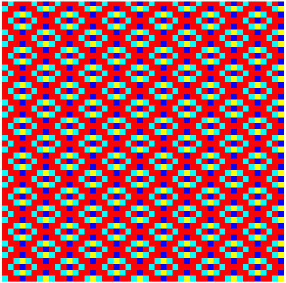
(16,1,8)

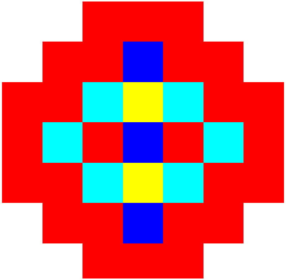






(24,17,24)



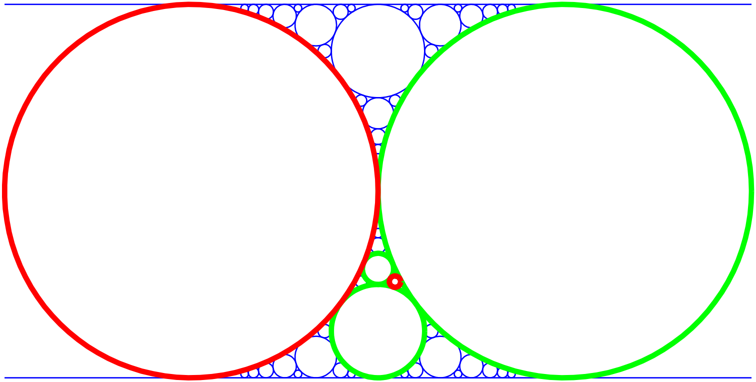


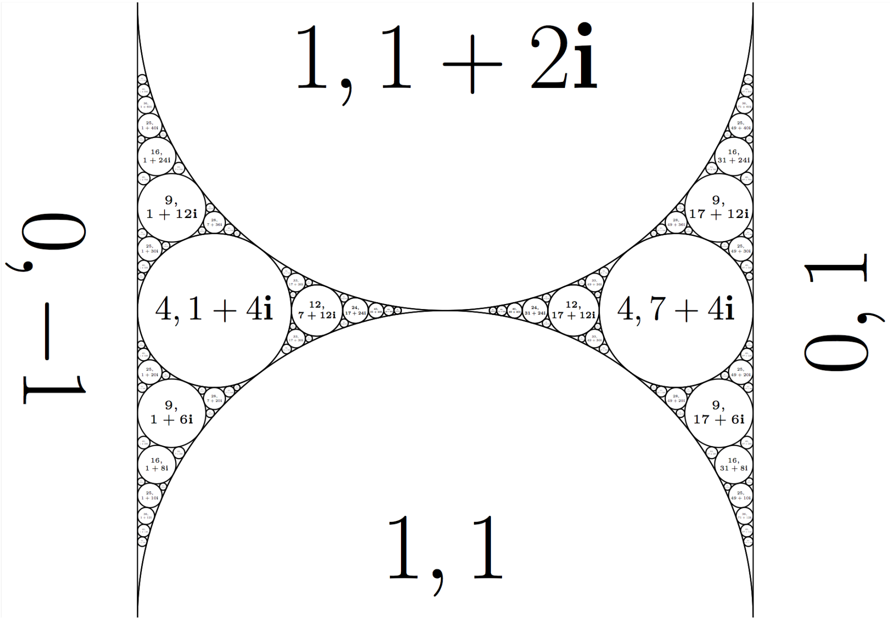
Curvature coordinates
(Descartes 1643; Lagarias-Mallows-Wilks 2002)

If C_0 has parents C_1, C_2, C_3 and grandparent C_4, then

$$C_0 = 2(C_1 + C_2 + C_3) - C_4$$

in curvature coordinates $C = (c, cz)$.





