Chip-Firing and Rotor-Routing on Trees

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Joint work with Itamar Landau and Yuval Peres.

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- For a general directed graph, fix a cyclic ordering of the outgoing neighbors.

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$$A_1 = \{o\}$$

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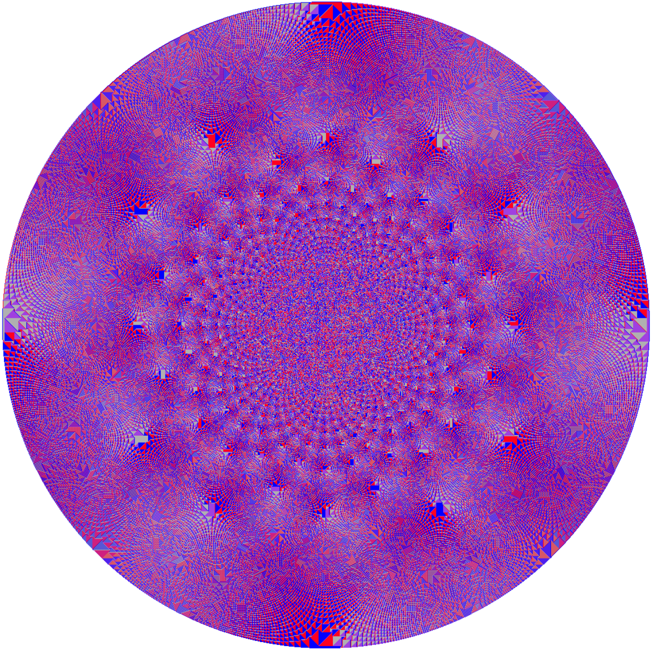
where $x_n \in \mathbb{Z}^d$ is the site at which rotor walk first leaves the region A_{n-1}.


ROTO-
ROOTER
PLUMBERS



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n	$R(n)$
10	0.822
10^2	1.588
10^3	1.637
10^4	1.683
10^5	1.724
10^6	1.741

Three Approaches to Circularity

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- Two ways to get sharper results:
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  3. Modify the underlying graph.
     - The tree is easier than the lattice.

Spherical Asymptotics

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- **Corollary**: Inradius/Outradius $\rightarrow 1$ as $n \rightarrow \infty$.

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  where B_n is the ball of radius n centered at the origin, and $b_n = \#B_n$.
- In particular, if $b_n < m < b_{n+1}$, then

$$B_n \subset A_m \subset B_{n+1}.$$

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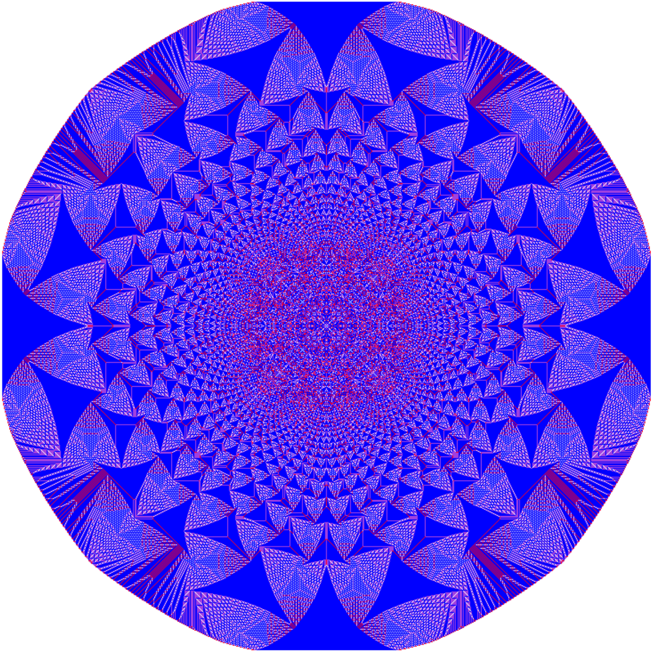
  - **Rotor**: Send extra particles according to the usual rotor rule.



