EXAMPLES THAT THE STRONG LEFSCHETZ PROPERTY DOES NOT SURVIVE SYMPLECTIC REDUCTION

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ABSTRACT. In this paper we construct a family of six dimensional compact Hamiltonian S^1-manifolds, each of which satisfies the strong Lefschetz property itself but has a non-Lefschetz symplectic quotient. In addition, we show that we can vary our construction such that none of these manifolds is homotopy equivalent to a closed Kähler manifold. As a byproduct, we give a characterization of the fundamental groups of six dimensional compact Hamiltonian strong Lefschetz circle manifolds.

1. INTRODUCTION

Brylinski defined in [Bry88] the notion of symplectic harmonic forms. He further conjectured on a compact symplectic manifold every cohomology class has a harmonic representative and proved this is the case for compact Kähler manifolds and certain other examples.

A symplectic manifold (M, ω) of dimension $2m$ is said to have the strong Lefschetz property or equivalently to be a strong Lefschetz manifold if and only for any $0 \leqslant k \leqslant m$, the Lefschetz type map

$$L^k_{[\omega]} : H^{m-k}(M) \to H^{m-k}(M), \ [\alpha] \to [\alpha \wedge \omega^k] \tag{1.1}$$

is onto. Mathieu [Mat95] proved the remarkable theorem that Brylinski conjecture is true for a symplectic manifold (M, ω) if and only if it has the strong Lefschetz property. This result was sharpened by Merkulov [Mer98] and Guillemin [Gui01], who independently established the symplectic d, δ-lemma for compact symplectic manifolds with the strong Lefschet property. As a consequence of the symplectic d, δ-lemma, they showed that strong Lefschetz manifolds are formal in a certain sense.

Lin and Sjamaar obtained an equivariant version of the above results in [L-S03]. In particular, it was proved in [L-S03] for a compact Hamiltonian G-manifold with the strong Lefschetz property any closed form has a canonical equivariantly closed extension unique up to coboundaries.

In this paper we investigated the question whether the strong Lefschetz property is preserved by the symplectic reduction. Suppose (M, ω, S^1, f) is a compact Hamiltonian manifold and suppose S^1 acts freely on $Z = f^{-1}(0)$. Marsden-Weinstein theorem asserts that the restriction of the symplectic form ω on Z descends to a symplectic form ω_0 on the quotient space $M_0 =$

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Z/S^1. If in addition we assume that ω is an invariant Kähler form on M, then ω_0 is a Kähler form on M_0. By Hard Lefschetz theorem any compact Kähler manifold, and in particular M_0, satisfies the strong Lefschetz property. Thus the above question has an affirmative answer for equivairant Kähler manifolds. One naturally wonders whether the symplectic reduced space (M_0, ω_0) will always have the strong Lefschetz property whenever (M, ω) has this property, even if (M, ω) is not an equivariant Kähler manifold.

In this paper we showed by counter examples that, in contrast with the equivariant Kähler case, the strong Lefschetz property does not survive the symplectic reduction in general. Our construction comes from investigating some interesting symplectic four manifolds discovered by Dong Yan in [Yan96], and Karshon's example [Ka96] on a compact Hamiltonian circle six manifold with non-$\log$ concave Duistermaat-Heckman function. Indeed we constructed a family of six dimensional compact Hamiltonian S^1-manifolds, each of which has the strong Lefschetz property itself but admits a non-Lefschetz symplectic quotient. In addition, we showed that we can vary our construction such that none of these manifolds is homotopy equivalent to a compact Kähler manifold, giving rise to examples of six dimensional compact non-Kähler Hamiltonian circle manifolds (c.f. [Le96], [T98]).

It is an important question with a rich history to which extent the symplectic category is larger than the Kähler category. Examples we constructed in this paper seem to suggest that even the category of strong Lefschetz symplectic manifolds with Hamiltonian circle actions is much larger than the category of Kähler manifolds with compatible Hamiltonian circle actions.

