The Scaling Limit of Diaconis-Fulton Addition

Lionel Levine

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Joint work with Yuval Peres

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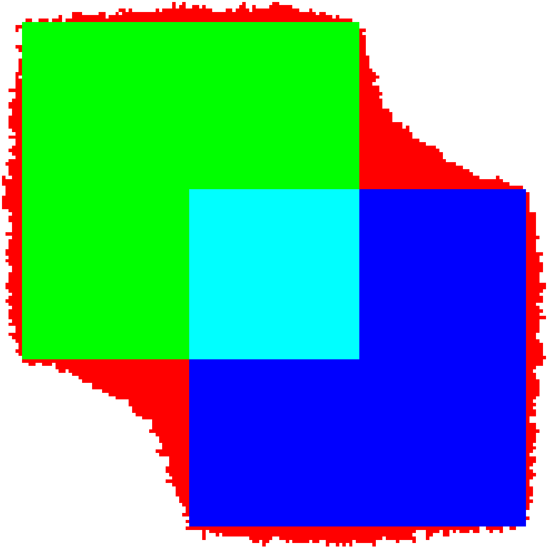
- Define $A + B = C_k$.
- Abeilan property: the law of $A + B$ does not depend on the ordering of $x_1, \ldots, x_k$.



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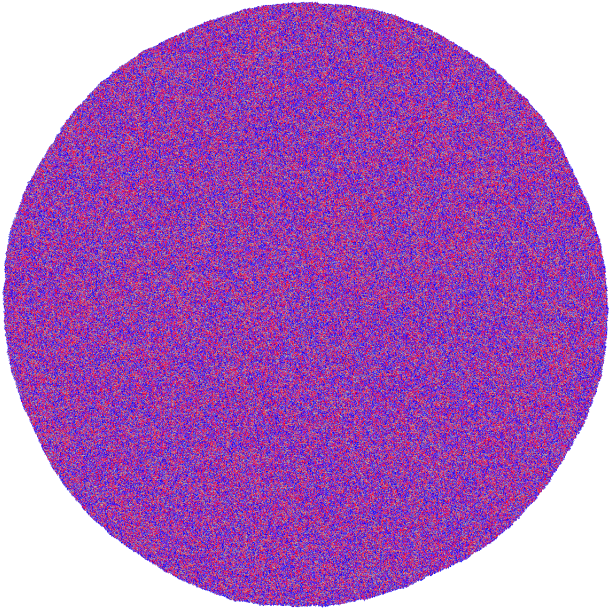
- Here $B_r = \{x \in \mathbb{Z}^d \ : \ |x| < r\}$, and ω_d is the volume of the unit ball in $\mathbb{R}^d$.



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- Each site $x \in \mathbb{Z}^2$ has a **rotor** pointing North, South, East or West.
  (Start all rotors pointing North, say.)
- A particle starts at the origin. At each site it comes to, it
  1. Turns the rotor clockwise by 90 degrees;
  2. Takes a step in direction of the rotor.

Rotor-Router Aggregation

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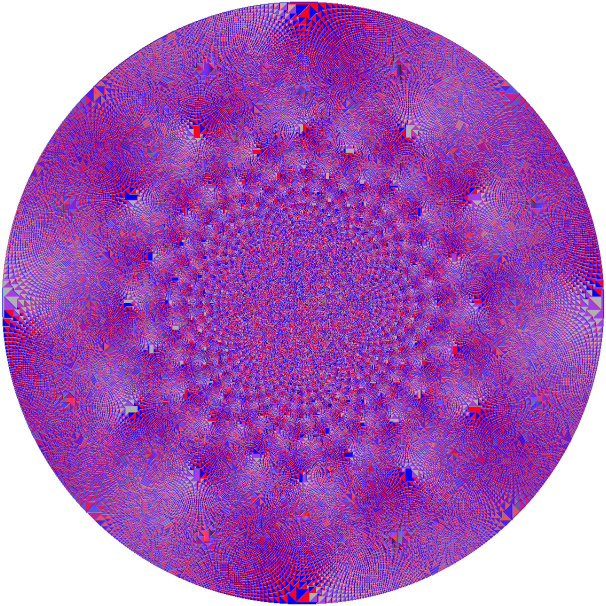
- Makes sense in $\mathbb{Z}^d$ for any d.