1, 1 + 2i
0, −1
0, 1
16, 1 + 24i
9, 1 + 12i
4, 1 + 4i
12, 7 + 12i
12, 17 + 12i
4, 7 + 4i
16, 31 + 24i
9, 17 + 12i
9, 1 + 6i
16, 1 + 8i
9, 17 + 6i
16, 31 + 8i
1, 1

Curvature coordinates
(Descartes 1643; Lagarias-Mallows-Wilks 2002)

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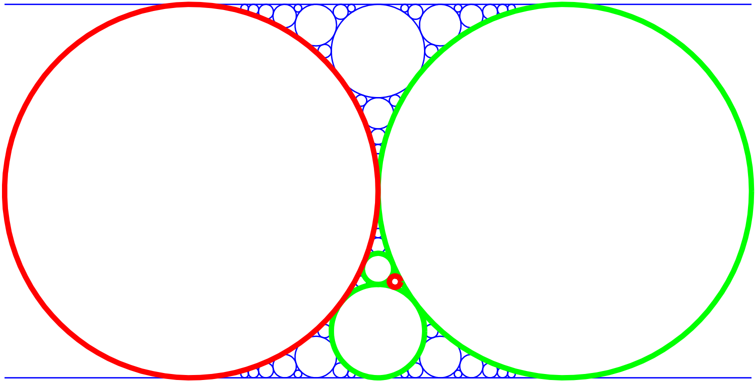
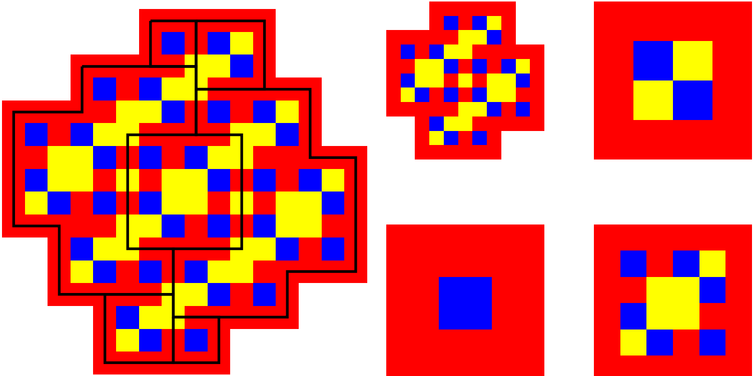
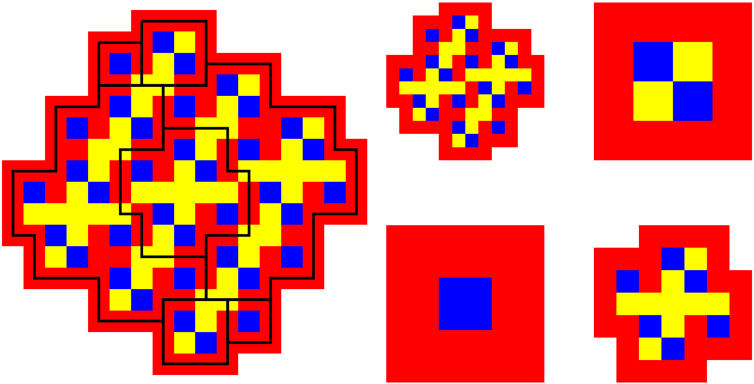
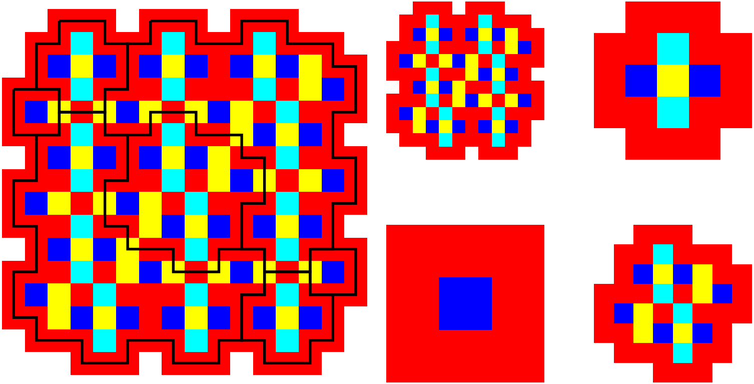
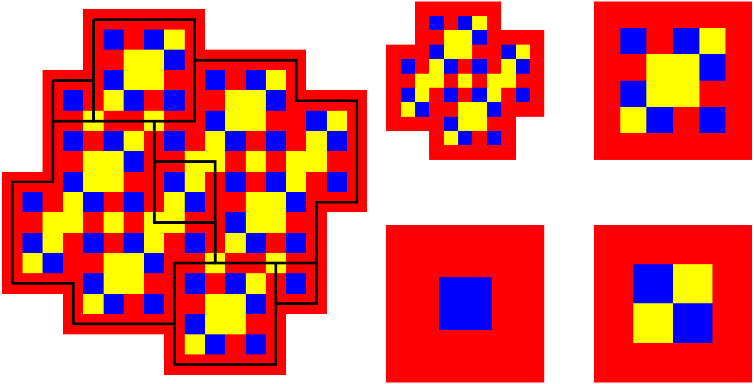