We briefly outline the content of this paper. In section 2 we give a quick review on symplectic cut and **Leray-Hirsch** theorem and proved one technical lemma. In section 3 we vary Dong Yan's construction to obtain symplectic four manifolds with certain properties we want. In section 4 we show from such symplectic four manifolds how to construct compact Hamiltonian strong Lefschetz circle manifolds with a non-Lefschetz symplectic quotient. As a byproduct, we present in section 5 a sufficient and necessary condition for a finitely presentable group G to be the fundamental group of a compact Hamiltonian strong Lefschetz circle manifolds.

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2. PRELIMINARIES

First we give with a quick review of the basic construction of symplectic cut as introduced by Lerman in [Le95]. Suppose (W, σ) is a symplectic manifold with a Hamiltonian circle action and a moment map $f : W \to \mathbf{R}$. If the circle S^1 acts freely on a level set $f^{-1}(a)$, then a is a regular value of the moment map $F(m, z) = f(m) - |z|^2$ arising from the action of S^1 on the product manifold $(W \times C, \sigma \oplus \frac{1}{\sqrt{-1}} dz \wedge d\bar{z})$, the action being $e^{i\theta}(m, w) = (e^{i\theta}m, e^{-i\theta}z)$. Then the manifold $M_{f>a}$ embeds as an open dense symplectic submanifold into the reduced space

$$\overline{W}_{f\geqslant a} := F^{-1}(a)/S^1 = \{(m, z) \in W \otimes C : f(m) - |z|^2 = a\}/S^1$$

and the difference $\overline{W}_{f\geqslant a} - W_{f\geqslant a}$ is symplecomorphic to the reduced space $f^{-1}(a)/S^1$. Topologically $\overline{W}_{f\geqslant a}$ is the quotient of the boundary manifold $W_{f\geqslant a}$ by the relation $\backsim$ where $m \backsim m'$ if and only if $m = e^{i\theta}m'$ for some $e^{i\theta} \in S^1$. A similar construction produces

$$\overline{W}_{f\leqslant a} = F^{-1}(a)/S^1 = \{(m, z) \in W \otimes C : f(m) + |z|^2 = a\}/S^1.$$

Next, since in section 4 we are going to make an extensive use of **Leray-Hirsch** theorem, we give its statement here and refer to [BT82] for details.

Theorem 2.1 (Leray-Hirsch theorem). *Let E be a fiber over M with fiber F. Suppose M has a finite good cover*[1]*. If there are global cohomology classes $e_1, e_2, \cdots, e_r$ which when restricted to each fiber freely generated the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{e_1, e_2, \cdots, e_r\}$, i.e.,*

$$H^*(E) \backsimeq H^* \otimes \mathbf{R}\{e_1, e_2, \cdots, e_r\} \backsimeq H^*(M) \otimes H^*(F).$$

We close this section with the following simple technical lemma which we need in section 4.

Lemma 2.2. *For any closed 2-form α on a compact orientable $2m$ dimensional manifold M, for any $0 \leqslant l \leqslant 2m$, define*

$$L_{[\alpha]} : H^l(M) \to H^{l+2}(M), [u] \to [\alpha] \wedge [u], \ \forall [u] \in H^l(M)$$

Suppose α_0, α_1 are two closed 2-form and suppose $L^k_{[\alpha_0]} : H^{m-k}(M) \to H^{m+k}(M)$ is an isomorphism. Then the set

$$\Lambda = \{t \in \mathbf{R} \mid L^k_{\alpha_1} + tL^k_{\alpha_0} : H^{m-k}(M) \to H^{m+k}(M) \text{ is an isomorphism}\}$$

is an open and dense subset of $\mathbf{R}$.

