Bott-Samelson varieties, subword complexes and brick polytopes

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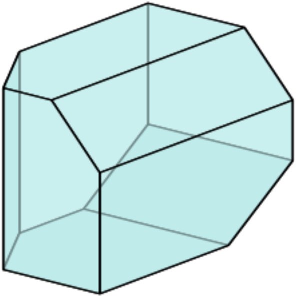
The toric variety of an associahedron

Theorem (E)

The toric variety of the associahedron can be described as a poset in which the ascending chains are flags of vector spaces.



$\longleftrightarrow$

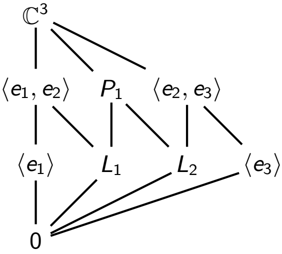


Figure : Picture from wikipedia

Outline

Outline

The subword complex

Definition (Knutson-Miller)

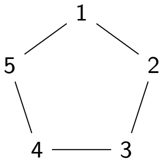
Let Q be a word in the generators of W and $w \in W$, where W is the Weyl group of a complex semisimple Lie group. Then $\Delta(Q, w)$ is the simplicial complex with

- vertices $= \{$position of the letters of $Q\}$
- facets $= \{$subwords $J \subset Q$ such that $\prod Q|_{J=(1,\ldots,1)} :=$ (product of the letters in $Q \setminus J$) is a reduced expression for $w\}$

Example

For $Q = (s_1, s_2, s_1, s_2, s_1)$

$\Delta(Q, w_0) =$



The subword complex is a simplicial complex that encondes which subwords of Q have product w.

Definition

Given a word Q, then the elements in W that can be obtained by multiplying the letters in subwords of Q form a poset with maximum. Let us denote this maximum by $\mathbf{Dem(Q)} \in W$.

Natural Question

Knutson and Miller prove that $\Delta(Q, w)$ is a sphere if and only if $w = Dem(Q)$
Question: Can $\Delta(Q, w)$ be realized as the boundary complex of a convex polytope?

An answer:

Theorem (Pilaud-Stump)

For certain Q then $\Delta(Q, w)$ is the boundary complex of the polytope dual to the brick polytope.

Outline

The Brick polytope

Let $\nabla(W) := \{w_{s_i} : s_i \in S\}$ be the fundamental weights of W.
Let F be a subword of Q and define

$$\omega(F,k) := (\prod^{k-1} Q|_{F=(1,\ldots,1)})(\omega_{q_k})$$

and

$$B(F) := \sum_{k\in[m]} \omega(F,k).$$

Definition

Given a subword complex $\Delta(Q,w)$ with $|Q| = m$, the *brick polytope* is the convex hull of the brick vectors of some faces of $\Delta(Q,w)$

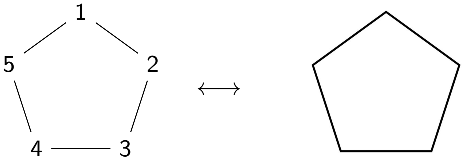
$$B(Q,w) := \text{conv}\{B(F) : F \in \Delta(Q,w) \text{ and } \prod Q|_{F=(1,\ldots,1)} = w\}.$$

Associahedra

Theorem

If we take $Q = \mathbf{c}\mathbf{w}_0(\mathbf{c})$ to be the word starting with a Coxeter element $\mathbf{c}$ and then followed by the expression of w_0 corresponding to c then the subword complex $\Delta(Q, w_0)$ is dual to the associahedron.

