# DUNKL WAVE EQUATION & REPRESENTATION THEORY OF SL(2)

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The classical wave equation $u_{tt} = \Delta u$ in $n+1$ dimensions, and its various modifications, have been studied for centuries, and one of the most important features concerns the *domain of dependence* of the solution $u$ on initial data: $u|_{t=0} = f$, $u_t|_{t=0} = g$. To take the simplest example, the solution for $n = 1$ is given by the ancient formula of d'Alembert:

$$u(t,x) = \frac{1}{2}\big(f(x-t) + f(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy \tag{1}$$

where one sees explicitly that the value of the solution at $(t,x)$ depends only on the initial data within the ball of radius $|t|$ centered at $x$. This is expressed as the finite speed of propagation, and is a feature of a large class of equations called hyperbolic equations. Remarkably, for $n \geq 3$ and odd, the solution to the wave equation depends *only* on an arbitrarily thin neighborhood of the boundary sphere. In fact, that's what ensures our ability (as far as classical physics goes) to see and hear sharp signals without residual vibrations. This phenomenon is termed **Huygens' principle**, and turns out to be extremely rare among hyperbolic equations. It was believed for a long time that the only such second-order linear hyperbolic equations (i.e., wave operator plus lower-order terms with non-constant coefficients) are the classical wave equation in odd spatial dimensions (excluding $n = 1$). Since 1950s, non-trivial examples have been found, and one interesting class arises from root systems (or rather, finite reflection groups) in $\mathbb{R}^n$, and has been explained by means of Dunkl operators (see below).

The proper setting to formulate Huygens' principle, at least for linear equations, is the theory of distributions. The solution to the classical wave equation can be expressed as

$$u(t,\cdot) = (\partial_t P_t) * f + P_t * g$$

where $P_t \in \mathcal{S}'(\mathbb{R}^n)$ is the time $t$ slice of a certain distribution $P \in \mathcal{S}'(\mathbb{R}^{n+1})$. One may write $P = E - E_-$, where $E$ is supported in the future cone $C = \{(t,x) \in \mathbb{R}\times\mathbb{R}^n : |x| \leq t\}$, and $E_-(t,x) = E(-t,-x)$ is supported in the past cone $-C$. Furthermore, $E$ and $E_-$ are both fundamental solutions of the wave operator: $(\partial_t^2 - \Delta)E = \delta$, and they are the unique ones with the prescribed support[1]. (For $n = 1$, d'Alembert's formula says that $E$ is simply $\frac{1}{2}$ of the characteristic function $1_C$.) The Huygens' principle now becomes a question of the precise support structure of $E$ (and one should take it as the definition):

$$\text{Huygens' principle holds if and only if } \operatorname{supp} E \subseteq \partial C = \{(t,x) : |x| = t\}.$$

[1] One way to express the uniqueness is to say that $\mathcal{D}'(C)$ of distributions supported in a sharply peaked convex cone $C$ is an *associative* algebra under convolution, for which $\delta$ is the identity element. A fundamental solution is then an inverse to $(\partial_t^2 - \Delta)\delta$ in this convolution algebra.

There is an explicit form for $E$, from which one can see immediately that Huygens' principle holds if and only if $n \geq 3$ and $n$ is odd.

If we add non-constant coefficient lower-order terms to the wave operator, the fundamental solution should be thought of as (a family of) distributions $E(\cdot,\xi) \in \mathcal{S}'(\mathbb{R}^{n+1})$, supported in the future cone $\xi + C$ with apex at $\xi \in \mathbb{R}^{n+1}$, such that $\mathcal{L}\,E(\cdot,\xi) = \delta(\cdot - \xi)$. For the wave operator $\mathcal{L} = \partial_t^2 - \Delta$, it relates to the previous case via $E(\cdot,\xi) = E(\cdot - \xi)$. It has been shown that Huygens' principle never occurs in even spatial dimensions [Hadamard], and the only one for $n < 5$ is the wave equation in $n = 3$ [Mathisson, Asgeirsson].

In 1950s, Stellmacher discovered the first nontrivial example (for $n = 5$), and generalizations of it point to root systems [Berest-Veselov]. For a root system $\mathscr{R} \subset \mathbb{R}^n$ (and parameters $k_\alpha \geq 0$), the operator

$$\mathcal{L} = \partial_t^2 - \Delta + \sum_{\alpha \in \mathscr{R}^+} \frac{k_\alpha(k_\alpha+1)(\alpha,\alpha)}{(\alpha,x)^2} \tag{2}$$

satisfies Huygens' principle if and only if $n$ is odd, $k_\alpha \in \mathbb{N}$ and

$$3 + 2\sum_{\alpha \in \mathscr{R}^+} k_\alpha \leq n \tag{3}$$

These operators actually arise from Dunkl operators (defined below), and also feature in integrable systems.

