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COMPARISON THEOREMS FOR REVERSIBLE MARKOV CHAINS

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We introduce geometric comparison inequalities that give bounds on the eigenvalues of a reversible Markov chain in terms of the eigenvalues of a second chain. The bounds are applied to get sharp results for the exclusion process.

1. Introduction. Let x be a finite set. Let $P(x, y)$ be an irreducible Markov kernel on X with stationary probability $\pi(x)$. Assume throughout that P, π is reversible:

$$\pi(x)P(x,y) = \pi(y)P(y,x).$$

By symmetry, P has eigenvalues $1 = \beta_0 > \beta_1 \geq \cdots \geq \beta_{|X|-1} \geq -1$. This paper develops methods for getting upper and lower bounds on β_i by comparison with a second reversible chain on the same state space. This extends the ideas introduced in Diaconis and Saloff-Coste (1993), where random walks on finite groups were considered. The bounds involve geometric properties such as the diameter and covering number of an associated graph along the lines of Diaconis and Stroock (1991).

The main application gives a sharp upper bound on the second eigenvalue of the symmetric exclusion process. Thus, let $\mathscr{G}_0$ be a connected undirected graph with n vertices. For simplicity, we assume in this introduction that $\mathscr{G}_0$ is regular. To start, r unlabelled particles are placed in an initial configuration, $1 \leq r \leq n$. At each step, a particle is chosen at random; then one of the neighboring sites of this particle is chosen at random. If the neighboring site is unoccupied, the chosen particle is moved there; if the neighboring site is occupied, the system stays as it was. This is a reversible Markov chain on the r-sets of $\{1, 2, \ldots, n\}$ with uniform stationary distribution. Liggett (1985) gives background and motivation (he focuses on infinite systems). Fill (1991) gives bounds on the second eigenvalue of the labeled exclusion process on the finite circle $\mathbb{Z}_n$.

We study this chain by comparison with a second Markov chain on r-sets that proceeds by picking a particle at random, picking an unoccupied site at random (not necessarily a neighboring site) and moving the particle to the unoccupied site. This is a well studied chain (the Bernoulli–Laplace model for diffusion). Its eigenvalues are known. We show that the comparison techniques apply to give upper bounds on the eigenvalues of the exclusion

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process. For example, suppose that $\mathscr{G}_0$ is an n-point segment with a loop at each end. Our results give $\beta_1 \le 1 - 4/rn^2$. We also prove a sharp lower bound that, for this example, yields $1 - \pi^2/2rn^2 \le 1 - (1 - \cos(\pi/n))/r \le \beta_1$. These bounds improve upon results of Fill (1991). These techniques also yield interesting bounds for more complicated graphs.

Simple exclusion is a well studied process related to a variety of other processes and to certain mechanical systems. Kipnis, Olla and Varadhan (1989) and Quastel (1992) are recent works on the limiting behavior of exclusion processes after an appropriate scaling, and they contain other references. Fill (1991) gives other motivations. Thomas (1980) connects the Hamiltonian of the quantum Heisenberg ferromagnets model in a finite box $\subset \mathbb{Z}^d$ with simple exclusion processes. More precisely, he shows that the restrictions of this Hamiltonian to certain subspaces of its natural Hilbert space are unitarily equivalent to the generators of simple exclusion processes. Our bounds can be interpreted in this context.

After a first draft of this paper was completed, Claude Kipnis informed us of the work of Quastel (1992). In his paper, Quastel studies the limiting behavior of a colored particle process on the d dimensional torus $\mathbb{Z}_n^d$. As a tool, he needs an upper bound on the second largest eigenvalue (i.e., a lower bound on the spectral gap) of simple exclusion. His approach to this question is very similar to ours and uses comparison with the Bernoulli–Laplace model of diffusion. The comparison argument is only a small part of his paper and we provide more details on this matter. The two works were done independently and take very different points of view.

In Section 2A, we set out preliminaries on eigenvalues and the two-quadratic forms we use. The comparison techniques are developed in Section 2B, which shows how they specialize to the results of Diaconis and Stroock (1991) and the comparison bounds for symmetric random walks on groups of Diaconis and Saloff-Coste (1993). A variant using multicommodity flows along the lines of the work of Sinclair (1991) is developed in Section 2C. The exclusion process is treated in Sections 3 (upper bound) and 4 (lower bound). Section 5 contains examples and Section 6 gives bounds on total variation in terms of eigenvalues and some final comments.