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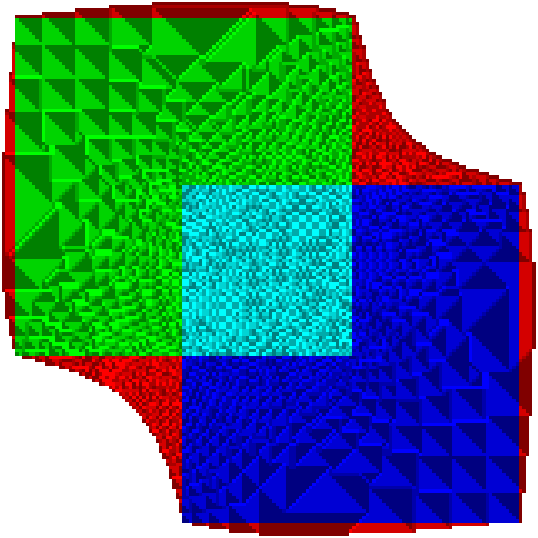
- **Corollary**: Inradius/Outradius $\to 1$ as $n \to \infty$.



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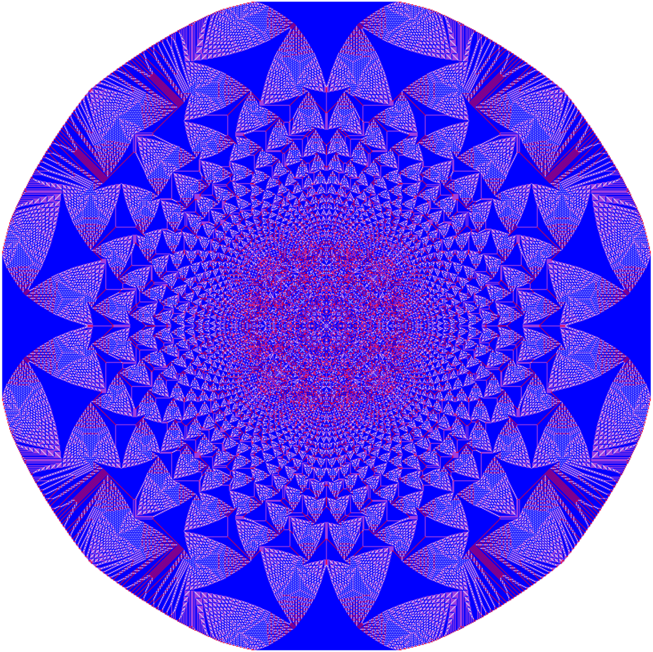
  - **Rotor**: Send extra particles according to the usual rotor rule.



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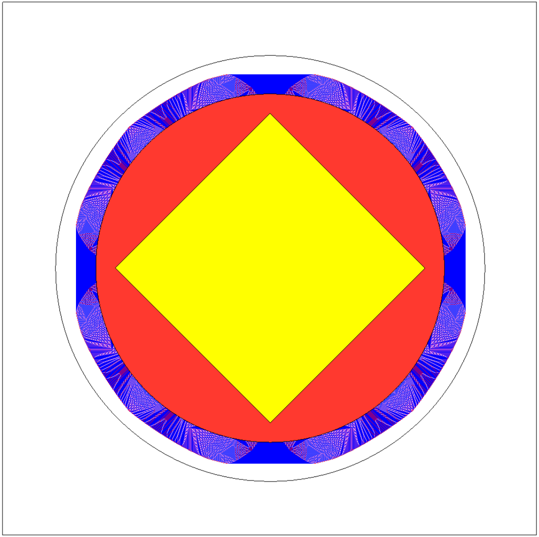
- Improves the bounds of Le Borgne and Rossin.



