Internal Diffusion-Limited Erosion

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Joint work with Yuval Peres

Internal Erosion of a Domain

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- Start a **simple random walk** at the origin.
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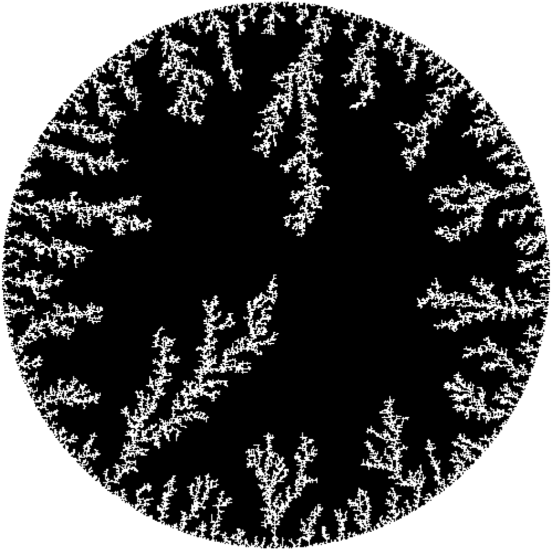
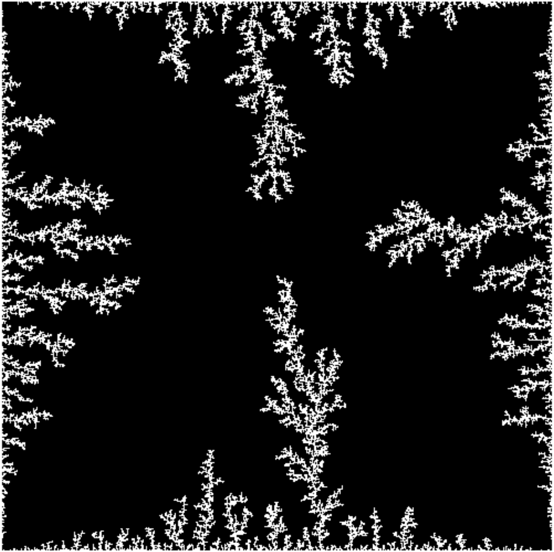
- Iterate this operation until the origin is eroded.
- The resulting **random set** is called the internal erosion of A.



Internal erosion of a disk of radius 250 in $\mathbb{Z}^2$.



Internal erosion of a box of side length 500 in $\mathbb{Z}^2$.

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    $$\mathbb{E}\,\#\,\text{eroded sites} = \Theta(n^{\alpha})$$
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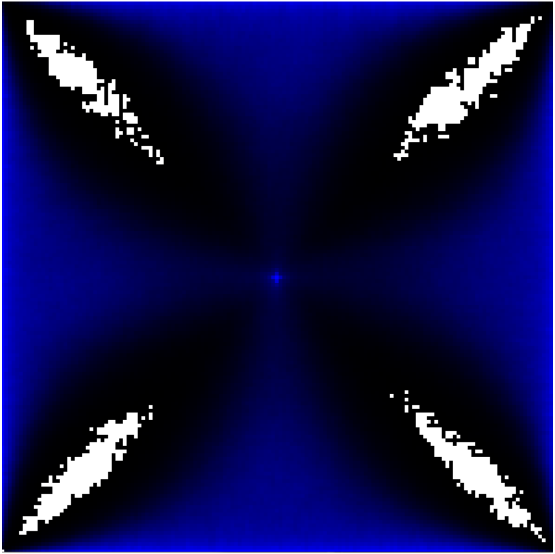
- Analogy with **diffusion-limited aggregation**.
- What is the probability that a given site x is eroded?
  - For some sites, is this probability $o(n^{\alpha-2})$?



Probability of a given site being eroded from a box in $\mathbb{Z}^2$.

Internal Erosion in One Dimension

- Interval $A = [-m, n] \subset \mathbb{Z}$ with $-m \leq 0 \leq n$.
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- How large is the remaining interval when the origin gets eroded?
- **Urn model**: choose a ball at random, then remove a ball from the other urn.
- **"OK Corral" Process**: Gunfight with m fighters on one side and n on the other. Williams-McIlroy '98, Kingman '99, Kingman-Volkov '03.