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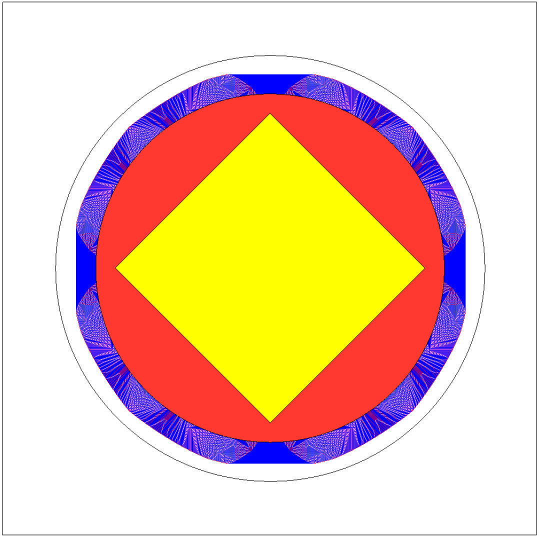
- Improves the bounds of Le Borgne and Rossin.



(Disk of area $n/3$) $\subset S_n \subset$ (Disk of area $n/2$)

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- The sink never topples.
- Order of topplings does not affect the final state σ°.

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- Babai-Toumpakari: The minimal ideal of $\mathcal{M}$ is a finite abelian group $SP(G)$ called the **sandpile group** of G.

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- $SP(G)$ is the group of **recurrent chip configurations** on G, i.e. configurations σ such that

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for some configuration $\tau \neq 0$.

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- Consider the chip configuration

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- **Proof**:

$$\beta = \sum_{v \neq s} \Delta_v.$$

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- Finite rooted tree T.
- Collapse the leaves to a single sink vertex.
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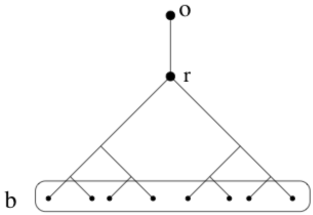

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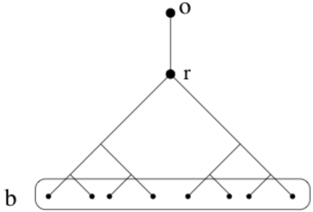

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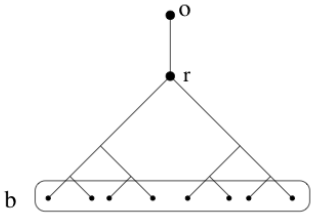

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- What is the structure of the sandpile group?

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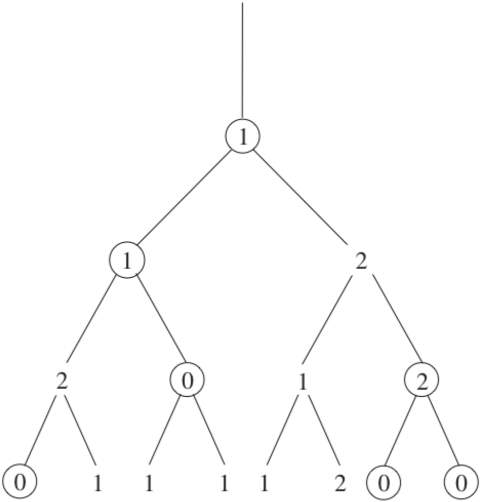
  - If strict inequality holds at x, then x will never be burned.

A Recurrent Configuration on the Regular Ternary Tree



- Critical vertices are circled.

Structure of the Sandpile Group

- **Theorem** (L.) Let T_n be a branch of the regular ternary tree of height n. Then

$$SP(T_n) \simeq \mathbb{Z}_{2^n-1} \oplus \mathbb{Z}_{2^{n-1}-1} \oplus \ldots \oplus (\mathbb{Z}_7)^{2^{n-4}} \oplus (\mathbb{Z}_3)^{2^{n-3}}.$$

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- Similar decomposition for the d-regular tree for any d.