2. Forms and eigenvalues.

A. *Preliminaries.* Let X be a finite set. Let $P(x,y)$, $\pi(x)$ be a reversible, irreducible Markov chain on X. Let $l^2(X)$ have scalar product $\langle f, g\rangle = \sum_{x\in X} f(x)g(x)\pi(x)$. Because of reversibility, the operator $f \mapsto Pf$, with $Pf(x) = \sum f(y)P(x,y)$, is self-adjoint on l^2 with eigenvalues $\beta_0 = 1 > \beta_1 \ge \beta_2 \ge \cdots \ge \beta_{|X|-1} \ge -1$. These eigenvalues can be characterized by the Dirichlet form $\mathscr{E}$ defined as

$$(2.1)\quad \mathscr{E}(f,f) = \langle (I-P)f, f\rangle = \tfrac{1}{2}\sum_{x,y}(f(x)-f(y))^2\pi(x)P(x,y).$$

We also use the form

$$(2.2)\quad \mathscr{F}(f,f) = \langle (I+P)f,f\rangle = \tfrac{1}{2}\sum_{x,y}(f(x)+f(y))^2\pi(x)P(x,y).$$

Given a subspace W of $L^2(X)$, set

$$M_{\mathscr{E}}(W) = \max\{\mathscr{E}(f,f);\ \|f\|_2 = 1,\ f\in W\},$$
$$m_{\mathscr{E}}(W) = \min\{\mathscr{E}(f,f);\ \|f\|_2 = 1,\ f\in W\}$$

and define $M_{\mathscr{F}}(W)$ and $m_{\mathscr{F}}(W)$ accordingly. The usual minimax characterization of eigenvalues [see, for instance, Horn and Johnson (1985)] gives, for $0\le i\le |X|-1$,

$$\begin{aligned} 1-\beta_i &= \min\{M_{\mathscr{E}}(W);\ \dim W = i+1\} = \max\{m_{\mathscr{E}}(W);\ \dim W^{\perp} = i\},\\ 1+\beta_i &= \min\{M_{\mathscr{F}}(W);\ \dim W = |X|-i\}\\ &= \max\{m_{\mathscr{F}}(W);\ \dim W^{\perp} = |X|-i-1\}. \end{aligned}$$

If $\tilde{P}(x,y)$, $\tilde{\pi}$ is a second reversible Markov chain on X, the minimax characterization yields, for $1\le i\le |X|-1$,

$$(2.3)\quad \begin{aligned} &\beta_i \le 1 - \frac{a}{A}\bigl(1-\tilde{\beta}_i\bigr), && \text{if } \tilde{\mathscr{E}}\le A\mathscr{E},\ \tilde{\pi}\ge a\pi,\\ &\beta_i \ge -1 + \frac{a}{A}\bigl(1+\tilde{\beta}_i\bigr), && \text{if } \tilde{\mathscr{F}}\le A\mathscr{F},\ \tilde{\pi}\ge a\pi. \end{aligned}$$

B. *Comparison of Dirichlet Forms.* This section develops a geometric bound between Dirichlet forms. Let $\tilde{P}, \tilde{\pi}$ and P,π be reversible Markov chains on the finite set X. In the applications, P,π is the chain of interest and $\tilde{P},\tilde{\pi}$ is a chain with known eigenvalues. Both π and $\tilde{\pi}$ are assumed to be supported on X. For each pair $x\ne y$ with $\tilde{P}(x,y)>0$, fix a sequence of steps $x_0 = x, x_1, x_2,\dots, x_k = y$ with $P(x_i,x_{i+1})>0$. This sequence of steps will be called a *path* γ_{xy} of length $|\gamma_{xy}| = k$. Set $E = \{(x,y);\ P(x,y)>0\}$, $\tilde{E} = \{(x,y);\ \tilde{P}(x,y)>0\}$ and $\tilde{E}(e) = \{(x,y)\in\tilde{E};\ e\in\gamma_{xy}\}$, where $e\in E$. In other words, E is the set of "edges" for P and $\tilde{E}(e)$ is the set of paths that contain e. Here is a convention that we fix once and for all in this paper. All graphs are undirected graphs. However, we describe such a graph as a set of vertices X and a symmetric set of directed edges $E\subset X\times X$.