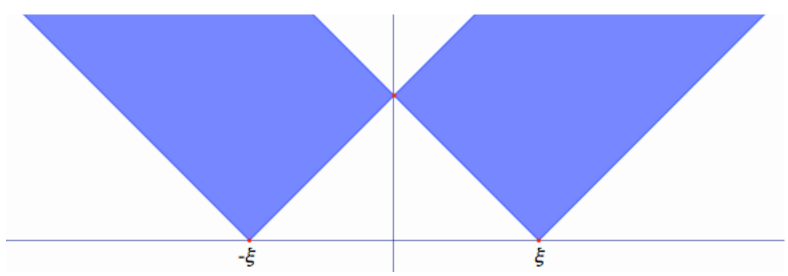
Note that Huygens' principle is still exceedingly rare, and for each $n$ there are only finitely many choices of $k_\alpha$. A remarkable recent result [Ben Saïd-Ørsted] has expanded the horizon, though we are to leave the realm of local operators (differential operators). For example, in one spatial dimension, the equation

$$u_{tt}(t,x) = u_{xx}(t,x) + \frac{2u_x(t,x)}{x} - \frac{u(t,x) - u(t,-x)}{x^2} \tag{4}$$

is solved by (take $f = 0$ for simplicity; compare with Eq. (1))

$$u(t,x) = \frac{1}{2x}\int_{x-t}^{x+t} \xi\, g(\xi)\, d\xi - \frac{1}{8x^2}\int_{x-t}^{x+t} \left(t^2 - (x-\xi)^2\right)\left(g(\xi) - g(-\xi)\right) d\xi \tag{5}$$

When expressed in terms of fundamental solution $E(\cdot,\xi)$, one can check that the support is actually the shaded region




Physically, if you stand at $x$ and wait for the signal emitted from $\xi$, you will hear an extended sound, but eventually it quiets down, as if the sound were canceled out by the "reflected" sound from $-\xi$. A similar phenomenon also happens with some higher-order hyperbolic equations with constant coefficients, and was studied as the problem of *lacuna* [Petrovsky, Atiyah-Bott-Gårding].

Where does such an equation as (4) come from? The answer is again Dunkl operators (with $\mathscr{R} = A_1$ and $k = 1$ in the notations below).

## 1. Dunkl operator

Let $\mathscr{R}$ be a root system in $\mathbb{R}^n$, where the (standard) inner product is denoted by $(\cdot, \cdot)$. Let $W = \langle s_\alpha \rangle_{\alpha \in \mathscr{R}} \subset O(n)$ be the associated finite reflection group, and choose (and fix) a $W$-invariant function $\alpha \mapsto k_\alpha \geq 0$ on $\mathscr{R}$. Define the **Dunkl operators** by

$$\partial_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathscr{R}^+} k_\alpha (\alpha, \xi) \frac{f(x) - f(s_\alpha x)}{(\alpha, x)}. \tag{6}$$

That is, the Dunkl operator $\partial_\xi$ differs from the usual directional derivative $\partial_\xi$ by a non-local part involving the difference of the function $f$ across each reflecting plane $(\alpha, x) = 0$. The fact that makes Dunkl operators so useful is that they commute: $\partial_\xi \partial_\eta = \partial_\eta \partial_\xi$ for any $\xi, \eta \in \mathbb{R}^n$. Note also that if $f$ is a polynomial, then $\partial_\xi f$ is still a polynomial, of one less the degree.

Thanks to commutativity, any polynomial $P(\xi) \in \mathbb{C}[\xi_1, \ldots, \xi_n]$ gives rise to a non-local linear operator $P(\partial)$ as the Dunkl version of $P(\partial)$. Of particular importance is the **Dunkl Laplacian**

$$\Delta f(x) = \sum_{i=1}^n \partial_{\xi_i}^2 f(x) = \Delta f(x) + \sum_{\alpha \in \mathscr{R}^+} k_\alpha \left[ \frac{2\,\partial_\alpha f(x)}{(\alpha, x)} - (\alpha, \alpha) \frac{f(x) - f(s_\alpha x)}{(\alpha, x)^2} \right].$$

The new class of equations in [Ben Saïd-Ørsted] is simply the Dunkl wave equation:

$$\begin{cases} u_{tt} = \Delta u \\ u(0, x) = f(x) \\ u_t(0, x) = g(x) \end{cases} \tag{7}$$

and we will solve it by developing a series of Dunkl analogues of basic notions in ordinary analysis, to make sure everything works out the same way [Dunkl, Opdam, de Jeu, Trimèche].

