INTRODUCING CALCULUS: THE CARDIOID

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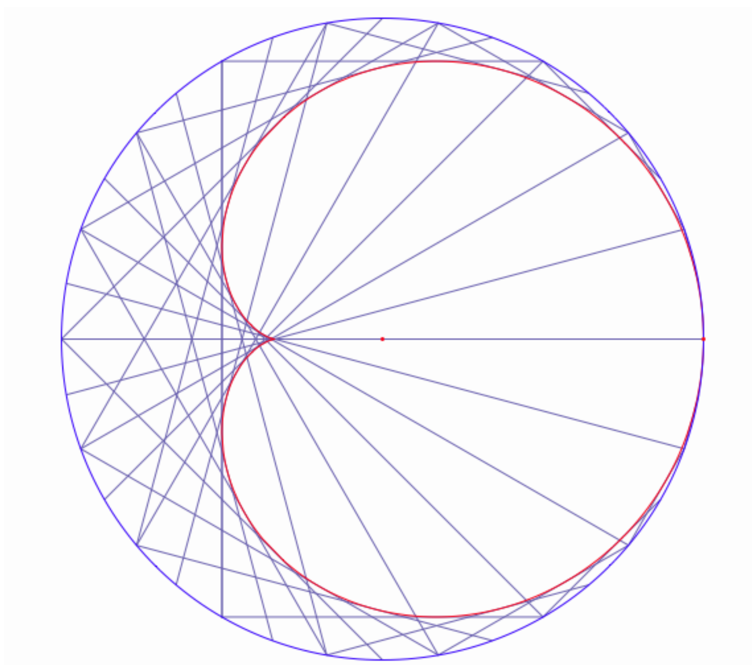
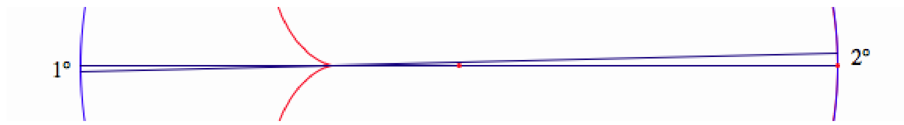
Let's start with a big circle and connect each point with twice the degree mark (as measured from, say, the red dot on the right) by a straight line. Of course we can't really do ALL the points, so we'll settle for doing it for every $10°$. Before long, you'll see an interesting shape coming out (note that $200°$ should be connected with $400° - 360° = 40°$, and so on), and the end result is something like this:



The more lines you draw, the "curve" that they make out, known as the *cardioid* (heart-shaped), will appear more smooth, except for that pinch on the left. The position of the pinch, more formally called the *cusp*, will be our focus.

Before that, let's take a closer look at what the lines are doing, and their relation with the curve. It appears that each line "just touches" or "kisses" the curve, much like the tangent to a circle touching the circle. In fact, people simply call it the *tangent* to the curve (at a particular point), and it describes the direction in which the curve is tracing at that point, if only momentarily. On the other hand, two nearby lines intersect at a point, and if we move the two lines closer together, the intersection will approach the point of tangency. This sort of “approaching” argument is at the heart of calculus.

If we want to figure out the location of the cusp, we should take a nearby line that connects, say, $181°$ to $362° = 2°$, and it looks like this (drawn to exact scale):



By the rules of similar triangles, we see that the intersection should be at one third the distance along the horizontal diameter. One may question that the two ends are not straight, but arcs. However, if we let the angle get smaller and smaller, the arcs are indistinguishable from straight vertical line segments. The beauty of calculus shines in situations in which we can work with such *infinitesimal* figures or quantities.

The same analysis can be done for other lines, and one finds that the point of tangency is always at the one third mark. Thus we know the exact location of all the points of the cardioid, such as the topmost and the leftmost points. In fact, we can easily write down a parametrization which will be useful for more detailed analysis (in the Appendix):

$$x = \frac{2}{3}\cos(\phi) + \frac{1}{3}\cos(2\phi)$$
$$y = \frac{2}{3}\sin(\phi) + \frac{1}{3}\sin(2\phi)$$

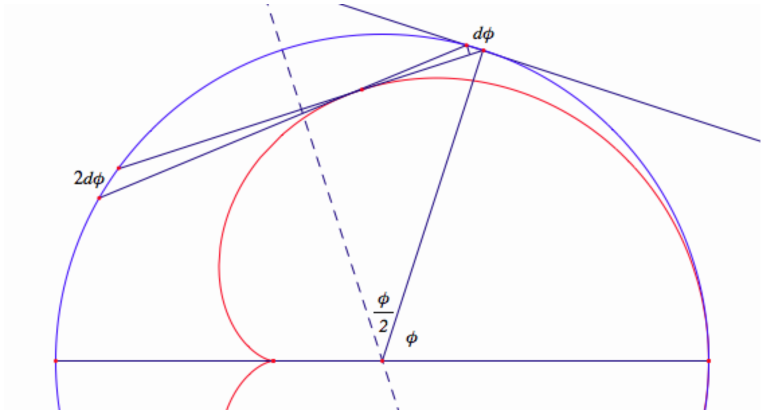
(The radius of the big circle is taken to be 1, and ϕ and 2ϕ are the angles — or the arclengths — along the circle.)

