Math 54 Spring 2005

Solutions for 3.3

February 9th

Section 3.3

Def. 3.19: *Let V be a set on which addition and scalar multiplication are defined. If the following axioms are satisfied for all $u, v, w \in V$ and $r, s \in \mathbf{R}$.*

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) *There is a special member 0 of V s.t.* $\mathbf{u} + 0 = \mathbf{u}$, $\forall \mathbf{u} \in V$.

(d) $\forall \mathbf{u} \in V\ there is a negative u in V s.t. \mathbf{u} + (-\mathbf{u}) = 0$

(e) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$

(f) $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$

(g) $(rs)\mathbf{u} = r(s\mathbf{u})$

(h) $1\mathbf{u} = u$

11. *Show that axioms (3.19a, e, g) are satisfied for P_n.*

Let $\mathbf{f}$ and $\mathbf{g}$ be polynomials in P_n. So, we can write

$$\mathbf{f} = a_0 + a_1 x + \ldots + a_n x^n$$
$$\mathbf{g} = b_0 + b_1 x + \ldots + b_n x^n.$$

Axiom 3.19a:

$$\begin{gathered}\mathbf{f} + \mathbf{g} = (a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n) = \\ = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n = \\ = (b_0 + a_0) + (b_1 + a_1)x + \ldots + (b_n + a_n)x^n = \\ = (b_0 + b_1 x + \ldots + b_n x^n) + (a_0 + a_1 x + \ldots + a_n x^n) = \mathbf{g} + \mathbf{f}.\end{gathered}$$

Axiom 3.19e:

$$\begin{gathered}r(\mathbf{f} + \mathbf{g}) = r((a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n) = \\ = r(a_0 + b_0) + r(a_1 + b_1)x + \ldots + r(a_n + b_n)x^n = \\ = (ra_0 + rb_0) + (ra_1 + rb_1)x + \ldots + (ra_n + rb_n)x^n = \\ = (ra_0 + ra_1 x + \ldots + ra_n x^n) + (rb_0 + rb_1 x + \ldots + rb_n x^n) = r\mathbf{f} + r\mathbf{g}.\end{gathered}$$

Axiom 3.19g:

$$\begin{gathered}(rs)\mathbf{f} = (rs)(a_0 + a_1x + \ldots + a_nx^n) = \\ = (rs)a_0 + (rs)a_1x + \ldots + (rs)a_nx^n = \\ = r(sa_0) + r(sa_1)x + \ldots + r(sa_n)x^n = \\ = r(sa_0 + sa_1x + \ldots + sa_nx^n) = \\ r(s(a_0 + a_1x + \ldots + a_nx^n)) = r(s\mathbf{f}).\end{gathered}$$

12. *Show that axioms (3.19b, f, h) are satisfied for P_n.*

Let $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ be polynomials in P_n. So, we can write

$$\begin{gathered}\mathbf{f} = a_0 + a_1x + \ldots + a_nx^n \\ \mathbf{g} = b_0 + b_1x + \ldots + b_nx^n \\ \mathbf{h} = c_0 + c_1x + \ldots + c_nx^n.\end{gathered}$$

Axiom 3.19b:

$$\begin{gathered}(\mathbf{f} + \mathbf{g}) + \mathbf{h} = ((a_0 + a_1x + \ldots + a_nx^n) + (b_0 + b_1x + \ldots + b_nx^n)) + (c_0 + c_1x + \ldots + c_nx^n) = \\ = ((a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n) + (c_0 + c_1x + \ldots + c_nx^n) = \\ = ((a_0 + b_0) + c_0) + ((a_1 + b_1) + c_1)x + \ldots + ((a_n + b_n) + c_n)x^n) = \\ = (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))x + \ldots + (a_n + (b_n + c_n))x^n) = \\ = (a_0 + a_1x + \ldots + a_nx^n) + ((b_0 + c_0) + (b_1 + c_1)x + \ldots + (b_n + c_n)x^n) + = \\ = (a_0 + a_1x + \ldots + a_nx^n) + ((b_0 + b_1x + \ldots + b_nx^n) + (c_0 + c_1x + \ldots + c_nx^n)) = \mathbf{f} + (\mathbf{g} + \mathbf{h}).\end{gathered}$$

Axiom 3.19f:

$$\begin{gathered}(r + s)\mathbf{f} = (r + s)(a_0 + a_1x + \ldots + a_nx^n) = \\ = (r + s)a_0 + (r + s)a_1x + \ldots + (r + s)a_nx^n = \\ = (ra_0 + sa_0) + (ra_1 + sa_1)x + \ldots + (ra_n + sa_n)x^n = \\ = (ra_0 + ra_1x + \ldots + ra_nx^n) + (sa_0 + sa_1x + \ldots + sa_nx^n) = \\ = r(a_0 + a_1x + \ldots + a_nx^n) + s(a_0 + a_1x + \ldots + a_nx^n) = r\mathbf{f} + s\mathbf{f}.\end{gathered}$$

Axiom 3.19h:

$$\begin{gathered}1\mathbf{f} = 1(a_0 + a_1x + \ldots + a_nx^n) = \\ = (1a_0) + (1a_1)x + \ldots + (1)a_nx^n = \\ = a_0 + a_1x + \ldots + a_nx^n = \mathbf{f}.\end{gathered}$$

18. *Is the following a vector space?. The set of all ordered triples of real numbers, (x_1, x_2, x_3), with usual addition, and the following scalar multiplication: $r(x_1, x_2, x_3) = (2rx_1, 2rx_2, 2rx_3)$.*

    No. This is not a vector space. For example

$$1(1,1,1) = (2,2,2) \neq (1,1,1),$$

    which contradicts axiom (h): $1\mathbf{u} = \mathbf{u}, \forall \mathbf{u} \in \mathbf{V}$.

28. *Is the following a vector space?. The set of all $2x2$ matrices of the form*

$$\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$$

    with usual addition and scalar multiplication from M_{22}.

    M_{22} is a vector space so it is enough to check if the given set is closed under addition and scalar multiplication.

    Addition:

$$\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} + \begin{bmatrix} c & c+d \\ c+d & d \end{bmatrix} = \begin{bmatrix} a+c & a+b+c+d \\ a+b+c+d & b+d \end{bmatrix}.$$

    We can see that the sum is again in our set (the elements on the anti-diagonal are equal to the sum of the elements on the diagonal).

    Scalar multiplication:

$$r\begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} = \begin{bmatrix} ra & ra+rb \\ ra+rb & rb \end{bmatrix}.$$

    We can see that the result is again in our set (the elements on the anti-diagonal are equal to the sum of the elements on the diagonal).

    So the given subset of M_{22} is closed under addition and scalar multiplication $\Rightarrow$ it is a vector space.

32. *Prove $(-1)\mathbf{u} = -\mathbf{u}$.*

    $(-1)\mathbf{u} = (-1)\mathbf{u} + 0 = (-1)\mathbf{u} + (\mathbf{u} + (-\mathbf{u})) = ((-1)\mathbf{u} + \mathbf{u}) + (-\mathbf{u}) = ((-1)\mathbf{u} + 1\mathbf{u}) + (-\mathbf{u}) = ((-1)+1)\mathbf{u} + (-\mathbf{u}) = 0\mathbf{u} + (-\mathbf{u}) = 0 + (-\mathbf{u}) = -\mathbf{u}$.