The Number of Surviving Gunners

- Let $m = n$ (an equal gunfight).
- **Theorem** (Kingman-Volkov '03) Starting from the interval $[-n, n]$, let $R(n)$ be the number of sites remaining when the origin is eroded. Then as $n \to \infty$

$$\frac{R(n)}{n^{3/4}} \Longrightarrow \left(\frac{8}{3}\right)^{1/4} \sqrt{|Z|} \tag{1}$$

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- **Claim**: The order in which segments of the rod finish burning has the same distribution as the order in which sites become eroded.
- This follows from the **memoryless** property of exponentials:

$$\mathbb{P}(X_j - Y_k \geq x | X_j \geq Y_k) = \mathbb{P}(X_j \geq x).$$

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- Thus the number of segments remaining in the rod is order $\Theta(\sqrt{L_n}) = \Theta(n^{3/4})$.

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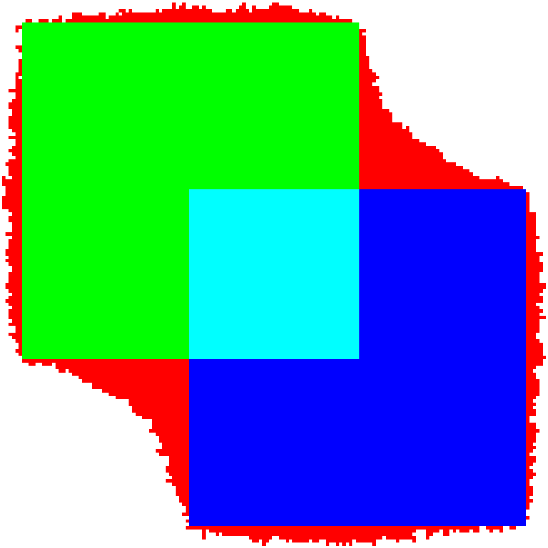
- Define $A + B = C_k$.
- **Abeilan property**: the law of $A + B$ does not depend on the ordering of $x_1, \dots, x_k$.



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- Not clear how to define dynamics in $\mathbb{R}^d$.

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where $D_\varepsilon, D^\varepsilon$ are the inner and outer ε-neighborhoods of D.

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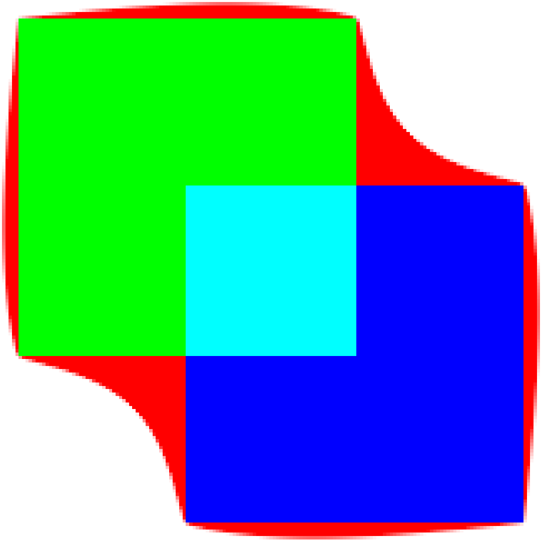
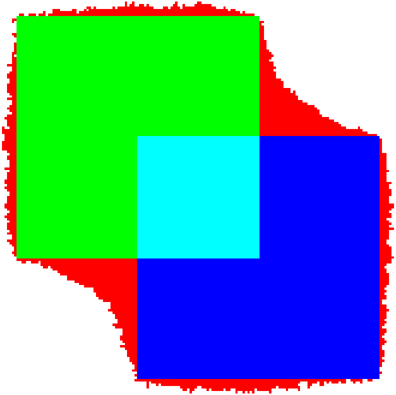
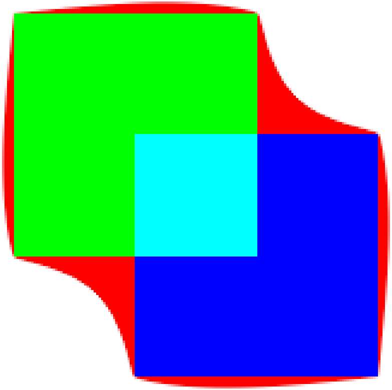




Diaconis-Fulton sum



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- Need additional information to determine the domain $A \oplus B$.