The **Dunkl transform** is the proper analogue of Fourier transform. In lieu of the usual exponential kernel $e^{(\cdot,\cdot)}$, we have a unique holomorphic function $e^{(\cdot,\cdot)}$ on $\mathbb{C}^n \times \mathbb{C}^n$ satisfying

$$\partial_\xi e^{(\cdot, w)} = (\xi, w) e^{(\cdot, w)} \qquad \text{and} \qquad e^{(0, w)} = 1$$

Now, the Dunkl transform of $f \in L^1 = L^1(\mathbb{R}^n, v(x)dx)$ is defined by

$$\mathscr{F} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{(x, -i\xi)} v(x) dx$$

where the measure is given by

$$v(x) = \prod_{\alpha \in \mathscr{R}_+} |(\alpha, x)|^{2k_\alpha}.$$

The Dunkl transform enjoys many of the properties of Fourier transform, such as a Plancherel formula in $L^2 = L^2(\mathbb{R}^n, v(x)dx)$, and that its inverse is given by

$$\mathscr{F}^{-1} f(x) = \frac{1}{c^2} \int_{\mathbb{R}^n} f(\xi) e^{(x, i\xi)} v(\xi) d\xi$$

where

$$c = \int_{\mathbb{R}^n} e^{-|x|^2/2} v(x) dx.$$

For purposes of solving Dunkl differential equations, we need the essential properties:

$$\mathscr{F} \partial_j = i\xi_j \mathscr{F} \qquad \partial_j \mathscr{F} = \mathscr{F}(ix_j)$$

Furthermore, $\mathscr{F}$ (and its properties) extends to tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ in the usual way: $\langle \mathscr{F} u, \phi \rangle = \langle u, \mathscr{F} \phi \rangle$ for $u \in \mathcal{S}'$ and $\phi \in \mathcal{S}$.

At the heart of the construction of $e^{(\cdot,\cdot)}$ is a certain linear isomorphism $V$ that intertwines the algebra generated by the Dunkl operators with the algebra of ordinary differential operators:

$$\partial_\xi V f = V \partial_\xi f \qquad f \in C^\infty(\mathbb{R}^n)$$

Now the **Dunkl translation** defined by

$$\tau_y f(x) = V_x V_y (V^{-1} f)(x - y)$$

makes Dunkl operators "translation-invariant": $\partial_\xi \tau_y f(x) = \tau_y \partial_\xi f(x)$. We will denote it suggestively as $\tau_y f(x) = f(x - y)$, though it's not to be taken that $x - y$ means anything.

The **Dunkl convolution** $*$ is defined by means of this translation operator:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y) v(y) dy$$

for functions $f$ and $g$ that make the integral converge. For distributions $\mathcal{S}$ and $\mathcal{T}$, their Dunkl convolution, if exists, is defined via

$$\langle \mathcal{S} * \mathcal{T}, \phi \rangle = \langle \mathcal{S} \otimes \mathcal{T}, \phi(x + y) \rangle$$

which agrees with the previous definition for regular distributions. (For purposes of solving linear PDEs, we are mostly interested in the convolution of a distribution with a smooth function.) The important property that further justifies all these is

$$\mathscr{F}(f * g) = (\mathscr{F} f)(\mathscr{F} g)$$

from which it follows that $f * g = g * f$.

Finally, we can solve the Dunkl wave equation (7) by

$$u(t, \cdot) = (\partial_t P_t) * f + P_t * g \tag{8}$$

where

$$P_t(x) = P(t,x) = \mathscr{F}^{-1}\frac{\sin(t|\xi|)}{|\xi|}. \tag{9}$$

An application of the Paley-Wiener theorem for Dunkl transform shows that supp $P \subseteq C \cup -C$, which makes the convolution in (8) sensible.

## 2. Huygens' Principle and Representation Theory of $\mathfrak{sl}_2$

Huygens' principle in general appears to be a rather rigid phenomenon and only happens for special parameter values, so it does not come as a surprise that representation theory has some bearing on it. In fact, we have Huygens' principle for Dunkl wave equation (7) in the sense that supp $E \subseteq \partial C$ if and only if (compare with (3))

$$\frac{n-3}{2} + \gamma \in \mathbb{N} = 0, 1, 2, \dots \tag{10}$$

where $\gamma = \sum_{\alpha \in \mathscr{R}^+} k_\alpha$.

One way to ensure that a distribution $u$ is supported in some set $X \subset \mathbb{R}^n$ is to have $fu = 0$, where $f$ is a smooth function that vanishes identically on $X$. For our case, $P$ is supported on $X = \partial C \cup -\partial C$ if and only if

$$(t^2 - |x|^2)^m P = 0 \qquad \text{for some} \quad m \in \mathbb{N}^+. \tag{11}$$

The necessity is due to the fact that every distribution is of some finite order. (Think about $x\delta' \neq 0$, but $x^2\delta' = 0$ in $\mathcal{S}'(\mathbb{R})$.)