$\Delta(Q, w_0) =$



The Toric variety of the Brick Polytope

Let $\Delta(W) := \{\alpha_{s_i} : s_i \in S\}$ be the simple roots of W

Definition

A word Q is *root independent* if for some vertex $B(F)$ of $B(Q, w)$ (or all vertices) we have that the multiset $r(F) := \{\{\mathrm{r}(F, i) : i \in F\}\}$ is linearly independent, where

$$\mathrm{r}(I, k) := \left(\prod^{k-1} Q|_{I=(1,\ldots,1)} \right) (\alpha_{q_k})$$

Theorem (E)

If Q is root independent then we can construct a toric variety associated to the brick polytope $B(Q, w)$. For $W = A_n$, this variety corresponds to a poset in which the ascending chains are flags of vector spaces.

Outline

Setup

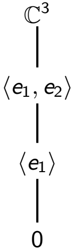
- $G = SL_n(\mathbb{C})$
- Fix $\{e_1, \ldots, e_n\}$, a basis of $\mathbb{C}^n$
- The Borel subgroup B consists of upper triangular matrices
- G/B is the flag manifold $= \{(V_1, \ldots, V_n) : V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n\}$
- The maximal torus T consists of diagonal matrices
- $W = A_{n-1}$ is the Weyl group of G generated by $s_1, \ldots, s_{n-1}$

Outline

Definition via an example

Example

Let $n = 3$ and $Q = (s_1, s_2, s_1, s_2, s_1)$ be a word on the generators of W. Then the **Bott-Samelson variety** of Q is
$BS^Q = \{(L_1, P_1, L_2, P_2, L_3) : \text{ the following incidences hold}\}$



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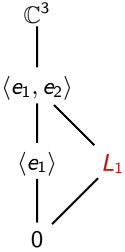

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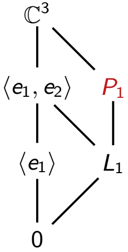

Definition via an example

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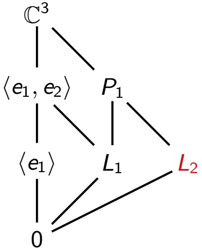

Definition via an example

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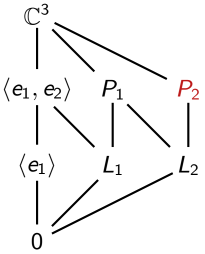

Definition via an example

Example

Let $n = 3$ and $Q = (s_1, s_2, s_1, s_2, s_1)$ be a word on the generators of W. Then the **Bott-Samelson variety** of Q is
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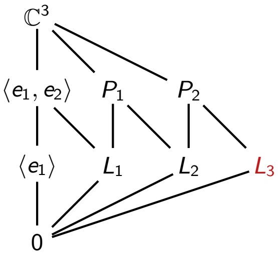
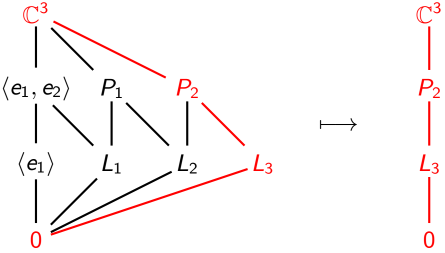


Natural Map

There is a natural map $m_Q : BS^Q \to G/B$ mapping BS to the rightmost flag.

Example

Consider $m_Q : BS^{(s_1,s_2,s_1,s_2,s_1)} \to G/B$

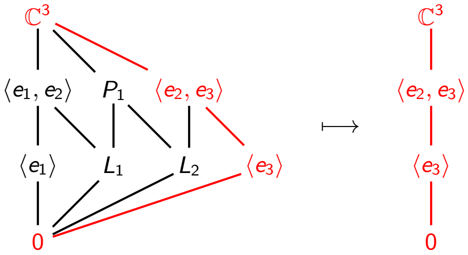


An important fiber

- Given $w \in W$, the fiber $m_Q^{-1}(wB)$ has dimension is $|Q| - \ell(w)$
- If $w = Dem(Q)$ then $m_Q^{-1}(wB)$ is smooth

Example

For $Q = (s_1, s_2, s_1, s_2, s_1)$ we have that the fiber is
$m_Q^{-1}(s_1 s_2 s_1 B) = \{(L_1, P_1, L_2) : \text{ the following incidences hold}\}$