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[1] An open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of a n dimensional manifold M is called a good cover if all non-empty finite intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is diffemorphic to $\mathbf{R}^n$. It is well-known that every compact manifold has a finite good cover. See e.g., [BT82]

Proof. By Poincaré duality $\dim H^{m-k}(M) = \dim H^{m+k}(M) = r$ for some non-negative integer r. The linear map

$$L^k_{\alpha_1} + tL^k_{\alpha_0} : H^{m-k}(M) \to H^{m+k}(M) \tag{2.1}$$

is an isomorphism if and only if it's determinant is non-zero. Since $L^k_{[\alpha_0]} : H^{m-k}(M) \to H^{m+k}(M)$ is an isomorphism, the determinant of the map 2.1 is a degree r real polynomial function of t. Thus the determinant of the map 2.1 is non-zero except for at most r distinct real numbers. □

3. Symplectic four manifolds with certain properties

In this section, we establish the existence of symplectic four manifolds with certain properties which we need in the next section for our construction of counter examples. This is proved in Lemma 3.2, which has appeared in different guises in [Yan96] and [Gm] and depends on an idea of Johnson and Rees [JR87].

Definition 3.1. *Let* G *be a discrete group. A (non-degenerate) skew structure on* G *is a (non-degenerate) skew bilinear form*

$$<,>: H^1(G, \mathbf{R}) \times H^1(G, \mathbf{R}) \to \mathbf{R}$$

which factors through the cup product, this is, for some linear functional $\sigma : H^2(G, \mathbf{R}) \to \mathbf{R}$, $\langle a, b\rangle = \sigma(a \cup b)$, *where* $a, b \in H^1(G, \mathbf{R})$.

A finitely presentable group G is called a Kähler group if it is the fundamental group of a closed kähler manifold; otherwise it will be called a non-Kähler group. It was proved in [JR87] any Kähler group admits a non-degenerate skew structure.

Lemma 3.2. *Let* G *be a finitely presentable group which admits a non-degenerate skew structure. Then there is a closed, symplectic 4-manifold* (N, ω) *with* $\pi_1(N) = G$ *such that the following two conditions are satisfied:*

1 *the Lefschetz map* $L_{[\omega]} : H^1(N) \to H^3(N)$ *is identically zero.*
2 *there exists a class* $c \in H^2(N)$ *such that the map* $L_c : H^1(N) \to H^3(N)$ *is an isomorphism.*

Proof. According to Gompf [Gm], there exists a closed symplectic 4-manifold (N, ω) with $\pi_1(N) = G$ such that the assertion (1) holds. To prove the assertion (2), note that there is a natural map $f : N \to K(G, 1)$ such that the induced homomorphism

$$f^* : H^*(G, \mathbf{R}) \to H^*(N, \mathbf{R})$$

is an isomorphism in dimension 1 and injective in dimension 2. Let $\langle , \rangle$ be a non-degenerate skew structure on G and σ be the corresponding functional on $H^2(G, \mathbf{R})$. Since $H^2(G, \mathbf{R})$ is a subspace of $H^2(N, \mathbf{R})$, σ extends to a functional $\tilde{\sigma}$ on $H^2(M, \mathbf{R})$. By Poincaré duality, there exists a class c such that

$$\tilde{\sigma}(a) = (a \wedge c, [N])$$

where $a \in H^2(N, \mathbf{R})$ and $[N]$ is the fundamental class of N. Suppose $x \in H^1(N, \mathbf{R})$ such that $L_c(x) = x \wedge c = 0 \in H^3(N, \mathbf{R})$. Then for any $y \in H^1(N, \mathbf{R})$ we have $\tilde{\sigma}(y \wedge x) = ((y \wedge x) \wedge c, [N]) = (y \wedge (x \wedge c), [N]) = 0$. Note $\tilde{\sigma}(y \wedge x) = \sigma(y \wedge x) = \langle y, x \rangle$ we conclude that $\langle y, x \rangle = 0$ for any $y \in H^1(M, R)$. It then follows from the non-degeneracy of $\langle \ , \ \rangle$ that $x = 0$. This shows that L_c is injective. Then by Poincaré duality L_c must be an isomorphism indeed. □

4. Examples that the strong Lefschetz property is not preserved by symplectic reduction

We start with the following proposition which enables us to construct six dimensional compact Hamiltonian strong Lefschetz manifolds from the symplectic four manifolds with properties given in Lemma 3.2.