(Disk of area $n/3$) $\subset S_n \subset$ (Disk of area $n/2$)

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- **Theorem** (L.-Peres): There are constants c and c' depending only on d, such that

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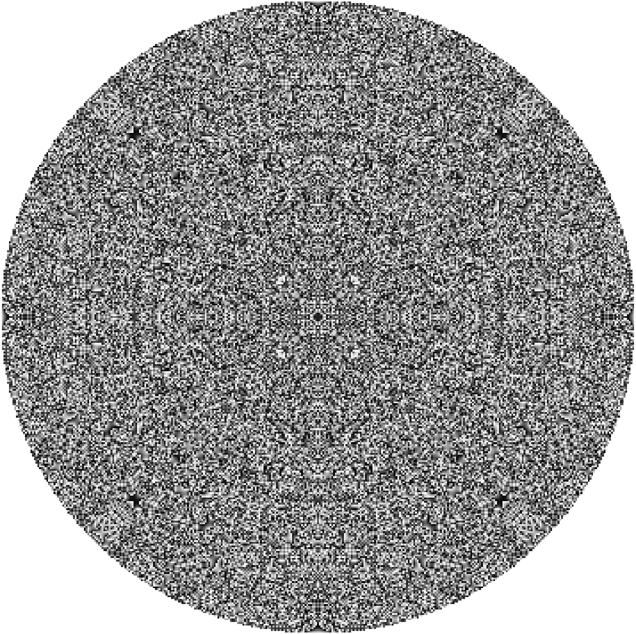
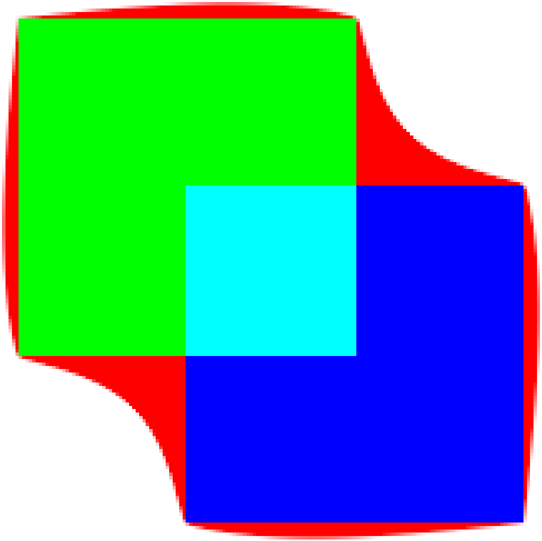
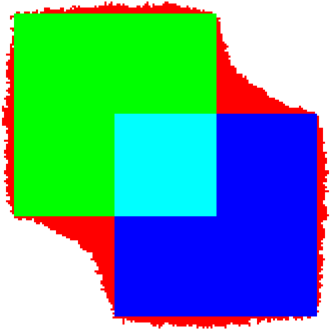
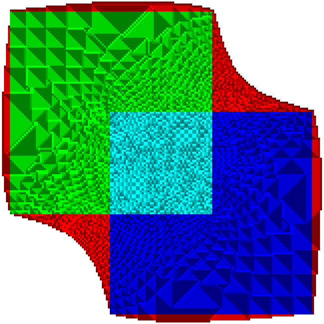
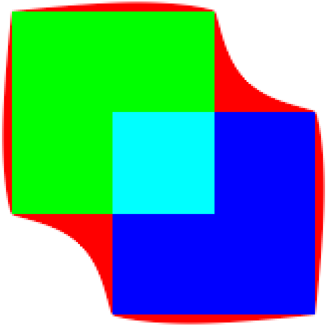
where $m = \omega_d r^d$.











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- Is it the same for all three models?
- Not clear how to define dynamics in $\mathbb{R}^d$.

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$$= \begin{cases} -1 & x \in A \cap B \\ 0 & x \in A \cup B - A \cap B \\ 1 & x \in A \oplus B - A \cup B. \end{cases}$$

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- **Claim**: odometer $= s - \gamma$.

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- Reverse inequality: $s-\gamma-u$ is superharmonic on $A \oplus B$ and nonnegative outside $A \oplus B$, hence nonnegative inside as well.