Theorem (E)

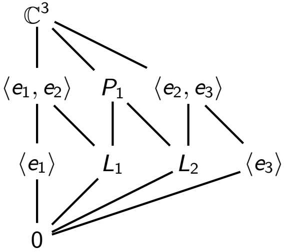
The fiber $m_Q^{-1}(wB)$ is a toric variety if and only if the following hold

- *Q is root independent,*
- *$\ell(w) \leq |Q| \leq \ell(w) + n$, and*
- *$Dem(Q) = w$.*

Moreover, $m_Q^{-1}(wB)$ is the toric variety associated to the brick polytope $B(Q, w)$.

Example

The toric variety of a 2-dimensional associahedron (a pentagon) is



How to prove this?

The theorem on the previous slide will follow from understanding the symplectic structure on $m_Q^{-1}((B)w)$.

General Toric Symplectic Geometry

- We will give a torus action on $m_Q^{-1}(wB)$
- This action will allow us to have a moment map $\mu : m_Q^{-1}(wB) \to \mathbf{R}^n$
- **Theorem**(Atiyah, Guillemin-Sternberg): The image of the moment map is the convex hull of the images of the T-fixed points under the moment map.
- Whenever we have a toric variety the image of moment map tells us what the polytope of this variety is.

Goal

Understand the T-fixed points to be able to describe the polytope corresponding to $m_Q^{-1}(wB)$.

General Theorem

These symplectic tools give us a more powerful theorem:

Theorem (E)

The image of the moment map of the symplectic manifold $m_Q^{-1}(wB)$ *is the brick polytope* $B(Q, w)$.

So let's find out more about the symplectic geometry of $m_Q^{-1}(\mathrm{B})$ and what its T-fixed points are!

Outline

Subword Complexes and Brick Polytopes

- Subword complexes
- Brick polytopes

Bott-Samelson varieties for SL_n

- Definition and properties
- Symplectic Structure on BS^Q
- Brick polytopes for $W = A_{n-1}$

The general Bott-Samelson story

Symplectic Structure on BS^Q

Torus action

- Note that the torus T acts on $\mathbb{C}^n$ by multiplication and this action can be extended to T acting on each of the basis of the vector spaces in $(V_1, \ldots, V_m) \in BS^Q$.
- The T-fixed points of this action are precisely the points $(V_1, \ldots, V_m)$ such that each vector space has as basis a subset of $\{e_1, \ldots, e_n\}$.
- Bott-Samelson varieties are symplectic manifolds with respect to this torus and they have a moment map

Moment map

Given $p = (V_1, \dots, V_m) \in BS^Q$ and $i \in [m]$ define

$$\mu(p,i) = (\mathrm{dim}_{e_1}(V_i), \dots, \mathrm{dim}_{e_m}(V_i)),$$

where $\mathrm{dim}_{e_j}(V)$ denotes the dimension of V on the e_j-th coordinate. Then the moment map is

$$BS^Q \xrightarrow{\mu} \mathbf{R}^n$$

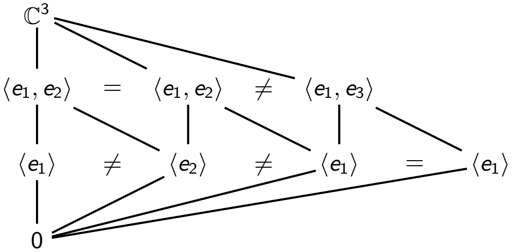
$$(V_1, \dots, V_m) \overset{\mu}{\longmapsto} \left(\sum_{i=1}^{n} \mathrm{dim}_{e_1}(V_i), \dots, \sum_{i=1}^{n} \mathrm{dim}_{e_m}(V_i) \right).$$

From Bott-Samelsons to Subword Complexes

There is a nice 1-1 correspondence between $T-$fixed points of BS^Q and subwords of Q. Moreover, the point corresponding to the subword J gets mapped by m to the flag $(\prod J)B$.

