Math 54 Spring 2005

Solutions for 3.4

February 11th

Section 3.4

1. $V = \mathbb{R}^2$, $S = \{(s, 2s)|s \text{ is a real number}\}$

   [a.] S is closed under addition. $(s, 2s) + (t, 2t) = (s + t, 2s + 2t) = (s + t, 2(s + t))$. The second coordinate is still twice the first one.

   [b.] S is closed under scalar multiplication.$r(s, 2s) = (rs, 2rs)$. The second coordinate is still twice the first one.

   [c.] S is a subspace, since it is closed under addition and scalar multiplication.

2. $V = \mathbb{R}^3$, $S = \{(0, s, t)|s, t \text{ are real numbers}\}$

   [a.] S is closed under addition. $(0, s_1, t_1) + (0, s_2, t_2) = (0, s_1 + s_2, t_1 + t_2)$. The first coordinate is still zero.

   [b.] S is closed under scalar multiplication.$r(0, s, t) = (0, rs, rt)$. The first coordinate is still zero.

   [c.] S is a subspace, since it is closed under addition and scalar multiplication.

3. $V = \mathbb{R}^2$, $S = \{(n, n)|n \text{ is an integer}\}$

   [a.] S is closed under addition. $(n, n) + (m, m) = (n + m, n + m)$. The coordinates are still equal and integers since the sum of two integers is an integer.

   [b.] S is not closed under scalar multiplication. For example $\frac{1}{2}(1, 1) = (\frac{1}{2}, \frac{1}{2})$. The coordinates are not integers anymore.

   [c.] S is not a subspace, since it is not closed under addition or scalar multiplication.

4. $V = \mathbb{R}^3$, $S = \{(x, y, z)|x, y, z \geq 0\}$

   [a.] S is closed under addition. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$. The coordinates are still positive since the sum of two positive real numbers is positive.

   [b.] S is not closed under scalar multiplication. For example $(-1)(1, 1, 1) = (-1, -1, -1)$. The coordinates are not positive anymore.

   [c.] S is not a subspace, since it is not closed under addition or scalar multiplication.

5. $V = \mathbb{R}^3$, $S = \{(x, y, z)|z = x + y\}$

   [a.] S is closed under addition. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$. If $z_1 = x_1 + y_1$ and $z_1 = x_1 + y_1$ then $z_1 + z_2 = x_1 + x_2 + y_1 + y_2$.

   [b.] S is closed under scalar multiplication. $r(x, y, z) = (rx, ry, rz)$. If $z = x + y$ then $rz = rx + ry$.

   [c.] S is a subspace, since it is closed under addition and scalar multiplication.

14. $V = \mathbf{M}_{22}$, $S = \{A | A \text{ is singular}\}$
We know that non singular is the same as invertible $\Rightarrow$ singular is the same as not invertible which for $2x2$ matrices means the determinant is zero.

[a.] S is not closed under addition. For example

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are singular but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not.

[b.] S is closed under scalar multiplication. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a singular matrix. Then $ad - bc = 0$. Now $r \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$, and $(ra)(rd) - (rb)(rc) = r^2(ad - bc) = 0$ so $\begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix}$ is singular.

[c.] S is not a subspace, since it is not closed under addition or scalar multiplication.

For Exercises 25-32, find $NS(A)$ for the given matrix A. For which n is $NS(A)$ a subspace of $\mathbb{R}^n$?

28. $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \end{bmatrix}$
$NS(A)$ is a subspace of $\mathbb{R}^3$. The second row of the matrix is 2 times the first row, so row reducing the matrix we obtain $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, we have a pivot in the first column, and two free variables. If we call the variables x_1, x_2, x_3, we set the free variables $x_2 = s$, $x_3 = t$, and then we see that $x_1 = -3s - 2t$. Thus $NS(A) = \left\{ \begin{bmatrix} -3s - 2t \\ s \\ t \end{bmatrix} \middle| s, t \in \mathbb{R} \right\} =$ $Span \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. This is a plane in $\mathbb{R}^3$.

32. $A = \begin{bmatrix} 3 & -1 & 1 \\ -6 & 2 & -2 \\ -3 & 1 & -1 \end{bmatrix}$
$NS(A)$ is a subspace of $\mathbb{R}^3$. The second row of the matrix is -2 times the first row and the third row is -1 times the first row, so row reducing the matrix we obtain $A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, we have a pivot in the first column, and two free variables. If we call the variables x_1, x_2, x_3, we set the free variables $x_2 = s$, $x_3 = t$, and then we see that $x_1 = \frac{1}{3}(s - t)$. Thus $NS(A) =$ $\left\{ \begin{bmatrix} \frac{1}{3}(s-t) \\ s \\ t \end{bmatrix} \middle| s, t \in \mathbb{R} \right\} = Span \left\{ \begin{bmatrix} \frac{1}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\}$. This is a plane in $\mathbb{R}^3$.

38. $p_1 = x + x^2, p_2 = x + x^3, p_3 = x + x^2 + x^3$
Define $V = Span \{p_1, p_2, p_3\}$.

[a.] $2x + x^2 = 2p_1 + p_2 - p_3$.

[b.] $2 - 3x + 4x^2 + x^3$ is not in the V, since any element of V has a constant term of zero.

[c.] $0 = 0p_1 + 0p_2 + 0p_3$.

[d.] $x = p_1 + p_2 - p_3$.

42. $Span\left\{\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} 4r \\ 0 \end{bmatrix} \middle| r \in \mathbb{R}\right\} = \left\{\begin{bmatrix} r \\ 0 \end{bmatrix} \middle| r \in \mathbb{R}\right\}$. This is the x-axis in $\mathbb{R}^2$.