To take this infinitesimal reasoning further, we may ask where on the curve is it the most curvy? It appears that the least curvy part is at $0°$, but the most curvy part is somewhere around the two lobes and the cusp, but it’s hard to tell exactly where. How would one measure curviness quantitatively? By definition, the *curvature* is the rate at which the tangent is turning, with respect to arclength, as you are walking along the curve, or in the language of infinitesimals, the ratio of the infinitesimal angle $d\theta$ (between two nearby tangents) with the corresponding infinitesimal arclength ds:

$$\frac{d\theta}{ds} = \text{curvature}$$

For instance, for a circle of radius R, the tangent rotates by an infinitesimal $d\theta$ over an arclength of $R\,d\theta$, so the curvature is $1/R$ everywhere along the circle.

For our cardioid, by analyzing the infinitesimal triangle,



we deduce that the angle $d\theta$ is given by

$$d\theta = \frac{d\phi \sin\frac{\phi}{2}}{\frac{2}{3}\sin\frac{\phi}{2}} = \frac{3}{2}\,d\phi$$

which means that the line segment is rotating at a constant rate of $\frac{3}{2}$ relative to that of ϕ. If you don't fully trust this argument, think about the rotation of the dashed line in the picture, which is perpendicular to the line segment.

To figure out ds is trickier. The two adjacent line segments differ in the positions of their two ends, in the tangential direction, by $d\phi \cos\frac{\phi}{2}$ and $2d\phi\cos\frac{\phi}{2}$, so the one-third marks differ by

$$\frac{2}{3}d\phi\cos\frac{\phi}{2} + \frac{1}{3}2d\phi\cos\frac{\phi}{2} = \frac{4}{3}d\phi\cos\frac{\phi}{2}$$

The radial direction is negligible, being a "higher differential," so we get

$$ds = \frac{4}{3}\cos\frac{\phi}{2}\,d\phi$$

and the curvature is thus

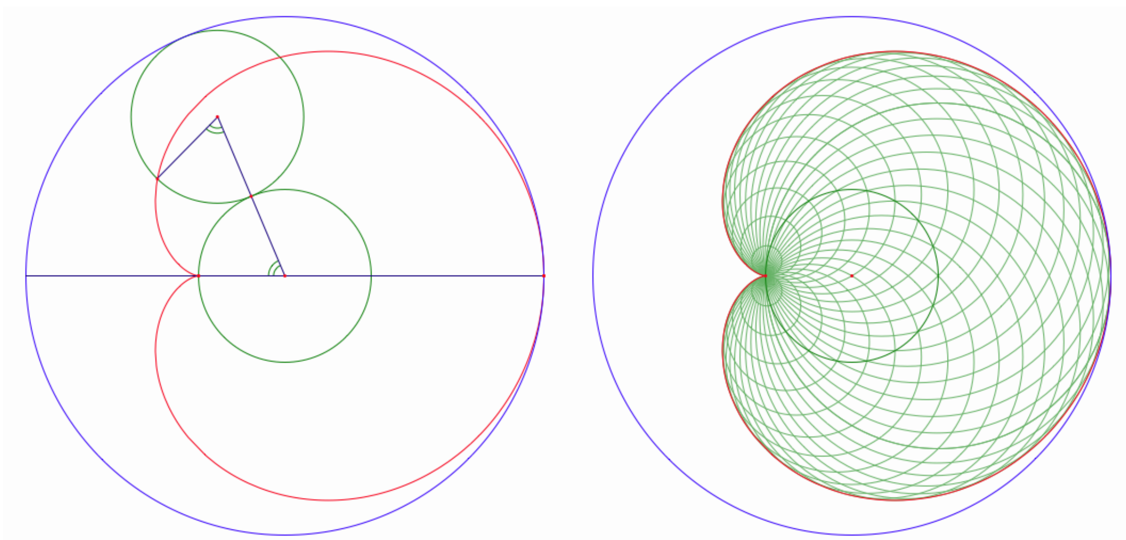
$$\frac{d\theta}{ds} = \frac{9/8}{\cos\frac{\phi}{2}}$$

At $\phi = 0$, the curvature is just $\frac{9}{8}$ (slightly more curvy than the unit circle), while at $\phi = \pi$, the curvature is infinity! The cusp is indeed the most curvy part of the cardioid, and is as curvy as it possibly can.