Proposition 4.1. *Suppose* (N, ω_0) *is a* 4*-dimensional compact symplectic manifold such that:*

(i) *the Lefschetz map* $L_{[\omega_0]} : H^1(N) \to H^3(N)$ *is identically zero.*
(ii) *there exists an integral cohomology class* $[c] \in H^2(N)$ *such that the map* $L_{[c]} : H^1(N) \to H^3(N)$ *is an isomorphism.*

Then there exists a non-trivial S^2 *bundle* $\pi : M \to N$ *which satisfies the following conditions:*

(i) *there exists a closed two form* η *on* M *such that the restriction of the cohomology class* $[\eta]$ *to each fiber* S^2 *generates the second cohomology group* $H^2(S^2)$.
(ii) *there exists an open interval* (A, B) *such that* $\omega := \pi^*(\omega_0 - t_0\epsilon c) + \epsilon\eta$ *is a symplectic structure on* M *for any* $t_0 \in (A, B)$ *and sufficiently small constant* $\epsilon > 0$*; furthermore, there is a* S^1 *action on* M *such that* (M, ω) *is a compact Hamiltonian* S^1*-manifold which has a non-Lefschetz symplectic quotient.*
(iii) (M, ω) *satisfies the strong Lefshetz property for some suitably chosen constants* $t_0 \in (A, B)$ *and* $\epsilon > 0$.

Proof. Let L be a complex line bundle over N with Chern class $[c]$. Denote by L^0 the complex line bundle L minus its zero section and by p the bundle map $L \to N$. Fix a Hermitian metric on the complex line bundle L and let the function $\mu : L \to \mathbf{R}$ be the norm squared with respect to this Hermitian metric. The Hermitian metric gives rise to a fiberwise circle action on L^0 and we denote by ξ the fundamental vector field on L^0 generated by this circle action. Let Θ be a connection 1-form with curvature p^*c. This means that Θ is defined on L^0 such that the restriction of Θ to a fiber of L^0 is $d\theta$ in polar coordinates on the fiber, and such that $d\Theta = p^*c$. Finally choose three positive constants $0 < A < B < T < \infty$ and have them fixed once and for all.

Set $\eta = \mu p^* c + d\mu \wedge \Theta$. Choose a constant $A < t_0 < B$ and consider the minimal coupling form (For details see for instance [AW77], [S77] and [GS84].)

$$(4.1) \qquad \gamma := p^*\omega_0 + (\mu - t_0)p^*c + d\mu \wedge \Theta = p^*(\omega_0 - t_0 c) + \eta$$

on L^0. This is a S^1-invariant closed 2-form such that $i_\xi \gamma = -d\mu$. However, γ is only symplectic on a tubular neighborhood of $\{z \in L^0 \mid \mu(z) = t_0\}$. To remedy this, we rescale the above minimal coupling form 4.1 to a symplectic form

$$(4.2) \qquad \gamma_\epsilon := p^*\omega_0 + \epsilon(\mu - t_0)p^*c + \epsilon d\mu \wedge \Theta = p^*(\omega_0 - t_0\epsilon c) + \epsilon\eta$$

on $0 < \mu < T$ by a small constant $\epsilon > 0$. In order not to cluster the the main ideas of our construction, here we will take for granted that for sufficiently small constant $\epsilon > 0$ the form 4.2 is symplectic on $0 < \mu < T$. However we will give a proof of this simple fact in Lemma 4.2 below.

