Math 54W Spring 2005

Solutions to Homework Section 5.3

April 1st, 2005

In exercises 1–3, n linearly independent eigenvectors and their corresponding eigenvalues of an $n \times n$ matrix A are given.

(a) Find a matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

(b) Find A.

1. $(1,1)$,2; $(1,0)$,-3.
   Λ is the diagonal matrix whose entries are the eigenvalues
   $$\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$
   and S is the matrix whose columns are the eigenvectors
   $$S = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$
   To find A, just multiply:
   $$A = S\Lambda S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -3 & 5 \\ 0 & 2 \end{pmatrix}.$$

3. $(1,0,1)$,-2; $(0,1,0)$,-1; $(1,0,-1)$,0.
   We have
   $$\Lambda = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$
   and
   $$A = S\Lambda S^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

In exercises 7–9, show that the given matrix is not diagonalizable.

7. $A = \begin{pmatrix} 4 & 0 \\ 1 & 4 \end{pmatrix}$.
   This matrix is already in Jordan-normal form, and it is not diagonal, so it is not diagonalizable.

9. $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

The characteristic polynomial is $(t-2)^2(t+3)$. The eigenspace corresponding to eigenvalue 2 is

$$W_2 = NS \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Since this is only one-dimensional, A is not diagonalizable.

13. *Let* $S = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}$. *Let* $A = S\Lambda S^{-1}$, *but do not calculate* A. *Find different* S_1 *and* Λ_1 *such that the same* A *satisfies* $A = S_1\Lambda_1 S_1^{-1}$.

The simplest solution is to switch the order of the eignvalues, i.e. swap the columns of Λ. Since the ordering of the eigenvectors in S must match the ordering of the eigenvalues in Λ, we must also swap the columns of S:

$$\Lambda_1 = \begin{pmatrix} 3 & \\ & 2 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

Another solution is to leave Λ unchanged, and rescale the columns of S. This works because any scalar multiple of an eigenvector is still an eigenvector with the same eigenvalue. For example, we could multiply the first column of S by 10 and the second column by 20 to get

$$\Lambda_1 = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 10 & 20 \\ 20 & -20 \end{pmatrix}.$$

In excercises 17–25, determine if the given matrix is diagonalizable. If so, find matrices S *and* Λ *such that the given matrix equals* $S\Lambda S^{-1}$.

17. $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$.

The characteristic polynomial is

$$\det \begin{pmatrix} 2-t & -4 \\ 1 & -2-t \end{pmatrix} = (2-t)(-2-t) + 4 = t^2,$$

so the only eigenvalue is zero. Since A is not the zero matrix, its null space is not all of $\mathbb{R}^2$, so it is not possible to find two linearly independent eigenvectors. Therefore A is not diagonalizable.

19. $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$.

The characteristic polynomial is

$$\det \begin{pmatrix} 1-t & 4 \\ 1 & -2-t \end{pmatrix} = (1-t)(-2-t) - 4 = t^2 + t - 6 = (t+3)(t-2),$$

so the eigenvalues are 2 and -3. Since A is a 2×2 matrix with 2 distinct eigenvalues, it is diagonalizable by Theorem 5.30. Λ is the diagonal matrix consisting of the eigenvalues:

$$\Lambda = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

To find S we need to find the eigenvectors:

$$W_2 = NS\begin{pmatrix} -1 & 4 \\ 1 & -4 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right\}.$$

$$W_{-3} = NS\begin{pmatrix} 4 & 4 \\ 1 & 1 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}.$$

Thus the eigenvectors are $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, so

$$S = \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}.$$

21. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$

This matrix is upper-triangular, so the diagonal entries, 1 and 2, are the eigenvalues. There are two linearly independent eigenvectors with eigenvalue 1, namely the standard basis vectors $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. By Theorem 5.31 any eigenvector will eigenvalue 2 will be linearly independent from e_1 and e_2, giving three linearly independent eigenvectors. So A is diagonalilzable. We have

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

To find S we need to find the eigenvector of eigenvalue 2:

$$W_2 = NS\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right\}.$$

Thus

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

23. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$

This matrix is lower-triangular, and all diagonal entries are equal to 1, so the only eigenvalue is 1. The corresponding eigenspace is

$$W_1 = NS\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since the matrix we are taking the null space of is not the zero matrix, the null space has dimension less than 3, so it is impossible to find 3 linearly independent eigenvectors. Therefore A is not diagonalizable.

25. $A = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 5 & -5 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

This matrix is upper-triangular, and from its diagonal entries we see that its eigenvalues are 3 and -2. The first two standard basis vectors e_1 and e_2 are linearly independent eigenvectors with eigenvalue 3. The other eigenspace is

$$W_{-2} = NS \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Since W_{-2} and W_3 are both 2-dimensional, A has four linearly independent eigenvectors and hence it is diagonalizable, with

$$\Lambda = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$