4D Polytopes

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We have seen some generalities about how to construct 4 dimensional polytopes. We will now try to draw them in a piece of paper. Let's start with the one having the least number of vertices. Before we dive into the mysterious world of 4 dimensions, let's try to find some patterns in 2 and 3 dimensions.

1. What is the polygon with the least number of vertices? How many vertices does it have? How many edges?

2. What is the polyhedron with the least number of vertices? How many vertices does it have? How many edges does it have? How many faces? (Hint: think about what would happen if it only had 2 vertices, or only 3, or only 4.)

3. How many vertices do you expect the 4D polytope with the least number of vertices to have? We will call this polytope a 4-simplex.

4. Notice that the triangle is made up of 3 edges and the tetrahedron (the 3D polyhedron with the least number of vertices) is made up of 4 triangles. The 4-simplex is made up of some polyhedra, which ones? How many?

5. The skeleton of a polytope a drawing that includes only the vertices and edges of the polytope. The skeleton of a polygon is just the hollow polygon. Draw the 1-skeleton of the cube.

6. Draw the 1-skeleton of the triangle and the tetrahedron. How many edges contain a particular vertex? Which vertices are connected?

7. Use the conclusions from 6 to draw the 1-skeleton of the 4-simplex.

8. In general, the n-dimensional simplest polytope is called the n-simplex. Do you think you can draw the skeleton of the 5-simplex? What about the 6-simplex?

We will now study the hypercube.

9. The 0-dimensional hypercube is a point, draw it. Now you obtain the 1-dimensional hypercube by moving this point one unit to the left and considering what it sweeps. Draw the 1-dimensional hypercube.

10. Now start with the 1-dimensional hypercube , pick a direction perpendicular to it and sweep it one unit in this direction. What polygon is the 2-dimensional hypercube?

11. By now you can probably guess what the 3-dimensional hypercube is. Just in case: to construct it, we start with the 2-dimensional hypercube and move it one unit in the direction perpendicular to the paper. What polyhedron is it? Draw its skeleton in a piece of paper (so we are drawing a 3-dimensional object in 2-dimensions).

12. The 4-dimensional hypercube is constructed starting with a cube, picking one direction in the 4th space and sweeping it. A cartoon of that situation is

obtained by drawing two cubes and connecting similar vertices, try it out.

13. How many vertices does the 4D hypercube have? How many facets (=cubes)? Challenge: how many squares and edges?