Note that $i_\xi \gamma_\epsilon = -d\epsilon\mu$, i.e., the circle action on L^0 is Hamiltonian with respect to the form 4.2. If we perform symplectic cut twice for the Hamiltonian symplectic manifold $(\mu^{-1}(0, T), \gamma_\epsilon)$ at $\mu = A$ and $\mu = B$, we get a six dimensional compact symplectic manifold M which is fibred over N with typical fiber S^2. Observe that the closed two form η on $W := \{z \in L^0 \mid A < \mu(z) < B\}$ extends to a closed two form on M, which will also be denoted by η for convenience. When restricted to each fiber the cohomology class of this closed two form η on M generates the second cohomology of S^2. Denote by π the bundle map $M \to N$, then

$$(4.3) \qquad \omega := \pi^*(\omega_0 - t_0\epsilon c) + \epsilon\eta$$

is the symplectic form on M given by the above symplectic cuttings and its restriction to W coincides with the form 4.2. It is fairly easy to see that S^1 acts in a Hamiltonian fashion on (M, ω) with moment map $\epsilon\mu : M \to [\epsilon A, \epsilon B]$. Moreover, the preimage in L and in M of the open interval (A, B) are equivariantly symplectomorphic. If we perform symplectic reduction at $\mu = t_0$ the symplectic reduced space is N with the reduced form ω_0. Clearly, (N, ω_0) does not satisfy the strong Lefschetz property since the Lefschetz map $L_{[\omega_0]} : H^1(N) \to H^3(N)$ is zero. This proves that (M, ω) satisfies the conditions (1) and (2).

Note that the construction of the symplectic form ω involves the choices of the constants $A < t_0 < B$ and $\epsilon > 0$. To complete the proof it remains to show that (M, ω) has the strong Lefschetz property for some carefully chosen constants $\epsilon > 0$ and $A < t_0 < B$. The upshot is that these constants can be successfully chosen such that the resulting symplectic manifold (M, ω) is a strong Lefschetz manifold. We leave the proof of this fact in Lemma 4.3 below. □

Lemma 4.2. *For sufficiently small positive number $\epsilon > 0$, the two form*

$$\gamma_\epsilon = p^*(\omega_0 - t_0\epsilon c) + \epsilon\eta$$

is non-degenerate on $0 < \mu < T$.

Proof. Set $P = \{z \in L^0 \mid \mu(z) = t_0\}$. Then it is easy to see that $S^1 \to P \to N$ is a principal S^1-bundle. Moreover there is a diffemorphism $\psi : L^0 \to P \times (0, \infty)$ such that $Pr_2 \circ \psi = \mu$, where $Pr_2 : P \times (0, \infty) \to (0, \infty)$ is the projection of the product space $P \times (0, \infty)$ to its second factor. We will identify L^0 with $P \times (0, \infty)$ using this diffemorphism. Since P is compact, we can assume that the minimal coupling form $p^*\omega_0 + (\mu - t_0)p^*c + d\mu \wedge \Theta$ is non-degenerate on a tubular neighborhood $\{z \in L^0 \mid t_0 - \delta < \mu(z) < t_0 + \delta\}$ of P for some tiny number $\delta > 0$. Then for sufficiently small constant $\epsilon > 0$, the map

$$h : P \times (0, \infty) \to P \times (0, \infty),\ (x, t) \to (x, \epsilon(t - t_0) + t_0),\ \forall (x, t) \in P \times (0, \infty)$$

maps $\mu^{-1}(0, T)$ into $\mu^{-1}(t_0 - \delta, t_0 + \delta)$ and is a diffemorphism from $\mu^{-1}(0, T)$ onto an open subset of $\mu^{-1}(t_0 - \delta, t_0 + \delta)$. In particular the pull back

$$h^*(p^*(\omega_0 - t_0 c) + \eta) = p^*(\omega_0 - t_0 \epsilon c) + \epsilon \eta$$

is non-degenerate at each point of $0 < \mu < T$. This completes the proof of the lemma. □