The shape of the cardioid probably looks familiar, as you may have seen it inside a coffee mug under a light source. That's no accident. If we placed a light source inside the mug against the wall, the light rays after the first bounce are exactly prescribed by the double angle rule, and where the rays converge at would appear brighter than elsewhere. We'd say that the cardioid is a *caustic* of the circle, and the cusp is where a lot of light rays converge, resulting in a very bright spot (the focus). The problem, though, is that it's never the case that the light source is inside the mug, but is somewhere far out that we may assume that the light comes in as parallel rays. Simple geometry reveals that the first bounce obeys a triple angle rule instead, and the resulting curve is called the nephroid (kidney-shaped), but since we only see half of it, it looks very similar to the cardioid. In fact, all the analysis done for the cardioid can be easily changed for the nephroid, in particular the focus is a fourth of the way (or half the radius) in from the wall.

You may also have seen the cardioid featuring prominently in possibly the most recognizable mathematical image in popular science, the Mandelbrot set. That's no accident either, and you may read (and see) more about it on wikipedia.

There are other ways to generate the cardioid. If we fix a quarter coin on the table, and roll another quarter around it without slipping, then a point on the edge of the rolling quarter traces out exactly the cardioid. (The question of how many turns the quarter makes never fails to puzzle people.)



Last but not least, the collection of all the circles that are centered on a given circle and passing through a fixed point on that given circle, make out the shape of the cardioid too.

Appendix

To give a little more taste and flavor of the calculus of differentials, and to substantiate the shaky calculation of the curvature, we shall give the standard calculation that would apply to other curves (but is less fun). Start with the parametrization

$$x = \frac{2}{3}\cos(\phi) + \frac{1}{3}\cos(2\phi)$$
$$y = \frac{2}{3}\sin(\phi) + \frac{1}{3}\sin(2\phi)$$

and with a few basic rules such as $d\sin\theta = \cos\theta\, d\theta$, $d\cos\theta = -\sin\theta\, d\theta$, we have

$$\begin{aligned} dx &= -\frac{2}{3}\sin(\phi)\, d\phi - \frac{2}{3}\sin(2\phi)\, d\phi \\ dy &= \frac{2}{3}\cos(\phi)\, d\phi + \frac{2}{3}\cos(2\phi)\, d\phi \end{aligned}$$

so that

$$\begin{aligned} ds = \sqrt{dx^2 + dy^2} &= \frac{2}{3}\sqrt{2 + 2\cos(\phi - 2\phi)}\, d\phi \\ &= \frac{4}{3}\sqrt{\frac{1+\cos\phi}{2}}\, d\phi \\ &= \frac{4}{3}\left|\cos\frac{\phi}{2}\right|\, d\phi \end{aligned}$$

As for $d\theta$,

$$\tan\theta = \frac{dy}{dx} = -\frac{\cos(\phi) + \cos(2\phi)}{\sin(\phi) + \sin(2\phi)} = -\frac{u}{v}$$

(Note that at $\phi = 0$, both dx and dy vanish and the curve momentarily stops at the cusp and does not really have a proper tangent there.) Taking differential of both sides,

$$\begin{aligned} \sec^2\theta\, d\theta &= -\frac{v\,du - u\,dv}{v^2} \\ (1 + \tan^2\theta)\, d\theta &= \frac{1}{v^2}\begin{pmatrix} (\cos(\phi) + \cos(2\phi))(\cos(\phi) + 2\cos(2\phi)) \\ \qquad - (\sin(\phi) + \sin(2\phi))(-\sin(\phi) - 2\sin(2\phi)) \end{pmatrix} d\phi \\ \frac{v^2 + u^2}{v^2}\, d\theta &= \frac{1 + 2 + 3\cos(\phi - 2\phi)}{v^2}\, d\phi \\ d\theta &= \frac{3 + 3\cos(\phi)}{2 + 2\cos(\phi)}\, d\phi = \frac{3}{2}\, d\phi \end{aligned}$$

Therefore, the curvature is

$$\frac{d\theta}{ds} = \frac{9/8}{|\cos\frac{\phi}{2}|}$$

You may calculate literally everything you'd like to know about the cardioid. For instance, the total length of the curve is given by integrating the infinitesimal ds:

$$\int ds = 2\int_0^{\pi} \frac{4}{3}\cos\frac{\phi}{2}\, d\phi = 2\left[\frac{8}{3}\sin\frac{\phi}{2}\right]_0^{\pi} = \frac{16}{3}$$

while the area is given by

$$\begin{aligned}
-\int y\,dx &= 2\int_0^\pi \left(\frac{2}{3}\sin(\phi)+\frac{1}{3}\sin(2\phi)\right)\left(\frac{2}{3}\sin(\phi)\,d\phi+\frac{2}{3}\sin(2\phi)\,d\phi\right)\\
&= \frac{4}{9}\int_0^\pi \left(2\sin^2\phi+\sin^2 2\phi+3\sin(\phi)\sin(2\phi)\right)\,d\phi\\
&= \frac{4}{9}\left(\pi+\frac{\pi}{2}+6\int_0^\pi \sin^2\phi\cos\phi\,d\phi\right)\\
&= \frac{4}{9}\left(\frac{3\pi}{2}+6\left[\frac{\sin^3\phi}{3}\right]_0^\pi\right)=\frac{6\pi}{9}
\end{aligned}$$