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- Multiples of the root in T_4:

$\hat{r}$	$2\hat{r}$	$3\hat{r}$	$4\hat{r}$	$5\hat{r}$	$6\hat{r}$	$7\hat{r}$	$8\hat{r}$	$9\hat{r}$	$10\hat{r}$	$11\hat{r}$	$12\hat{r}$	$13\hat{r}$	$14\hat{r}$	$15\hat{r} = e$
2	0	1	2	0	1	2	2	2	0	1	2	0	1	2
0	1	1	1	2	2	2	2	0	1	1	1	2	2	2
2	2	2	2	2	2	2	0	1	1	1	1	1	1	1

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- In particular, $\hat{r}$ has order

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

The Sandpile Group of a Tree, In Terms of its Branches

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  - After modding out by $\hat{r}$, the branches become independent.
  - Since $(k+1)\hat{r} \mapsto (\hat{r}_1, \ldots, \hat{r}_k)$ we have to mod out by this on the right.

Strengthening to a Direct Sum

- **Lemma**: Let T_n be the regular ternary tree of height n. Then

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$$p(u)(x) = 2^{n+1-|x|} \sum_{|y|=|x|} u(y).$$

- Note if u is already constant on levels, then

$$p(u) = 2^n u = u$$

since $u = k\hat{r}$ and $\hat{r}$ has order $2^n - 1$. $\square$

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- $SP(T_n) = \mathbb{Z}_{2^n-1} \oplus \mathbb{Z}_{2^{n-1}-1} \oplus \ldots \oplus (\mathbb{Z}_7)^{2^{n-4}} \oplus (\mathbb{Z}_3)^{2^{n-3}}$.

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- So the probability that a stable state is recurrent is about

$$\left(\frac{2^c}{3}\right)^{2^n} = (0.858)^{2^n}.$$

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  - $R =$ diameter of the set of sites that topple.
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- **Fact**: $RR(G) \simeq SP(G)$.

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- Starting chip configuration u, ending configuration v.
- **Lemma**: If H is harmonic, and the initial and final rotor configurations are the same, then

$$\sum_{x \in T} H(x)u(x) = \sum_{x \in T} H(x)v(x).$$

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- Enough to show $A'_{c_n} = B_n$ for some sequence c_n.

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- Thus $A'_{c_n} = B_n$, where

$$c_n = c_{n-1} + 3(2^n - 1).$$

Escape Sequences

- Escape sequence

$$a_j = \begin{cases} 0, & \text{if the } j^{th} \text{ chip returns to the origin;} \\ 1, & \text{if the } j^{th} \text{ chip escapes to infinity.} \end{cases}$$

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  - Every subword of $a^{(j)}$ of length $2^k - 1$ contains at most 2^{k-1} ones.

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  - Known to exist for $\mathbb{Z}^2$ (Jim Propp) and for the d-regular tree.

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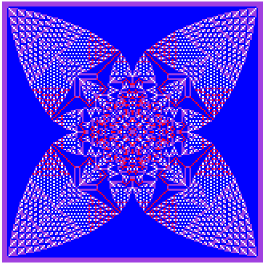

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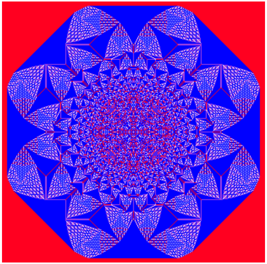

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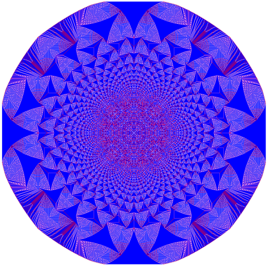

$h = 0$