THEOREM 2.1. *Let $\tilde{P},\tilde{\pi}$ and P,π be reversible Markov chains on a finite set X. For the Dirichlet forms defined in* (2.1),

$$\tilde{\mathscr{E}}\le A\mathscr{E}$$

with

$$(2.4)\quad A = \max_{(z,w)\in E}\left\{\frac{1}{\pi(z)P(z,w)}\sum_{\tilde{E}(z,w)}|\gamma_{xy}|\tilde{\pi}(x)\tilde{P}(x,y)\right\}.$$

PROOF. Clearly, we can assume that none of the paths γ_{xy} contains loops. For an edge $e = (z, w) \in E$, let $f(e) = f(z) - f(w)$. Then

$$\begin{aligned}
\tilde{\mathscr{E}} &= \frac{1}{2} \sum_{x, y \in X} (f(x) - f(y))^2 \tilde{\pi}(x) \tilde{P}(x, y) \\
&= \frac{1}{2} \sum_{x, y} \left\{ \sum_{e \in \gamma_{xy}} f(e) \right\}^2 \tilde{\pi}(x) \tilde{P}(x, y) \\
&\le \frac{1}{2} \sum_{x, y \in X} |\gamma_{xy}| \tilde{\pi}(x) \tilde{P}(x, y) \sum_{e \in \gamma_{xy}} |f(e)|^2 \\
&\le \frac{1}{2} \sum_{e = (z, w)} |f(e)|^2 \frac{\pi(z) P(z, w)}{\pi(z) P(z, w)} \sum_{\gamma_{xy} \ni e} |\gamma_{xy}| \tilde{\pi}(x) \tilde{P}(x, y) \\
&\le A \mathscr{E}(f, f). \qquad \square
\end{aligned}$$

To state a companion result, for $x, y \in X$ with $P(x, y) > 0$, let γ_{xy}^* be a path with $|\gamma_{xy}^*|$ odd. For $e \in E$, set $\tilde{E}^*(e) = \{(x, y) \in \tilde{E};\ e \in \gamma_{x,y}^*\}$. Now, we cannot rule out the possibility of repeated edges along γ_{xy}^*. Thus, we set

$$r_{xy}(e) = \#\{(b_i, b_{i+1}) \in \gamma_{xy}^*;\ (b_i, b_{i+1}) = e\}. \tag{2.5}$$

Note that we can always assume that $r_{xy}(e) \le 2$. The "sum along the path argument" of Theorem 2.1 can be used to write $f(x) + f(y) = (f(x) + f(x_1)) - (f(x_1) + f(x_2)) + \cdots + (f(x_{k-1}) + f(x_k))$. The argument yields the following theorem.

THEOREM 2.2. *Let $\tilde{P}, \tilde{\pi}$ and P, π be reversible Markov chains on a finite state space X. For the quadratic form $\mathscr{F}$ defined in* (2.2),

$$\tilde{\mathscr{F}} \le A^* \mathscr{F}$$

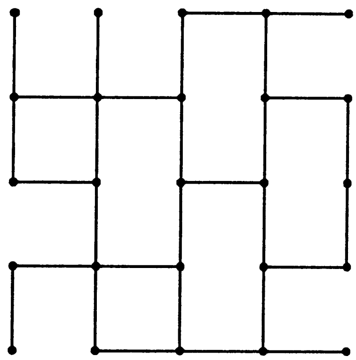
with

$$A^* = \max_{(z, w) \in E} \left\{ \frac{1}{\pi(z) P(z, w)} \sum_{\tilde{E}^*(z, w)} r_{xy}(z, w) |\gamma_{xy}^*| \tilde{\pi}(x) \tilde{P}(x, y) \right\} \tag{2.6}$$

and r_{xy} defined in (2.5).

We begin with a simple example to demystify the notation.

EXAMPLE 2.1. As a chain P of interest, consider the natural graph structure of the $l \times m$ grid $X = \{1, \ldots, l\} \times \{1, \ldots, m\}$ modified by deleting a number of edges from the grid. To keep things simple, suppose no basic square has more than one edge deleted. An example with $l = m = 5$ is



Let the resulting graph be called $\mathscr{G} = (X, E)$. This is a connected graph with $|E|$ edges. Take P to be the usual nearest-neighbor walk on $\mathscr{G}$. Let $d(i,j)$ be the degree of the vertex (i,j). The stationary distribution is $\pi(i,j) = d(i,j)/|E|$. The problem is to bound the eigenvalues of P. Note that these eigenvalues are not known in closed form even in the simplest case where no edge is deleted from the grid.