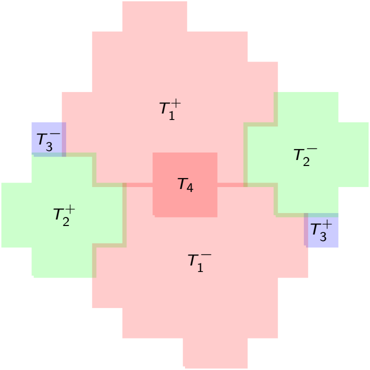






Inductive tile construction

We build tiles from copies of earlier tiles, using ideas from Katherine Stange 2012 “The Sensual Apollonian Circle Packing” to keep track of the tile interfaces.



The magic identities

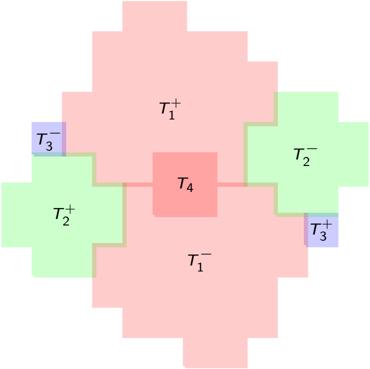
We associate an offset vector $v(C, C') \in \mathbb{Z}[\mathbf{i}]$ to each pair of tangent circles.
If (C_0, C_1, C_2, C_3) is a proper Descartes quadruple, then the offset vectors $v_{ij} = v(C_i, C_j)$ satisfy

$$v_{10} = v_{13} - \mathbf{i}v_{21}$$
$$v_{01} = \mathbf{i}v_{10}$$
$$v_{32} + v_{13} + v_{21} = 0$$
$$v_{32}^2 = c_3 c_2 (z_3 - z_2)$$
$$\bar{v}_{13} v_{21} + v_{13} \bar{v}_{21} = -2c_1$$

where $(c_i, c_i z_i)$ are the curvature coordinates of the circle C_i.

Inductive tile construction

Given tiles of the parent circles C_1, C_2, C_3 and grandparent circle C_4, arrange them using the offset vectors v_{14}, v_{24}, v_{34}:



A topological lemma

- Define a *tile* to be a finite union of squares of the form

$$[x, x+1] \times [y, y+1], \qquad x, y \in \mathbb{Z}$$

  whose interior is a topological disk.

- Suppose $\mathcal{T}$ is a collection of tiles such that
  - We *suspect* that $\mathcal{T}$ is a tiling of the plane:

$$\mathbb{C} = \bigcup_{T \in \mathcal{T}} T \quad \text{(disjoint union)}$$

  - *and* we can verify a lot of adjacencies between tiles;
  - *but* we have no simple way to verify disjointness.

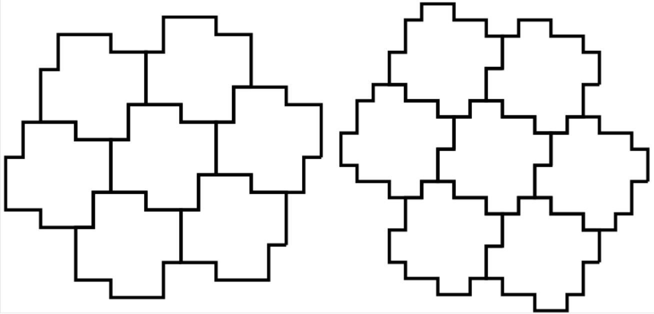
A topological lemma

Let $\mathcal{T}$ be an infinite collection of tiles,
and $G = (\mathcal{T}, \mathcal{E})$ be a graph with vertex set $\mathcal{T}$.
If the following hold, then $\mathcal{T}$ is a tiling of the plane.

