TEACHING STATEMENT: YAO LIU

This semester I am teaching Calculus II for the second time (and fifth time overall for any calculus) after more than a year of hiatus, but I found myself questioning more about the current standard of calculus teaching. I used to think that, as ancient a subject as calculus is, there hardly is anything we could improve in the overall design of the curriculum. After all, the students ought to come out knowing limits (if not the ϵ-δ definition), continuity, differentiability, being able to integrate a variety of functions and determine the convergence of infinite series (usually taught in this order), as a calculus course on their transcript is meant to certify. I'm starting to believe that we could do better, while fully aware of the existence of extensive studies on math education that I'm ignorant of.

It all started in the first week when I was reviewing the basic derivatives and rules, which I rewrote in terms of differentials as I thought they would come handy once we started on integration techniques. I felt that the students might be less familiar with this perspective, or they might have been told outright never to regard dy/dx as a quotient (or worse still to use Leibniz's notation dy for linear approximation, which kills all the intuition). So I decided to write a detailed yet brief note from this approach[1] for their convenience, and to my pleasant surprise I was able to get through all the rules of differentials under three pages, and even included an intuitive explanation of the definite integral. Granted this was not meant to be a first introduction, it nonetheless re-affirms my belief that the standard recourse through limits is merely — and should be presented as — one justification of the basic *calculus* of differentials and integrals, which I believe is the core of the subject that the students should acquire and be able to apply, reasoning with infinitesimals, in concrete and meaningful problems from geometry and physics, *without* first going through the logical sequence of the modern calculus curriculum.

As I recall, I used to begin the first lecture of Calculus I with a general overview that included my personal view of the two distinct yet interwoven themes throughout the course: the practical aspect that deals with concrete functions (mostly elementary functions) on the one hand, and the general concepts, definitions and theorems on the other. I knew it probably meant very little to the students without seeing examples, and like everyone else I was of the mind that we couldn't really separate the two, which in some sense is what I'm now proposing, with the help of infinitesimal reasoning.

The standard objection is that calculus with infinitesimals would be on shaky grounds; for instance, we have to be hand-wavy when we drop higher differentials. However, that would mostly be *our* perception, acquired after many years of mathematical training, while the students probably would feel just as comfortable (or uncomfortable) with infinitesimals as they do limits, and it may be even less susceptible to misuse, given the widely observed difficulty with limits among all but the best students. However much we feel impelled to point out the "faulty" reasoning

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[1]Available at `http://www.math.cornell.edu/~liuyao/1120/calculus.pdf`

with infinitesimals, it is not as effective pedagogically as first letting them play with it, and the astute students may come to that conclusion themselves, and thereby appreciate the need and the virtue of a more rigorous treatment. Meanwhile, the rest of the class is not being left behind, as they too would be able to come to a quick grasp, if not mastery, of the basic workings of calculus, without the distraction of generalities of functions and theorems. In fact this is not all that radical: the standard undergraduate real analysis course presupposes familiarity with real numbers (at the very least $\sqrt{2}$ and π), as well as sequences and series from standard calculus. No mortals can possibly learn real numbers by way of Dedekind cuts or Cauchy sequences, right after being taught the addition of fractions.

To me, the real advantage of this approach lies in the sheer beauty of infinitesimals, its simplicity and effectiveness in analyzing problems that range far beyond the standard calculus problems that we teach. I am particularly fond of the cardioid as the envelope of straight lines, which I proposed to use last year in an inquiry-based course intended for those who hated math, who never thought they'd take calculus — yet they all enjoyed the hands-on activity and discussion immensely (unfortunately we stopped short of mentioning calculus). It's a pity that such classical problems are deemed too hard for most calculus students of our time, precisely because we have given up the infinitesimal reasoning in favor of the rigorous modern approach. Incidentally, I have tried to incorporate other concrete and historically important problems in my lectures in general, including the cycloid, continued fractions, the gravitational force exerted by a spherical body, ruled surfaces, to name a few, doing my part in mitigating the adverse effects of those contrived and uninspiring problems that we give.

Another advantage is that it gives a perfect opportunity, at the point of shifting gears, to have a class discussion of the need for more care and rigor, various possible approaches (including non-standard analysis), as well as the general concept of functions — a deceptively simple concept which is taken for granted in modern calculus after Dirichlet, rightly so for its ubiquity in mathematics — and how it would greatly empower our toolkit. I believe these kinds of discussions are more valuable than to force upon them "the" correct way at the outset, as so often happens in our educational system. This, I hope, would give the students a more correct view of what mathematics is all about.

The only real challenge, which is part of a bigger problem, is that it is difficult to assess this kind of infinitesimal reasoning in a fair way, especially for a large lecture course. Ideally we'd like to give a problem that they have not encountered before, forcing them to think on their own, and the assessment would be more on their thought process than their ability to produce the correct answer. This is largely incongruous to their previous mathematical experience, and is bound to generate complaints. But, as the first math course in college, we have every reason to break free from the style of high school math and all its misfortunes, and one may even argue that in the information age that we live in, technical skills and factual knowledge could be acquired more easily than before as the need arises — it's not a secret that the students know to go to wikipedia or the countless online video tutorials — yet there has been little to no change to the way we teach calculus.

Although I have not had a chance to implement this approach (except for a few classical problems that I managed to include in my lectures or extra handouts), I did have some experience as the TA for an undergraduate course on Matrix Groups, which delves right into the concrete examples of the classical groups without any prerequisite in differentiable manifolds, or even topology. In fact, the last chapter of the text (*Naive Lie Theory* by Stillwell) introduces basic topology as necessitated by these examples. Although I'd rather introduce topology from some other motivation, this example-drive approach, with the concepts only coming after proper motivation, is certainly appropriate for Lie theory, and I believe it is, to varying extent, applicable to many courses at the undergraduate level.

I am also a firm believer in the unity of mathematics, and a lot could be done in making the connections more accessible than the traditional literature. To take a calculus example, it would be wonderful if the curious students could find out what exactly is involved in proving the extreme value theorem, without having to read half of a textbook on point-set topology[2]. More than any other discipline, mathematics holds the promise of making its entirety into a single collaborative repository for everyone in the world: theorems don't become obsolete, only to have simplified or alternative proofs, and true mathematicians would embrace them against their own idiosyncrasies. It would be a real privilege if I could play a small role in making that day come sooner.

[2]I have experimented a little with the idea at `http://www.math.cornell.edu/~liuyao/Test/math/FTC.html`