A chain $\tilde{P}$ on X, with known eigenvalues, can be constructed as follows. For each integer n, let $\tilde{P}_n$ be the nearest-neighbor chain on the n-point segment with a loop at each end. The eigenvalues of $\tilde{P}_n$ are given in Feller [(1968), page 436] and are equal to

$$\cos \frac{\pi j}{n}, \qquad 0 \le j \le n - 1.$$

Now, set

$$\tilde{P} = \tfrac{1}{2}\left(\tilde{P}_l \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{P}_m\right).$$

In other words,

$$\tilde{P}((x,y)(u,v)) = \tfrac{1}{2}\left(\tilde{P}_l(x,u)\,\delta_{yv} + \delta_{xu}\tilde{P}_m(y,v)\right).$$

This has stationary distribution $\tilde{\pi} \equiv 1/lm$. Its eigenvalues are the numbers

$$\frac{1}{2}\left(\cos\frac{\pi i}{l} + \cos\frac{\pi j}{m}\right).$$

In particular, assuming $l \ge m$, we have $\tilde{\beta}_1 = \frac{1}{2}(1 + \cos(\pi/l))$ and $\tilde{\beta}_{\min} \ge -\cos(\pi/l)$.

Observe that

$$\frac{lm}{|E|}\tilde{\pi}(i,j) \le \pi(i,j) \le \frac{4lm}{|E|}\tilde{\pi}(i,j). \tag{2.7}$$

The pairs $(x, y) \in X \times X$ with $x \ne y$ and $\tilde{P}(x, y) > 0$ are exactly the edges of X as a usual grid. Using the notation of Theorem 1, choose a path connecting them in $\mathscr{G}$. This path will be of length 1 if the connecting edge has not been deleted. It will be of length 3 otherwise. Consider the comparison constant A in (2.4). For any edge $e \in E$, there are at most two paths of length 3 and one path of length 1 using e. Using (2.4), $\tilde{\mathscr{E}} \le 7(|E|/4lm)\mathscr{E}$. Hence, (2.3) and (2.7) yield

$$\beta_i \le 1 - \tfrac{1}{7}\big(1 - \tilde{\beta}_i\big), \qquad 0 \le i \le lm - 1.$$

It is even easier to carry out a comparison in the other direction. Reverse the roles of P and $\tilde{P}$. Now, all paths can be chosen of length 1. We get $\mathscr{E} \le (4lm/|E|)\tilde{\mathscr{E}}$. Thus,

$$1 - 4\big(1 - \tilde{\beta}_i\big) \le \beta_i, \qquad 0 \le i \le lm - 1.$$

Combining bounds, the second largest eigenvalue of P satisfies

$$1 - \frac{\pi^2}{l^2} \le -1 + 2\cos\frac{\pi}{l} \le \beta_1 \le \frac{13}{14} - \frac{1}{14}\cos\frac{\pi}{l} = 1 - \frac{\pi^2}{28l^2} + O\left(\frac{1}{l^4}\right)$$

when $l \ge m$.

These inequalities show that the positive parts of the spectrums are quite close for the two processes. Here, the comparison constant A^* between $\tilde{\mathscr{F}}$ and $\mathscr{F}$ is $A^* = \infty$ because there is no path of odd length in $\mathscr{G}$ from a corner to itself. Indeed, $\tilde{\beta}_{\min} \ge -\cos(\pi/l)$, whereas $\beta_{\min} = -1$.

This example generalizes to higher dimension. Let X be a finite box of size $l_1 \times \cdots \times l_d = n$ in $\mathbb{Z}^d$ and set $l = \max_i\{l_i\}$. Consider the simple random walk for the natural graph structure of the box X (for simplicity we do not delete edges here). Comparing with a product walk shows that the second largest eigenvalue of the simple random walk in this box satisfies

$$\begin{aligned} 1 - \frac{\pi^2}{dl^2} \le 1 - \frac{2}{d}\left(1 - \cos\frac{\pi}{l}\right) \le \beta_1 &\le 1 - \frac{1}{d}\left(1 - \cos\frac{\pi}{l}\right) \\ &= 1 - \frac{\pi^2}{2\,dl^2} + O\left(\frac{1}{dl^4}\right). \end{aligned}$$

