Quiz 6 Solution
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1. Describe the span in $\mathbb{R}^3$ of the vectors $\mathbf{u} = (1, -1, 0)$, $\mathbf{v} = (0, 1, -1)$ and $\mathbf{w} = (-1, 0, 1)$.

A vector (x, y, z) lies in the span of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ if and only if

$$\begin{aligned} (x, y, z) &= a\mathbf{u} + b\mathbf{v} + c\mathbf{w} \\ &= (a - c, -a + b, -b + c) \end{aligned}$$

for some real numbers a, b and c. This happens if and only if there is a solution (a, b, c) to the system

$$\begin{array}{rcrcrcl} a & & & - & c & = & x \\ -a & + & b & & & = & y \\ & & -b & + & c & = & z \end{array} .$$

Row-reduce the associated matrix to get

$$\begin{aligned} \left(\begin{array}{ccc|c} 1 & 0 & -1 & x \\ -1 & 1 & 0 & y \\ 0 & -1 & 1 & z \end{array}\right) &\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 1 & -1 & x + y \\ 0 & -1 & 1 & z \end{array}\right) \\ &\to \left(\begin{array}{ccc|c} 1 & 0 & -1 & x \\ 0 & 1 & -1 & x + y \\ 0 & 0 & 0 & x + y + z \end{array}\right). \end{aligned}$$

From the bottom row we read $x + y + z = 0$. There are pivots in the first two columns, so the system is solvable provided $x + y + z = 0$. Therefore,

$$\mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$

which is a plane.