Lemma 4.3. *In the proof of Proposition 4.1,* t_0 *and* ϵ *can be successfully chosen such that the resulting symplectic manifold* (M, ω) *has the strong Lefschetz property.*

Proof. Consider the closed 2-form η on M. Its restriction to $\mu^{-1}(A, B)$ is $\mu p^*c + d\mu \wedge \Theta$. It follows easily the cohomology class of η to each fiber S^2 generates the second cohomology group $H^2(S^2)$ of S^2. Write $H(S^2) = \mathbf{R}[x]/(x^2)$, where $\mathbf{R}[x]$ is the real polynomial ring and (x^2) is the ideal of $\mathbf{R}[x]$ generated by the quadratic polynomial x^2. By **Leray-Hirsch** theorem there is an additive isomorphism

$$H(N) \otimes \mathbf{R}[x]/(x^2) \to H(M),\ [\alpha] \otimes x^i \to [\pi^*\alpha \wedge \eta^i],\ i = 0, 1.$$

As a result we have $[\eta^2] = [\pi^*\beta_2 \wedge \eta] + [\pi^*\beta_4]$, where β_2 is a closed two form and β_4 is a closed four form. By Lemma 2.2 we can choose a real number $A < t_0 < B$ such that the map

$$L_{[-2t_0 c + \beta_2]} : H^1(N) \to H^3(N) \text{ is an isomorphism.} \tag{4.4}$$

Then choose a $\epsilon > 0$ which is sufficiently small such that the rescaled minimal coupling form $p^*\omega_0 + \epsilon(\mu - t_0)p^*c + \epsilon d\mu \wedge \Theta$ is symplectic on $\mu^{-1}(0, T)$ and such that

$$[\omega_0^2 - t_0 \epsilon c]^2 \neq -\epsilon^2[\beta_4] + \epsilon[(\omega_0 - t_0 \epsilon c) \wedge \beta_2], \tag{4.5}$$

We claim the resulting symplectic manifold (M, ω) has to satisfy the strong Lefschetz property. By Poincaré duality it suffices to show the two Lefschetz maps

$$L_{[\omega^2]} : H^1(M) \to H^5(M) \tag{4.6}$$

$$L_{[\omega]} : H^2(M) \to H^4(M) \tag{4.7}$$

are injective. We will give a proof in two steps below.

(i) It follows from **Leray-Hirsch** theorem that $H^1(N) \xrightarrow[\pi^*]{\cong} H^1(M)$. Thus to show map 4.6 is injective we need only to show for any $[\lambda] \in H^1(N)$ if $L_{[\omega^2]}(\pi^*[\lambda]) = 0$ then we have $[\lambda] = 0$. Since $\omega = \pi^*(\omega_0 - t_0\epsilon c) + \epsilon\eta$, $[\eta^2] = [\pi^*\beta_2 \wedge \eta] + [\pi^*\beta_4]$ and any forms on N with degree greater than 4 vanishes, we have

$$\begin{aligned} 0 &= L_{[\omega^2]}([\pi^*\lambda]) \\ &= \pi^*\left(2\epsilon[\omega_0 - t_0\epsilon c] + \epsilon^2[\beta_2]\right) \wedge [\pi^*\lambda] \wedge [\eta]. \end{aligned} \tag{4.8}$$

However by **Leray-Hirsch** theorem $H(M)$ is free over 1 and $[\eta]$, we get that

$$\pi^*(2\epsilon[\omega_0] - 2t_0\epsilon^2[c] + \epsilon^2[\beta_2]) \wedge [\lambda] = 0.$$

By 4.4 $L_{[2\epsilon\omega_0 - 2\epsilon^2 t_0 c + \epsilon^2\beta_2]} = \epsilon^2 L_{[-2t_0 c + \beta_2]} : H^1(N) \to H^3(N)$ is an isomorphism, we conclude that $[\lambda] = 0$.