EXAMPLE 2.2. This example shows how present bounds include some previous results. Let P, π be a reversible Markov chain. Let $\tilde{P}(x, y) = \pi(y)$ for all x. This is a Markov chain with stationary distribution $\tilde{\pi}(x) = \pi(x)$. Then $\tilde{\mathscr{E}}(f, f) = \mathrm{Var}(f) = \sum_{x \in X}(f(x) - \bar{f})^2\pi(x)$ with $\bar{f} = \sum_{x \in X} f(x)\pi(x)$. The bound of Theorem 2.1 reduces to the geometric bound, $\beta_1 \le 1 - 1/A$, of Diaconis and Stroock [(1991), Proposition 1′, page 38]. These authors, along with Fill (1991) and Sinclair (1991) have shown that this bound can be usefully applied in a wide variety of problems. See also Example 2.5 and Corollary 2.1.

Now, take $\tilde{P}(x, x) = 1$ and $\tilde{P} \equiv 0$ otherwise. This trivial chain ($\tilde{P} = \mathrm{Id}$) has any probability measure as invariant measure. Choose $\tilde{\pi} = \pi$ and apply Theorem 2.2. Because $\tilde{P}(x, y) = 0$ unless $x = y$, the paths we consider are loops of odd length σ_x, $x \in X$. In this case, Theorem 2.2 yields a variant of

Proposition 2 in Diaconis and Stroock [(1991), page 40]: The smallest eigenvalue $\beta_{\min}$ of P is bounded by

$$\beta_{\min} \geq -1 + 2/A^{\#},$$

where

$$A^{\#} = \max_{(z,w)\in E} \left\{ \frac{1}{P(z,w)\pi(z)} \sum_{\sigma_x \ni (z,w)} r_x(z,w)|\sigma_x|\pi(x) \right\}$$

with r_x as in (2.5). When $P(z,z) \geq \varepsilon > 0$ for all $z \in X$, we can take σ_x to be the trivial loop at x and get

$$\beta_{\min} \geq -1 + 2\varepsilon.$$

EXAMPLE 2.3. Diaconis and Saloff-Coste (1993) developed comparison techniques of similar flavor for symmetric random walks on finite groups. Fill has pointed out that Theorem 2.1 specializes to give exactly the previous bounds. This is useful because the geometric flavor of the bound was not apparent in the group case. To develop the details, suppose $X = G$ is a finite group and $\Gamma = \{s_1, \ldots, s_m\}$ is a symmetric set of generators of G. Let q be a symmetric probability on G supported on Γ. Let $\tilde{q}$ be a second symmetric probability on G. These probabilities define Markov chains $P(x,y) = q(x^{-1}y)$ and $\tilde{P}(x,y) = \tilde{q}(x^{-1}y)$. Assume that each of these chains has the uniform distribution as its unique stationary distribution.

For each $\omega \in G$, choose a representation $\omega = s_1 s_2 \cdots s_l$ with $s_i \in \Gamma$ and set $|\omega| = l$. Let $N(s, \omega)$ be the number of times that a given $s \in \Gamma$ appears in this representation. Then, for any $x, y \in G$, set $\gamma_{xy} = (x, xs_1, xs_1s_2, \ldots, xs_1 \cdots s_l)$, where $\omega = x^{-1}y$. Now, the edges that occur are of the form $e = (z, zs)$ for $z \in G$ and $s \in \Gamma$. For such an edge,

$$\sum_{\gamma_{xy} \ni e} |\gamma_{xy}|\tilde{\pi}(x)\tilde{P}(x,y) = \frac{1}{|G|} \sum_{(x,\omega)\in\Omega} |\omega|\tilde{q}(\omega),$$

where the sum is over

$$\Omega = \{(x,\omega) : \exists\, i \in \{1,\ldots,l\} \text{ such that } xs_1 \cdots s_{i-1} = z,\ xs_1 \cdots s_i = zs\}.$$

For any fixed ω, the number of $x \in G$ such that $(x,\omega) \in \Omega$ is exactly $N(s,\omega)$, so

$$A = \max_s \frac{1}{q(s)} \sum_{\omega \in G} |\omega| N(s,\omega)\tilde{q}(\omega).$$

A similar analysis works for A^*. Diaconis and Saloff-Coste (1992b, 1993) give many examples of the use of this bound. The connection will be useful here as well because bounds for random walks on graphs can be used to bound the eigenvalues of the exclusion process on these graphs; see Theorem 2.3.