Example

The subword $J = (s_1, -, s_1, s_2, -)$ of $Q = (s_1, s_2, s_1, s_2, s_1)$ corresponds to the point on the right and the image of $m : BS^Q \to G/B$ is $(s_1 s_1 s_2)B = (s_2)B$.



From Bott-Samelsons to Subword Complexes

There is a nice 1-1 correspondence between $T-$fixed points of BS^Q and subwords of Q. Moreover, the point corresponding to the subword J gets mapped by m to the flag $(\prod J)B$.

Thus, the $T-$fixed points of the fiber $m^{-1}(wB)$ are encoded by the subwords J of Q such that $\prod J = w$.

Therefore, the subword complex encodes the T-fixed points of $m_Q^{-1}(wB)$

Outline

Brick polytopes for $W = A_{n-1}$

In the case $W = A_{n-1}$, Pilaud and Santos defined the brick polytope in terms of pseudoline arrangements.

Example

Let $Q = (s_1, s_2, s_1, s_2, s_1)$, then $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ and we have the brick configuration:

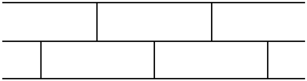


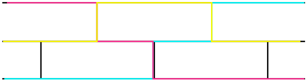
Figure : Bricks!

Brick polytopes for $W = A_{n-1}$

In the case $W = A_{n-1}$, Pilaud and Santos defined the brick polytope in terms of pseudoline arrangements.

Example

Let $Q = (s_1, s_2, s_1, s_2, s_1)$, then $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ and we have the pseudoline arrangement corresponding to the subword $J = (-, s_2, s_1, s_2, -)$:

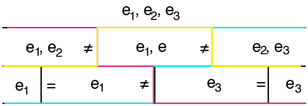


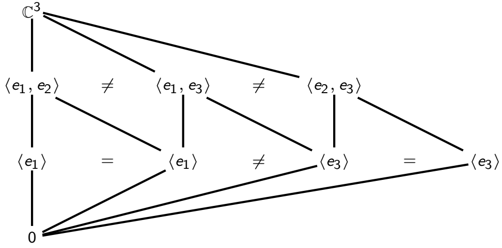
This pseudoline arrangement gives the vector $B(J) = (2, 0, 2)$ obtained by counting bricks above each line.
The **brick polytope** of Q is the convex hull of all the brick vectors $B(J)$ where J is the complement of a facet of $\Delta(Q, w_0)$.

Bott-Samelsons, Cluster Complexes and Brick polytopes

Example

The pseudoline arrangement corresponding to the word $J = (-, s_2, s_1, s_2, -)$ gives a T-fixed point of $BS^{(s_1,s_2,s_1,s_2,s_1)}$





Therefore the Bott-Samelson and the Brick stories coincide!

Outline

Introduction

Idea

Let G be a complex semi simple Lie group, B be a Borel subgroup of G and T be the maximal torus contained in B. Bott-Samelson varieties factor G/B into a product of $\mathbb{CP}^1$'s

A tiny bit of history

- Defined by Bott and Samelson in 1950's to study the cohomology ring of G/T
- Provide desingularizations for Schubert varieties

Definition

Let G be a complex semisimple Lie group, let B be a Borel subgroup of G, and T be a maximal torus contained in B. Let W be the Weyl group of G with generators $S = \{s_1, \ldots, s_n\}$, which correspond to the simple roots $\Delta(W) = \{\alpha_1, \ldots, \alpha_n\}$. Let P be a parabolic subgroup of G. We denote by P_i the minimal parabolic subgroup corresponding to s_i, we then have that $P_i/B \cong \mathbb{CP}^1$

Definition

Let $Q = (s_{i_1}, \ldots, s_{i_m})$ be a word in the generators of W. Then the product $P_{i_1} \times \cdots \times P_{i_m}$ has an action of B^m given by:

$$(b_1, \ldots, b_m) \cdot (p_1, \ldots, p_m) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{m-1}^{-1} p_m b_m)$$

The *Bott-Samelson variety* of Q is the quotient of the product of the P_i's by this action

$$BS^Q := (P_{i_1} \times \cdots \times P_{i_m})/B^m$$

Bott-Samelson varieties are smooth, irreducible and $|Q|$-dimensional algebraic varieties.