(ii) By **Leray-Hirsch** theorem, to show that map 4.7 is injective we need only to show if $L_{[\omega]}(\pi^*[\varphi] + k[\eta]) = 0$ for arbitrarily chosen scalar k and second cohomology class $[\varphi] \in H^2(N)$, then we have $[\varphi] = 0$ and $k = 0$. Since $\omega = (\pi^*\omega_0 - t_0\epsilon c) + \epsilon\eta$ and $[\eta^2] = [\pi^*\beta_2 \wedge \eta] + [\pi^*\beta_4]$, we have

$$\begin{aligned} 0 &= L_{[\omega]}(\pi^*[\varphi] + k[\eta]) \\ &= (\pi^*[(\omega_0 - t_0\epsilon c) \wedge \varphi] + \epsilon k\pi^*[\beta_4]) + \\ &\quad (k\pi^*[(\omega_0 - t_0\epsilon c)] + \epsilon\pi^*[\varphi] + \epsilon k\pi^*[\beta_2]) \wedge \eta \end{aligned} \tag{4.9}$$

By **Leray-Hirsch** theorem $H(M)$ is a free module over 1 and $[\eta]$, we get that

$$\pi^*[(\omega_0 - t_0\epsilon c) \wedge \varphi] + \epsilon k\pi^*[\beta_4] = 0 \tag{4.10}$$

$$k\pi^*[(\omega_0 - t_0\epsilon c)] + \epsilon\pi^*[\varphi] + \epsilon k\pi^*[\beta_2] = 0 \tag{4.11}$$

If $k = 0$, it follows easily from the equation 4.11 that $[\varphi] = 0$. Assume $k \neq 0$. Substitute $\pi^*[\varphi] = -\frac{1}{\epsilon}k\pi^*[(\omega_0 - t_0\epsilon c)] - k\pi^*[\beta_2]$ into the equation 4.10 we get

$$\pi^*[(\omega_0 - t_0\epsilon c)] \wedge (-k\pi^*[(\omega_0 - t_0\epsilon c)] - \epsilon k\pi^*[\beta_2]) + \epsilon^2 k\pi^*[\beta_4] = 0$$

Since $k \neq 0$, we get

$$\pi^*([\omega_0 - t_0\epsilon c])^2 = -\epsilon^2\pi^*[\beta_4] + \epsilon\pi^*[(\omega_0 - t_0\epsilon c) \wedge \beta_2]$$

This contradicts the equation 4.5.

□

Example 4.4. Note that $G = Z \times Z$ is a Kähler group and thus admits a non-degenerate skew structure. Clearly, by Lemma 3.2 there is a closed, symplectic 4-manifold (N, ω_0) such that $\pi_1(N) = Z \times Z$ and has all the properties stated in Proposition 4.1. It follows from Proposition 4.1 that there exists a compact six dimensional Hamiltonian manifold which has the strong Lefshetz property itself but admits a non-Lefschetz symplectic quotient.

Observe that by our construction $M \to N$ is a fibration with fiber S^2. Therefore we have an exact sequence of homotopy groups

$$\cdots \to \pi_1(S^2) \to \pi_1(M) \to \pi_1(N) \to \pi_0(S^2) \to \cdots .$$

It follows immediately that $\pi_1(M) = \pi_1(N)$. Instead of choosing $G = Z \times Z$, we could choose any finitely presentable group G which admits a non-degenerate skew structure. Since the six dimensional Hamiltonian S^1-manifold (M, ω) constructed by the above procedure has a non-Lefschetz symplectic quotient, ω can not be an invariant Kähler form. But in general we do not know whether M supports any Kähler form or not. However, as we are going to show in Lemma 4.6, there exist finitely presentable non-Kähler groups G which admit a non-degenerate skew structure. For any such a group G, the corresponding Hamiltonian manifold M has a non-Kähler fundamental group G and therefore does not support any Kähler structure.

We need the following theorem which is due to Johnson and Rees.