EXAMPLE 2.4. This example shows that removing a single edge can lead to bounds that are "off." Let $X = \{0, 1, 2, \ldots, n-1\}$. Let $\tilde{P}$ be the nearest-neighbor random walk on the circle $\mathbb{Z}_n = X$. Thus $\tilde{P}(i, i+1) = \tilde{P}(i, i-1) = \frac{1}{2}$ with all entries mod n. Here $\tilde{\pi}(i) = 1/n$ and the eigenvalues are easily shown to be $\tilde{\beta}_j = \cos(2\pi j/n)$, $0 \le j \le n-1$. As a different chain of interest, take the nearest-neighbor walk on the segment X with a loop at each end: $P(0,0) = P(0,1) = \frac{1}{2} = P(n-1, n-1) = P(n-1, n-2)$ with $P(i,j) = \tilde{P}(i,j)$ for $i \neq 0, n-1$. This also has $\pi(i) = 1/n$.

For the comparison, for each edge $(i, i+1)$, $0 \le i \le n-2$, let $\gamma_{i,i+1} = (i, i+1)$. Take $\gamma_{0,n-1} = 0, 1, \dots, n-1$. The maximum in (2.4) is taken on at $(0,1)$ with $A = n$. Here, the bound on eigenvalues is

$$\beta_j \le 1 - \frac{\left(1 - \tilde{\beta}_j\right)}{A} = 1 - \left(1 - \cos\left(\frac{2\pi j}{n}\right)\right)/n.$$

For $j = 1$, this gives

$$\cos\frac{\pi}{n} = \beta_1 \le 1 - \frac{2\pi^2}{n^3} + O\left(\frac{1}{n^5}\right),$$

which is clearly off by a factor of n.

EXAMPLE 2.5. It is of interest to specialize Theorems 2.1 and 2.2 to the case where P and $\tilde{P}$ are simple random walks associated with two nonoriented graphs $\mathscr{G} = (X, E)$ and $\tilde{\mathscr{G}} = (X, \tilde{E})$ on the same underlying finite set X. Then, if $d(x)$ and $\tilde{d}(x)$ are the degrees of $x \in X$, we have $\pi(x) = d(x)/|E|$ and $P(x,y) = 1/d(x)$ if $(x,y) \in E$, $P = 0$ otherwise, and $\mathscr{E}(f,f) = (1/2|E|)\sum_{x,y \in E}|f(x) - f(y)|^2$. It follows that the constant A in (2.4) is $A = (|E|/|\tilde{E}|)\Delta$ with

$$\Delta = \Delta(P, \tilde{P}) \equiv \max_{e \in E}\left\{\sum_{\tilde{E}(e)} |\gamma_{xy}|\right\}. \tag{2.8}$$

More generally, this is a reasonable way to bound A whenever $P(z,w)\pi(z)$ does not depend too strongly on z, w. A similar analysis can be used for A^* if we consider

$$\Delta^* = \Delta^*(P, \tilde{P}) \equiv \max_{e \in E}\left\{\sum_{\tilde{E}^*(e)} r_{xy}(e)|\gamma_{xy}^*|\right\}. \tag{2.9}$$

Setting $\delta = \min_{x \in X}\{\tilde{d}(x)/d(x)\}$, the estimate (2.3) on eigenvalues yields

$$-1 + \frac{\delta}{\Delta^*}\left(1 + \tilde{\beta}_i\right) \le \beta_i \le 1 - \frac{\delta}{\Delta}\left(1 - \tilde{\beta}_i\right). \tag{2.10}$$

Note that we did not need to compare $|E|$ and $|\tilde{E}|$. For instance, applying (2.10) with $\tilde{P} \equiv 1/|X|$ (i.e., with $\tilde{\mathscr{G}}$ the complete graph) we get the following corollary.