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- Alternative formulation:

$$\Delta u = 1 - 1_A - 1_B \qquad \text{on } D;$$
$$u = \nabla u = 0 \qquad \text{on } \partial D.$$

Least Superharmonic Majorant

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- Reverse inequality: $s - \gamma - u$ is superharmonic on $A \oplus B$ and nonnegative outside $A \oplus B$, hence nonnegative inside as well.

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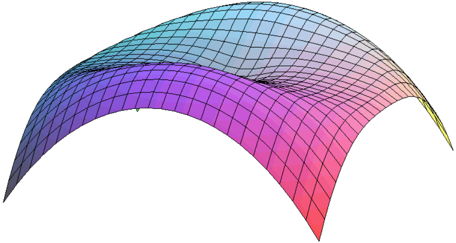

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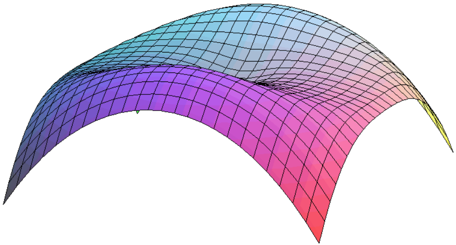

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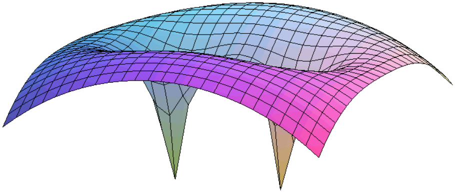

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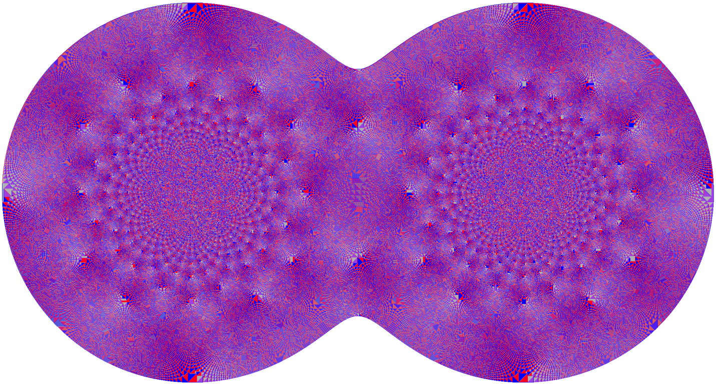
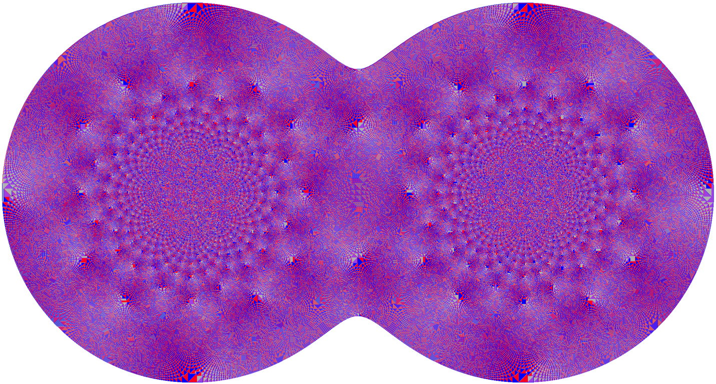


The domain $D = \{s > \gamma\}$ for two overlapping disks in $\mathbb{R}^2$.



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The boundary ∂D is given by the algebraic curve

$$\left(x^2+y^2\right)^2 - 2r^2\left(x^2+y^2\right) - 2(x^2-y^2) = 0.$$

Main Result

- Let $A, B \subset \mathbb{R}^d$ be bounded open sets such that ∂A, ∂B have measure zero.