1. G is a 3-connected planar triangulation.
2. G is invariant under translation by some full-rank lattice $L \subseteq \mathbb{Z}[\mathbf{i}]$, and $\sum_{T \in \mathcal{T}/L} \text{area}(T) = |\det L|$.
3. If $(T_1, T_2) \in \mathcal{E}$ then $T_1 \cap T_2$ contains at least 2 integer points.
4. For each face $F = \{T_1, T_2, T_3\}$ of G we can select an integer point
$$\rho(F) \in T_1 \cap T_2 \cap T_3$$
such that for each adjacent face $F' = \{T_1, T_2, T_4\}$ there is a path in $T_1 \cap T_2$ from $\rho(F)$ to $\rho(F')$.

Extra 90° symmetry

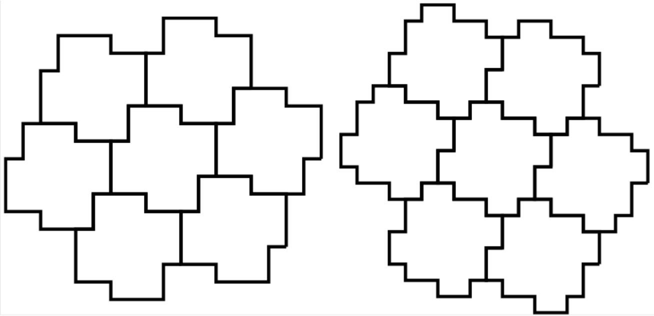
For each circle $C \in \mathcal{A}$ we build a tile T_C that tiles the plane.



A pleasant surprise: Each tile T_C has 90° rotational symmetry!

An open problem: classify such tilings.

If $\mathcal{T}$ is a primitive, periodic, hexagonal tiling of the plane by identical 90° symmetric tiles, must its fundamental tile be T_C for some circle $C \in \mathcal{A}$?



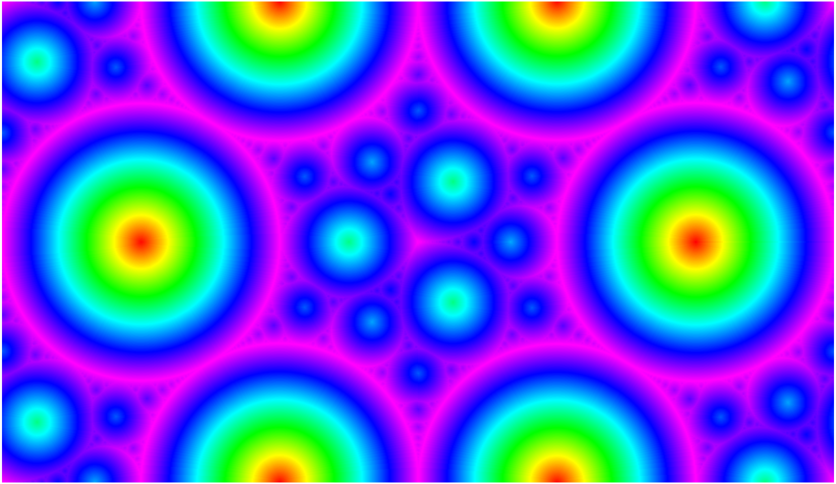
Other lattices, higher dimensions

We have described the set $\Gamma(\mathbb{Z}^2)$ in terms of an Apollonian circle packing of $\mathbb{R}^2$.

What about $\Gamma(\mathbb{Z}^d)$ for $d \geq 3$?

In general any periodic graph G embedded in $\mathbb{R}^d$ has an associated set of $d \times d$ symmetric matrices $\Gamma(G)$, which captures some aspect of the infinitessimal geometry of $\frac{1}{n}G$ as $n \to \infty$.

Γ for the triangular lattice



Thank you!

Reference:
L.-Pegden-Smart, arXiv:1309.3267
The Apollonian structure of integer superharmonic matrices