Quiz 7 Solution
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1. Which of the following sets are vector spaces? For the ones that aren't vector spaces, find one of the vector space axioms that fails to hold.

   (a) The set of all pairs (x, y), where x and y are real numbers, with the usual addition, but scalar multiplication defined by

   $$r(x, y) = (r^2 x, r^2 y)$$

   for all real numbers r.

   (b) The set of all 2×2 matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying

   $$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

   with the usual addition and scalar multiplication.

   (c) The set of all invertible 2×2 matrices, with the usual addition and scalar multiplication.

(a) Since $(r+s)^2 \neq r^2 + s^2$, the distributive law fails to hold:

$$\begin{aligned} (r+s)(x, y) &= ((r+s)^2 x, (r+s)^2 y) \\ &\neq ((r^2+s^2)x, (r^2+s^2)y) \\ &= (r^2 x, r^2 y) + (s^2 x, s^2 y) \\ &= r(x, y) + s(x, y). \end{aligned}$$

So this is not a vector space.

(b) If A and B are two matrices satisfying

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then

$$(A+B)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Likewise if r is any real number, then

$$(rA)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = r \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This shows that the set in question is a subspace of M_{22}, the space of all 2×2 matrices, and hence it is a vector space.

(c) Since the zero matrix is not invertible, this is not a vector space.