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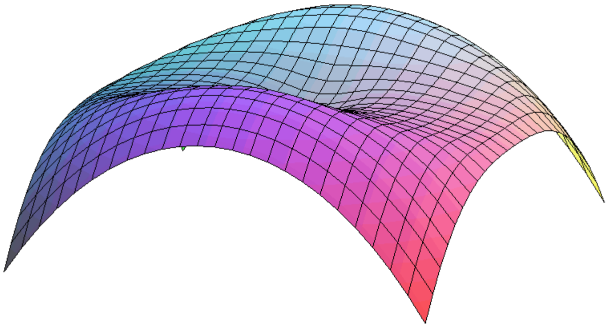
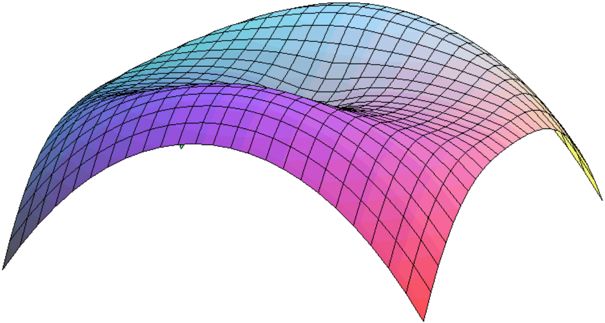
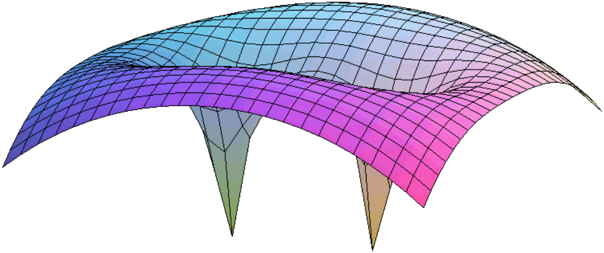
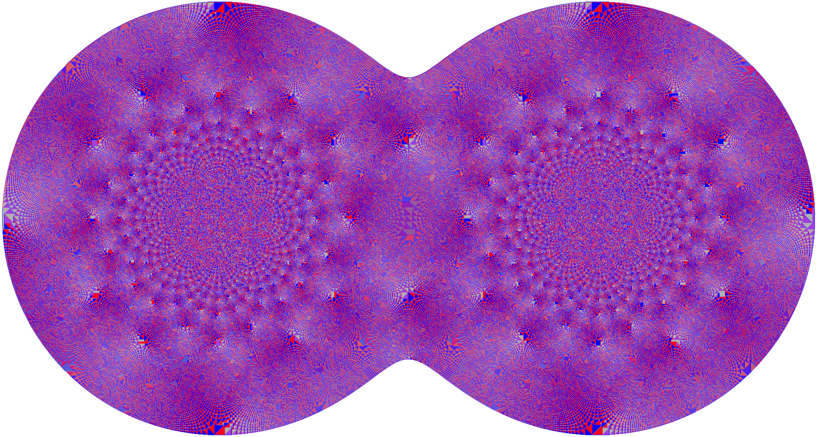
- Odometer: $u = s - \gamma$.









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  - $D_\varepsilon, D^\varepsilon$ are the inner and outer ε-neighborhoods of D.

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for all sufficiently large n, where

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- Follows from the main result and the case of a single point source.

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$$\Downarrow$$

convergence of obstacles

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- $\mathbb{P}(z \notin I_n) = \mathbb{P}(L = M)$.

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- Strategy: show $\mathbb{E}\tilde{L} < \mathbb{E}M$ and use concentration of measure.

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Concentration of Measure

- Using the fact that $D_n \to D$, $u_n \to u$, and the positivity of u, can show that
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- Large deviations:
$$\mathbb{P}\left(\tilde{L} > (1+\lambda)\,\mathbb{E}\,\tilde{L}\right) < 2e^{-c_\lambda \mathbb{E}\tilde{L}};$$
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- Conclude that $\mathbb{P}\left(\tilde{L} \geq M\right) < 4e^{-c'_\varepsilon \delta_n^{-2}}$.

Finishing Up

- Summing over $z \in D_\varepsilon^{::}$ and over n, by Borel-Cantelli only finitely many of the events $\{z \notin I_n\}$ occur, a.s.
- Hence $D_\varepsilon^{::} \subset I_n$ for sufficiently large n.

Circularity for the Divisible Sandpile

- Dirichlet problem for the odometer function

$$\begin{aligned} \Delta u &= 1 && \text{on } A_m - \{o\} \\ \Delta u(o) &= 1 - m && \\ u &= 0 && \text{on } \partial A_m. \end{aligned}$$

- Idea: Compare u to the function

$$\gamma(x) = |x|^2 - ma(x).$$

where a is the potential kernel

$$a(x) = \lim_{n\to\infty} (G_n(o) - G_n(x))$$

and $G_n(x)$ is the expected number of visits to x by SRW before time n.

