Quiz 18 Solution
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4/18/05

1. Find a solution to the system of differential equations

$$\begin{array}{rcrcr} f' & = & -7f & - & 9g \\ g' & = & 6f & + & 8g \end{array}$$

satisfying the initial conditions

$$f(0) = 1, \qquad g(0) = 0.$$

In matrix form, this system is

$$\begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} -7 & -9 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

The characteristic polynomial of the associated matrix is

$$\lambda^2 - (-7+8)\lambda + (-7 \cdot 8 - 6 \cdot -9) = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2).$$

Thus the eigenvalues are $\lambda = -1, 2$ and the eigenspaces are

$$W_{-1} = NS \begin{pmatrix} 6 & 9 \\ -6 & -9 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\};$$

$$W_2 = NS \begin{pmatrix} 9 & 9 \\ -6 & -6 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

So the eigenvectors are $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. The general solutuion is therefore

$$\begin{pmatrix} f \\ g \end{pmatrix} = ae^{-t} \begin{pmatrix} 3 \\ -2 \end{pmatrix} + be^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad a, b \in \mathbb{R}.$$

To satisfy the initial conditions, plug in $t = 0$ to get the system of equations

$$\begin{array}{rcrcr} 3a & + & b & = & 1 \\ -2a & - & b & = & 0. \end{array}$$

Solving this gives $a = 1$, $b = -2$, so the solution is

$$f(t) = 3e^{-t} - 2e^{2t}, \quad g(t) = -2e^{-t} + 2e^{2t}.$$