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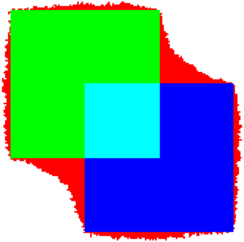
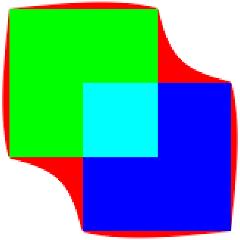
  - $D = A \cup B \cup \{s > \gamma\}$.
  - Convergence is in the sense of ε-neighborhoods: for all $\varepsilon > 0$

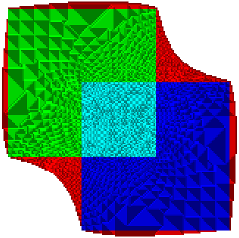
$$D_\varepsilon^{::} \subset D_n, R_n, I_n \subset D^{\varepsilon ::} \qquad \text{for all sufficiently large } n.$$



Internal DLA



Divisible Sandpile



Rotor-Router Model

Steps of the Proof

convergence of densities

$$\Downarrow$$

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- Follows from the main result and the case of a single point source.

A Quadrature Identity

- If h is harmonic on $\delta_n \mathbb{Z}^d$, then

$$M_t = \sum_j h(X_t^j)$$

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- Therefore if $I_n \to D$, we expect the limiting domain $D \subset \mathbb{R}^d$ to satisfy

$$\int_D h(x) dx = \sum_{i=1}^{k} \lambda_i h(x_i).$$

for all harmonic functions h on D.

Quadrature Domains

- Given $x_1, \dots x_k \in \mathbb{R}^d$ and $\lambda_1, \dots, \lambda_k > 0$.
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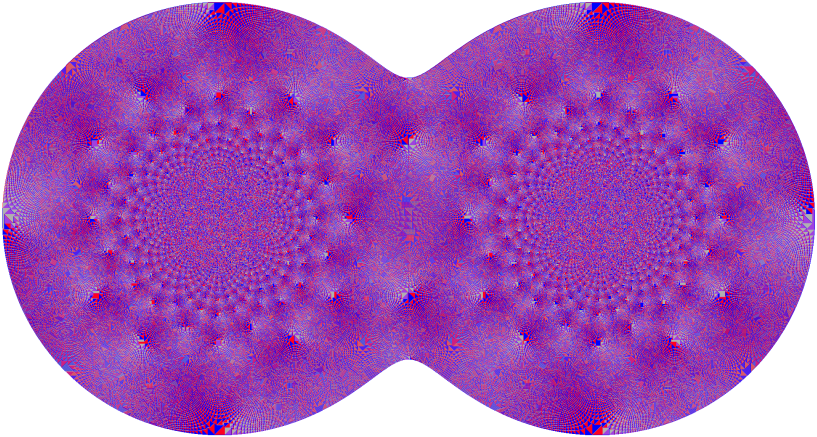
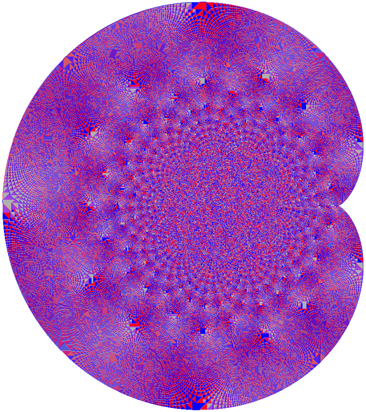
- The smash sum $B_1 \oplus \ldots \oplus B_k$ is such a domain, where B_i is the ball of volume λ_i centered at x_i.
- Generalizes the mean value property of superharmonic functions.
- The boundary of $B_1 \oplus \ldots \oplus B_k$ lies on an algebraic curve of degree $2k$.

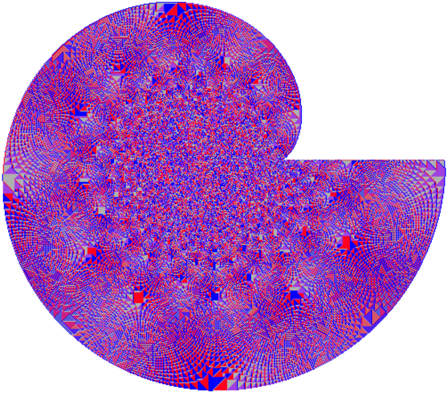


$$\iint_D h(x,y)\,dx\,dy = h(-1,0) + h(1,0)$$

Two Mystery Shapes



Cardioid?



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