- $a(x)$ is harmonic off o, and $\Delta a(o) = 1$.
- $\Delta |x|^2 = 1$

Taylor expansion

- Standard estimate:

$$a(x) = \frac{2}{\pi} \log |x| + k + O(|x|^{-2})$$

gives

$$\gamma(x) = |x|^2 - \frac{2m}{\pi} \log |x| + km + O(m|x|^{-2}).$$

- Get a constant $K = K(m)$ such that
  - If $r \le |x| < r+1$, then $\gamma(x) = K + O(1)$.
  - $\gamma(x) \ge K + (r - |x|)^2 + O\left(\frac{r^2}{|x|^2}\right)$.

Inner Radius

- $u - \gamma$ is superharmonic in B_r
- $u - \gamma \geq -K + O(1)$ on the boundary, hence on all of B_r.
- γ grows quadratically as we move away from the boundary
- $\therefore u > 0$ on B_{r-c}.

Outer Radius

- $u-\gamma$ is harmonic in A_m
- $u-\gamma \leq -K + O(1)$ on the boundary, hence on all of A_m.
- If $x \in A_m$ with $r \leq |x| < r+1$, then $u(x) \leq c'$.
- **Lemma:** If $y \in A_m - \{o\}$ there exists $z \sim y$ with $u(z) \geq u(y)+1$.
- **Proof**. For some neighbor z,

$$u(z) \geq \frac{1}{4} \sum_{w \sim y} u(w) = u(y)+1.$$

- $\therefore A_m \subset B_{r+c'}$.

Adapting the Proof for Rotors

- Rotor-router odometer:

$$u(x) = \text{total number of particles emitted from } x.$$

- Instead of $\Delta u = 1$, we only know $-2 \leq \Delta u \leq 4$.
- Repeating the argument only gives

$$B_{cr} \subset A_n \subset B_{c'r}.$$

Smoothing

- To do better, let

$$v(x) = \frac{1}{4k^2} \sum_{y \in S_k(x)} u(y)$$

where $S_k(x)$ is a box of side length $2k$ centered at x.

- Using $\Delta =$ div grad, we get

$$\Delta v(x) = \frac{1}{4k^2} \sum_{(y,z) \in \partial S_k(x)} \frac{u(z) - u(y)}{4}$$
$$= 1 + O\left(\frac{1}{k}\right)$$

if $o \notin S_k(x)$ and all sites in $S_k(x)$ are occupied.

Fancier Smoothing

- Let T be the first exit time of B_r, and

$$v(x) = \mathbb{E}_x u(X_T) - \mathbb{E}_x T + n \mathbb{E} \#\{j < T | X_j = o\}.$$

- Boundary value problem:

$$\begin{aligned} \Delta v &= 1 && \text{on } A_n \cap B_r - \{o\} \\ \Delta v(o) &= 1 - n \\ v &= 0 && \text{on } \partial A_n. \end{aligned}$$

- Want to show $u \approx v$.

Green's Function

- End up getting

$$u(x) \geq v(x) - \sum_{y \in B_r} \sum_{z \sim y} |G_{B_r}(x,y) - G_{B_r}(x,z)|.$$

- Error gets smaller as x approaches the boundary, and we can show $B_{r - C \log r} \subset A_n$.
- But for the outer radius, the error is

$$\sum_{y \in A_n} \sum_{z \sim y} |G_{A_n}(x,y) - G_{A_n}(x,z)|.$$

- Can't control this, so we need another approach.

Spreading Out

- Spherical shells

$$S_k = \{x \in \mathbb{Z}^d \ : \ k \le |x| < k+1\}.$$

- Lawler, Bramson, and Griffeath (1992): If $j < k$, $x \in S_j$, $y \in S_k$, then

$$\mathbb{P}_x(X_{T_k} = y) \le C/(k-j)^{d-1}.$$

- Want to show the same holds for rotor-router walk, with frequency replacing probability.