Natural map

Bott-Samelson varieties come equipped with a natural map

$$BS^Q \xrightarrow{\mathrm{m_Q}} G/B$$
$$(p_1, \dots, p_m) \longmapsto (p_1 \cdots p_m)B.$$

- The image of this map is the opposite Schubert cell $X^w := \overline{BwB}$, where $w = \mathrm{Dem}(Q)$.
- In the case in which Q is reduced, this map is a resolution of singularities of X^w.
- However, I have been concentrating on cases in which Q is not reduced.

Symplectic Structure on BS^Q

Let T act on BS^Q by $t \cdot (p_1, p_2, \dots, p_m) = (t \cdot p_1, p_2, \dots, p_m)$.

The nice 1-1 correspondence we saw before holds between $T-$fixed points of BS^Q and subwords of Q. Moreover, the point corresponding to the subword J gets mapped by m to the flag $(\prod J)B \in G/B$.

Thus, the $T-$fixed points of the fiber $m^{-1}(wB)$ are encoded by the subwords J of Q such that $\prod J = w$.

Again, we see that the subword complex encodes the T-fixed points of $m_Q^{-1}(wB)$

Moment Map

Given $Q = (q_1, \ldots, q_m)$, we describe the image of the T-fixed points under the moment map using the composition of the maps

$$BS^Q \xhookrightarrow{\varphi} \prod_{i:s_i \in Q} G/P_{\hat{i}} \longrightarrow \mathfrak{t}^*,$$

where $P_{\hat{i}}$ is the maximal parabolic subgroup of G corresponding to $S_{\hat{i}} := \{s_1, \ldots, \hat{s}_i, \ldots, s_n\}$.
The map $\varphi = (\varphi_1, \ldots, \varphi_m)$ where the k-th component is

$$\begin{aligned} BS^Q &\xrightarrow{\varphi_k} G/P_{\hat{k}} \\ (p_1, \ldots, p_m) &\longmapsto (\prod_{i<j} p_i) P_{\hat{k}}. \end{aligned}$$

For each k we have the moment map

$$\mu_k : G/P_{\hat{k}} \longrightarrow \mathfrak{t}^*,$$

where $\mu_k(P_{\hat{k}}) = \omega_{s_k}$, the fundamental weight corresponding to s_k.

Moment Map

The moment map of BS^Q is then

$$\sum_{k=1}^{m} \varphi_k \circ \mu_k.$$

Moreover, given a subword F and

$$p_F = \text{ the fixed point corresponding to } F$$

$$BS^Q \xrightarrow{\varphi_k \circ \mu_k} \mathfrak{t}^*$$
$$p_F \longmapsto \Big(\prod^{k-1} Q|_{F=(1,\ldots,1)}\Big)(\omega_{s_{q_k}}).$$

It then follows that $\mu(p_F) = B(F)$, the brick vector of F.

Theorems

Theorem (E)

The fiber $m_Q^{-1}(wB)$ is a toric variety if and only if the following hold

- *Q is root independent,*
- *$\ell(w) \leq |Q| \leq \ell(w) + n$, and*
- *$Dem(Q) = w$.*

Moreover, $m_Q^{-1}(wB)$ is the toric variety associated to the brick polytope $B(Q, w)$.

Theorem (E)

The fiber $m_Q^{-1}(wB)$ is a symplectic manifold with a Torus action and the image of its moment map is the brick polytope $B(Q, w)$.

Thank you!