Math 54 Spring 2005

Solutions to Homework Section 4.1

February 25th, 2005

2. $\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 0 & -3 \end{bmatrix}$. $f_A(\mathbf{x}) = \mathbf{A}\mathbf{x}$. More explicitly, $f_A(x_1,\ x_2,\ x_3) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ 3x_1 - 3x_3 \end{bmatrix}$. The domain is $\mathbb{R}^3$ and the range is $\mathbb{R}^2$ because f_A maps vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^2$. (The author uses range to mean a set containing the values the function takes rather than the set $\{f_A(\mathbf{x}) \mid \mathbf{x} \in D\}$ which is often called the image of f_A. In this case the image is also $\mathbb{R}^2$.)

4. $\mathbf{A} = \begin{bmatrix} 8 & -1 & 1 & 2 \\ 1 & 0 & -1 & 3 \end{bmatrix}$. $f_A(x_1, x_2, x_3, x_4) = \begin{bmatrix} 8x_1 - x_2 + x_3 + 2x_4 \\ x_1 - x_3 + 3x_4 \end{bmatrix}$. The domain is $\mathbb{R}^4$ and the range is $\mathbb{R}^2$.

8. Let $\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$. f_A reflects the x-axis and expands it by a factor of two, and reflects the y-axis and contracts it by a factor of two.

9. Let $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. f_A interchanges the x- and y-axes. This reflects any vector about the line $y = x$.

10. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Since $\mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$, the induced function f_A is given by $f_A(x, y) = (x, 0)$. Geometrically $\mathbf{A}$ projects a vector $\mathbf{x}$ onto its component parallel to the x-axis.

11. $\mathbf{A} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$. This is a special case of example 5 with $\theta = 45^\circ$. f_A rotates each vector in the plane 45° counterclockwise about the origin.

15. *Find a 3 by 3 matrix* $\mathbf{A}$ *that rotates the yz-plane by* 45°*, and reflects the x-axis.*

    We'll take "rotate" to mean "rotate counterclockwise". We calculate that $\mathbf{A} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{A} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cos(45^\circ) \\ \sin(45^\circ) \end{bmatrix}$, $\mathbf{A} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\sin(45^\circ) \\ \cos(45^\circ) \end{bmatrix}$. So $\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos(45^\circ) & -\sin(45^\circ) \\ 0 & \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$.

18. *Consider the map* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $T(x, y) = (y, x)$ *Is* T *linear?*

    T is linear. We check that

    $$T(k(x, y)) = T(kx, ky) = (ky, kx) = k(y, x) = kT(x, y)$$

    and

    $$T((x, y) + (x', y')) = T(x + x', y + y') = (y + y', x + x') = (y, x) + (y', x') = T(x, y) + T(x', y')$$

    $T = f_A$ with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ because $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$.

19. *Consider the map* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by* $T(x, y) = (x^2, y)$. *Is* T *a linear map?*

T is not linear. For example, $2T(1, 0) = 2(1, 0) = (2, 0)$, whereas $T(2, 0) = (4, 0)$.

22. *Consider the map* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *defined by* $T(x, y, z) = (z - x, z - y)$ *Is* T *linear?*

T is linear. We check that

$$T(kx, ky, kz) = (kz - kx, kz - ky) = k(z - x, z - y) = kT(x, y, z)$$

$$\text{and} \quad T(x + x', y + y', z + z') = (z + z' - (x + x'), z + z' - (y + y'))$$
$$= (z - x, z - y) + (z' - x', z' - y') = T(x, y, z) + T(x', y', z')$$

$T = f_A$ with $\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ because $\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z - x \\ z - y \end{bmatrix}$.

30. *Consider the map* $T : M_{22} \to \mathbf{R}$ *defined by* $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = a + b + c + d$. *Is* T *a linear map?*

T is linear. We check that $T(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}) = ka + kb + kc + kd = kT(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$ and

$$T(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) + T(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}) = a + b + c + d + a' + b' + c' + d' = T(\begin{bmatrix} a + a' & b + b' \\ c + c' & d + d' \end{bmatrix})$$

If we identify M_{22} with $\mathbb{R}^4$ by putting the matrix entries into a 4-dimensional column vector, then $T = f_A$ with $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$.

36. *Let* $\mathbf{A}$ *be a fixed* 2×3 *matrix. Define a map* $T : M_{33} \to M_{23}$ *by* $T(\mathbf{C}) = \mathbf{AC}$. *Is* T *a linear map?*

This T is linear. We check that $T(k\mathbf{C}) = \mathbf{A}(k\mathbf{C}) = k(\mathbf{AC}) = kT(\mathbf{C})$, and

$$T(\mathbf{C}_1 + \mathbf{C}_2) = \mathbf{A}(\mathbf{C}_1 + \mathbf{C}_2) = \mathbf{AC}_1 + \mathbf{AC}_2 = T(\mathbf{C}_1) + T(\mathbf{C}_2)$$