It turns out that we actually have a representation of $\mathfrak{sl}_2$ on $\mathcal{S}'(\mathbb{R}^{n+1})$ that is just right for this kind of consideration. Let $e, f, h$ be the standard basis of $\mathfrak{sl}_2$, i.e.,

$$[e,f] = h \qquad [h,e] = 2e \qquad [h,f] = -2f$$

and define first a representation on $\mathcal{S}(\mathbb{R}^{n+1})$, or for that matter $C^\infty(\mathbb{R}^{n+1})$ [Heckman]:

$$\begin{aligned}
\omega(e) &= \frac{1}{2}\left(t^2 - |x|^2\right) \\
\omega(f) &= -\frac{1}{2}\left(\partial_t^2 - \Delta\right) \\
\omega(h) &= \frac{n+1}{2} + \gamma + t\partial_t + \sum_{j=1}^n x_j \partial_j
\end{aligned}$$

The correct representation $\omega'$ on $\mathcal{S}'(\mathbb{R}^{n+1})$ is given by the same expression for $e$ and $f$, but

$$\omega'(h) = \frac{n+1}{2} - \gamma + t\partial_t + \sum_{j=1}^n x_j \partial_j$$

with a sign change[2]. Note that our $P \in \mathcal{S}'(\mathbb{R}^{n+1})$ satisfies $\omega'(f)P = (\partial_t^2 - \Delta)(E - E_-) = 0$, and (11) becomes $\omega'(e)^m P = 0$ for some $m$. This will be a finite-dimensional $\mathfrak{sl}_2$-module,

[2]What seems a contradiction is not so: the (natural) embedding $\mathcal{T} : C^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is given by $\langle \mathcal{T}_f, \phi\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)v(x)dx$, and $v(x)$ is homogeneous of degree $2\gamma$.

if we can make sure that $P$ is an eigenvector of $\omega'(h)$, which is nothing but the Euler operator (up to an additive constant). The best way to see that is via $\delta \in \mathcal{S}'(\mathbb{R}^{n+1})$, which is homogeneous of degree $-(n+1)$, or

$$\omega'(h)\delta = -\left(\frac{n+1}{2} + \gamma\right)\delta$$

so the fundamental solution $E$ (and in turn $P$) must also be an eigenvector of $\omega'(h)$, but with eigenvalue 2 more than that of $\delta$, i.e.,

$$\omega'(h)P = -\left(\frac{n-3}{2} + \gamma\right)P.$$

That means $P$ is indeed a lowest weight vector in $\mathcal{S}'(\mathbb{R}^{n+1})$, so the necessary condition for it to generate a finite-dimensional representation is precisely (10):

$$\frac{n-3}{2} + \gamma \in \mathbb{N}.$$

That's almost the whole story, except for a caveat that it is possible that $P$ generates an infinite dimensional representation, isomorphic to the Verma module, that has a finite-dimensional quotient. For that, one needs more representation theory; in fact it seems unavoidable to invoke the Lie group $Mp(2,\mathbb{R})$, the double cover of $SL(2,\mathbb{R})$.

The action of $W$ (instead of $O(n)$ in the classical case) commutes with $Mp(2,\mathbb{R})$ on $\mathcal{S}(\mathbb{R}^n)$ (the spatial part), and we have the decomposition as $W \times Mp(2,\mathbb{R})$ representations

$$\mathcal{S}(\mathbb{R}^n)_{W\times\mathfrak{t}} = \bigoplus_{j=0}^{\infty} \mathcal{H}_j^n \otimes V_{\frac{n}{2}+\gamma+j}$$

where $\mathcal{H}_j^n = \ker \Delta$ is the space of Dunkl harmonic polynomials of degree $j$ on $\mathbb{R}^n$, and $V_\lambda$ is the lowest weight $\mathfrak{sl}_2$-module of lowest weight $\lambda$.

The temporal part needs to have highest weight modules:

$$S(\mathbb{R}) = \overline{V}_{-\frac{1}{2}} \oplus \overline{V}_{-\frac{3}{2}}$$

in order for the tensor product to be our representation $\omega$ on $\mathcal{S}(\mathbb{R}^{n+1})$. Now, it's pure representation theory to show that, under the condition (10), the "formal vector completion" of the $W$-invariant piece

$$V_{\frac{n}{2}+\gamma} \otimes \overline{V}_{-\frac{3}{2}} \qquad \text{(with } j=0\text{)}$$

indeed contains finite-dimensional representations, and the one generated by $P$ is in here.