Quiz 13 Solution
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1. Find an orthonormal basis for the column space of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -2 & 2 & 0 \end{pmatrix}$$

First find a basis the usual way, by row reduction:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -2 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since only the first two columns have pivots, a basis for the column space is found by taking the first two columns of the *original* matrix:

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

To make this into an orthonormal basis, use Gram-Schmidt:

$$u_1 = v_1/||v_1|| = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}.$$

$$w_2 = v_2 - (u_1 \cdot v_2)u_1 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Since $||w_2|| = 1$ we have $u_2 = w_2$, so the orthonormal basis is

$$u_1 = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad u_2 = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$