Math 54 Spring 2005

Solutions to Homework Section 3.8

February 23th, 2005

1. We want to find real numbers a and b so that $\mathbf{x} = \begin{bmatrix} 5 \\ -9 \end{bmatrix} = a \begin{bmatrix} 3 \\ -1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. We do this by solving the system

$$\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

This has the unique solution $a = 3$, $b = -2$. These are the coordinates for $\mathbf{x}$ in the bases B and C. Order the coordinates so that they correspond to the ordering of the vectors in the ordered bases B and C.

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, [\mathbf{x}]_C = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

2. We want to find real numbers a and b so that $\mathbf{x} = \begin{bmatrix} 8 \\ 9 \end{bmatrix} = a \begin{bmatrix} .5 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. We do this by solving the system

$$\begin{bmatrix} .5 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

This has the unique solution $a = 10$, $b = 1$. These are the coordinates for $\mathbf{x}$ in the bases B and C. Ordering them appropriately, we have

$$[\mathbf{x}]_B = \begin{bmatrix} 10 \\ 1 \end{bmatrix}, [\mathbf{x}]_C = \begin{bmatrix} 1 \\ 10 \end{bmatrix}.$$

4. We want to find real numbers a and b so that $\mathbf{x} = \begin{bmatrix} -8 \\ 9 \end{bmatrix} = a \begin{bmatrix} -3 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. We do this by solving the system

$$\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$$

This has the unique solution $a = \frac{6}{5}$, $b = -\frac{11}{5}$. These are the coordinates for $\mathbf{x}$ in the bases B and C. Ordering them appropriately, we have

$$[\mathbf{x}]_B = \begin{bmatrix} \frac{6}{5} \\ -\frac{11}{5} \end{bmatrix}, [\mathbf{x}]_C = \begin{bmatrix} -\frac{11}{5} \\ \frac{6}{5} \end{bmatrix}.$$

8. We want to find real numbers a, b, and c so that $\mathbf{x} = \begin{bmatrix} -1 \\ -13 \\ 9 \end{bmatrix} = a \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} + c \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$.

We do this by solving the system

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -5 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -13 \\ 9 \end{bmatrix}$$

This has the unique solution $a = \frac{15}{13}$, $b = -\frac{46}{13}$, and $c = \frac{55}{13}$. These are the coordinates for $\mathbf{x}$ in the bases B and C. Ordering them appropriately, we have

$$[\mathbf{x}]_B = \begin{bmatrix} \frac{15}{13} \\ -\frac{46}{13} \\ \frac{55}{13} \end{bmatrix}, [\mathbf{x}]_C = \begin{bmatrix} -\frac{46}{13} \\ \frac{15}{13} \\ \frac{55}{13} \end{bmatrix}.$$

9. We solve $\mathbf{x} = 3 - 2x^2 = a(1+2x-x^2) + b(1-3x) + c(2)$ using the following system of equations (the first row corresponds to the x^0 coefficient, the second to the x^1 coefficient, etc.)

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

This has the unique solution $a = 2$, $b = \frac{4}{3}$, $c = -\frac{1}{6}$. These are the coordinates of $\mathbf{x}$ in the bases B and C. Ordering appropriately, we have

$$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ \frac{4}{3} \\ -\frac{1}{6} \end{bmatrix}, [\mathbf{x}]_C = \begin{bmatrix} -\frac{1}{6} \\ \frac{4}{3} \\ 2 \end{bmatrix}.$$

10. We solve $\mathbf{x} = -5 - 5x - 3x^2 = a(2+x) + b(3-2x) + c(2+3x+x^2)$ using the following system of equations (the first row corresponds to the x^0 coefficient, the second to the x^1 coefficient, etc.)

$$\begin{bmatrix} 2 & 3 & 2 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ -3 \end{bmatrix}$$

This has the unique solution $a = 2$, $b = -1$, $c = -3$. These are the coordinates of $\mathbf{x}$ in the bases B and C. Ordering appropriately, we have

$$[\mathbf{x}]_B = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}, [\mathbf{x}]_C = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}.$$

12. We solve $\mathbf{x} = \begin{bmatrix} 2 & -2 \\ 4 & -1 \end{bmatrix} = a \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. This gives us the following system of equations in a, b, c, d :

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \\ -1 \end{bmatrix}$$

which we solve to find $a = -1$, $b = -2$, $c = 3$, $d = 2$. The bases B and C are just reorderings of one another, so this calculation will do for both of them:

$$[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 2 \end{bmatrix}, [\mathbf{x}]_C = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 3 \end{bmatrix}$$

14. Expressions of $\mathbf{x}$ as a linear combination of the v_i's can be found by solving the following system of equations:

$$\begin{bmatrix} -1 & 3 & -1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Putting this matrix in row echelon form, we see that a and b are leading variables and c is a free variable. Solving by back substitution shows that the set of solutions is $\{(\frac{4}{5}s - 1, \frac{3}{5}s, s) | s \in \mathbb{R}\}$. Any three distinct values of s give three different ways of writing $\mathbf{x}$ as a linear combination of the v_i's. The linear combinations

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{5}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

correspond to the choices 0, $\frac{5}{4}$, and 5 for s. To express $\mathbf{y}$ as a linear combination we solve the system

$$\begin{bmatrix} -1 & 3 & -1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The set of solutions is $\{(\frac{4}{5}s, \frac{3}{5}s, s) | s \in \mathbb{R}\}$. The linear combinations

$$\mathbf{y} = 0 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + 0 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 8 \begin{bmatrix} -1 \\ -2 \end{bmatrix} + 6 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 10 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

correspond to the choices 0, 5, and 10 for s.