Holroyd-Propp Bound

- recurrent graph G
- $Y \subset Z$ sets of vertices
- $s(x)$ particles start at x
- Stop walks when they hit Z; how many land in Y?
- Let $H(x) = \mathbb{P}_x(X_T \in Y)$. Then

$$|RR(s,Y) - RW(s,Y)| \leq \sum_{u \in G} \sum_{v \sim u} |H(u) - H(v)|$$

independent of s and the initial rotor positions!

Outer Radius

- $N_j = \#$ particles that ever reach shell S_j.
- If $r < j < k$ with $N_k > N_j/2$, then

$$\frac{CN_j}{(k-j)^{d-1}} \#(S_k \cap A_n) \geq \frac{N_j}{2}$$

hence

$$\sum_{i=j}^{k} \#(S_i \cap A_n) \geq C(k-j)^d.$$

- Since $B_{r - C\log r}$ is fully occupied,

$$k \leq j + C(r^{d-1} \log r)^{1/d}$$

which gives

$$A_n \subset B_{r(1+Cr^{-1/d}(\log r)^{1+1/d})}.$$

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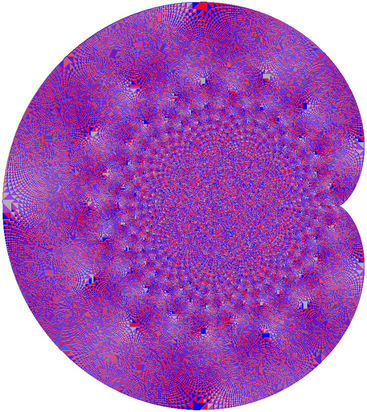
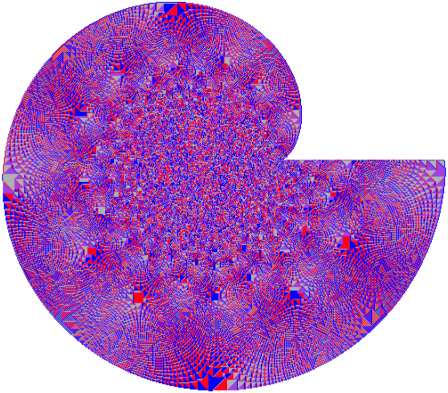
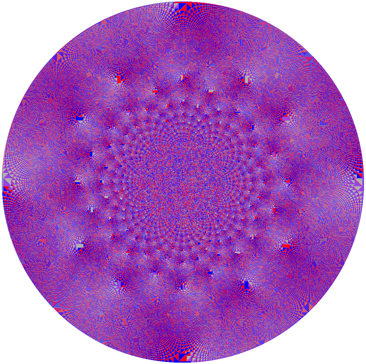
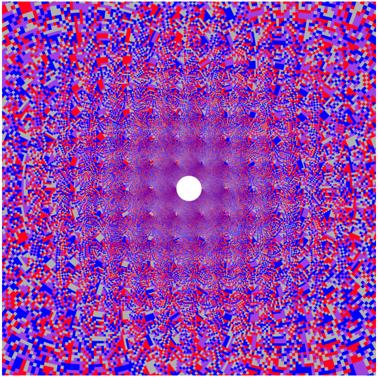
- Understand the patterns in the picture of rotor directions.
- Identify the limiting shape of the "broken rotor" models.









$$z \mapsto 1/z^2$$

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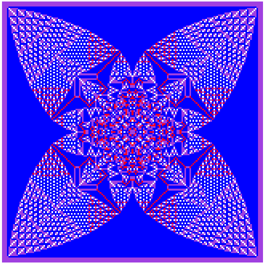
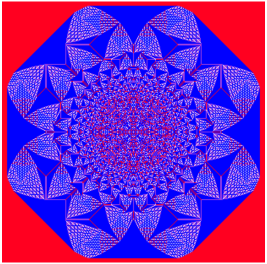
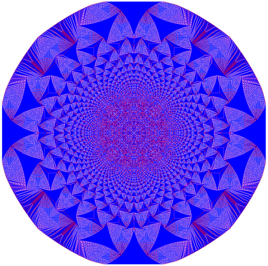
- Even for $h = 2$, the rate of growth of the square is not known.



$h = 2$



$h = 1$



$h = 0$