Theorem 4.5 ([JR87]). *Let G_1, G_2 be groups which each has at least one nontrivial finite quotient. Then for any group H, $(G_1 * G_2) \times H$ is not a Kähler group.*

Lemma 4.6. *For any positive composite number m, n, the group $G_{m,n} = (Z_m * Z_n) \times (Z \times Z)$ is a non-Kähler group which admits a non-degenerate skew structure.*

Proof. Since m, n are composite numbers, both Z_m and Z_n have nontrivial finite quotient. It follows from 4.5 that the group $G_{m,n}$ is not a Kähler group for any positive composite number m, n. Note by corollary 6.2.10 and exercise 6.2.5 of [CAW94], $H^i(Z_m * Z_n, \mathbf{R}) = H^i(Z_n, \mathbf{R}) \oplus H^i(Z_m, \mathbf{R}) = 0$ for $i \geqslant 1$. Then it follows from the Künneth formula in group cohomology(see for instance exercise 6.1.10 of [CAW94]) that $H^i(G_{m,n}, \mathbf{R}) = H^i(Z \times Z, \mathbf{R})$ for $i \geqslant 1$. Since $(Z \times Z)$ is a Kähler group, $(Z \times Z)$ must have non-degenerate skew structure. It follows that $G_{m,n}$ also has such a structure. □

Thus we have thus proved the following theorem:

Theorem 4.7. *There exists an infinite family of six dimensional compact Hamiltonian symplectic S^1-manifolds, each of which satisfies the following conditions:*

(i) *the strong Lefschetz property,*
(ii) *admitting a non-Lefschetz symplectic quotient,*
(iii) *not homotopy equivalent to any compact Kähler manifold.*

5. A REMARK ON THE FUNDAMENTAL GROUPS OF HAMILTONIAN STRONG LEFSCHETZ MANIFOLDS

We conclude this paper with an interesting observation on the fundamental groups of strong Lefschetz manifolds. Using Hard Lefschetz theorem, Johnson and Rees proved in [JR87] if a finitely presentable group G is the fundamental group of a compact Kähler manifold, then G has to admit a non-degenerate skew structure. We note that the fundamental groups of strong Lefschetz manifolds also have to admit a non-degenerate skew structure, and Johnson and Rees's argument applies verbatim to our situation. On the other hand, if G is a finitely presentable group which supports a non-degenerate skew structure, then our construction in section 4 shows clearly that it has to be the fundamental group of a compact strong Lefschetz four manifold; moreover, it has to be the fundamental group of a compact six dimensional Hamiltonian strong Lefschetz S^1-manifold. In summary, we have

Theorem 5.1. *Suppose* G *is a finitely presentable group. Then the following statements are equivalent:*

(i) G *admits a non-degenerate skew structure.*
(ii) G *is the fundamental group of a compact four dimensional strong Lefschetz sympelctic manifold .*
(iii) G *is the fundamental group of a compact six dimensional Hamiltonian strong Lefschetz* S^1*-manifold.*

Gompf proved in [Gm] the remarkable result that any finitely presentable group can be realized as the fundamental group of a symplectic four manifold. In contrast, Theorem 5.1 imposes a rather stringent restriction on the fundamental groups of compact strong Lefschetz manifolds. For example, if a finitely presentable group G satisfies $b_1(G) \neq 0$ and $b_2(G) = 0$, then it can not admit any non-degenerate skew structure and thus can not be the fundamental group of any strong Lefschetz manifolds. In particular any non-trivial finitely presentable free group can not be the fundamental group of a compact strong Lefschetz manifold (c.f., page 592-593 of [Gm]). In addition, Theorem 5.1 also asserts that, different from the fundamental groups of compact Kähler manifolds to which far more rich restrictions apply (see e.g., [AB96]), the fundamental groups of six dimensional compact Hamiltonian strong Lefschetz S^1-manifolds has only one restriction as we stated in Theorem 5.1.

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