COROLLARY 2.1. *Let $\mathscr{G} = (X, E)$ be an undirected connected graph. The nontrivial eigenvalues of the nearest-neighbor random walk on (X, E) satisfy*

$$(2.11) \qquad -1 + \frac{|X|}{d\Delta^*} \le \beta_i \le 1 - \frac{|X|}{d\Delta},$$

where

$$\Delta^* = \max_{e \in E}\left\{ \sum_{\gamma_{xy} \ni e} r_{xy}(e)|\gamma^*_{xy}| \right\}, \qquad \Delta = \max_{e \in E}\left\{ \sum_{\gamma_{xy} \ni e} |\gamma_{xy}| \right\}$$

with r_{xy} given by (2.5) *and $d = \max_x d(x)$.*

Diaconis and Stroock (1991) consider

$$\gamma = \max_{x, y} |\gamma_{xy}|,$$
$$b = \max_{e \in E} \#\{\gamma_{xy};\, e \in \gamma_{xy}\},$$

which have simpler geometric interpretations than $\Delta \le \gamma b$. The preceding upper bound is a slight improvement on their bound [Diaconis and Stroock (1991), Corollary 1, Section 1, page 39]

$$(2.12) \qquad \beta_1 \le 1 - |E|/d^2\gamma b.$$

Finding bounds on Δ (or γ, b) can turn out to be a hard combinatorial problem. However, it is feasible for "simple" graphs and some more sophisticated ones; see Diaconis and Stroock (1991), Sinclair (1991) and Fill (1991). Babai, Hetyii, Kantor, Lubotzky and Seress (1990) discuss bounding γ for Cayley graphs of finite groups. The main point in the method used in this paper and in Diaconis and Saloff-Coste (1993) is to compare P with a *nontrivial* known $\tilde{P}$. This reduces the complexity of the combinatorics of paths: Instead of having to deal with paths from any $x \in X$ to any $y \in X$, one just needs to consider paths that link x and y when $\tilde{P}(x, y) > 0$. This is well illustrated in the study of simple exclusion; see Section 3.

C. Comparison using multicommodity flows. Many variations on Theorems 2.1 and 2.2 are possible. We now describe one of them that will be applied later to exclusion processes. We adapt an idea of Sinclair (1991).

Suppose we are in the situation of Theorem 2.1 and want to compare the Dirichlet forms $\mathscr{E}$ and $\tilde{\mathscr{E}}$ of two reversible Markov chains P, π and $\tilde{P}, \tilde{\pi}$. It often happens that there is more than one path $x = x_0, x_1, \dots, x_k = y$ with $P(x_i, x_{i+1}) > 0$ between x and y such that $\tilde{P}(x, y) > 0$ [i.e., $(x, y) \in \tilde{E}$]. Let $\mathscr{P}_{xy}$ be the set of all simple paths connecting x to y as before and set $\mathscr{P} = \bigcup_{(x, y) \in \tilde{E}} \mathscr{P}_{xy}$. Also, for $e \in E$, let $\mathscr{P}(e) = \{\gamma \in \mathscr{P}, e \in \gamma]$. A function f on $\mathscr{P}$ is called a flow or more precisely a $(P, \tilde{P})$ flow if

$$\sum_{\gamma \in \mathscr{P}_{xy}} f(\gamma) = \tilde{P}(x, y)\tilde{\pi}(x).$$

The proof of Theorem 2.1 yields immediately the following theorem.

THEOREM 2.3. *Let $\tilde{P}, \tilde{\pi}$ and P, π be reversible Markov chains on a finite set X. For any $(P, \tilde{P})$ flow f, the Dirichlet forms defined in* (2.1) *satisfy*

$$\tilde{\mathscr{E}} \le A(f)\mathscr{E}$$

with

$$A(f) = \max_{(z,w)\in E} \left\{ \frac{1}{\pi(z)P(z,w)} \sum_{\mathscr{P}(z,w)} |\gamma| f(\gamma) \right\}. \tag{2.13}$$

Clearly, Theorem 2.3 contains Theorem 2.1: Take f to be the flow defined by $f(\gamma) = 0$ unless $\gamma = \gamma_{xy}$ is the chosen path for a pair $(x, y) \in \tilde{E}$ in which case $f(\gamma_{xy}) = \tilde{P}(x, y)\tilde{\pi}(x)$. The same idea yields a variant of Theorem 2.2, which we will not write down.

EXAMPLE 2.6. Let $(X, E) = \mathscr{G}$ be a graph with automorphism group acting transitively on the set E of the oriented edges. This implies that $\mathscr{G}$ is vertex transitive and thus regular; see Biggs (1974) for examples and more details. For the simple random walk on such a graph, Proposition 4 in Diaconis and Stroock [(1991), page 46] gives

$$\beta_1 \le 1 - \frac{1}{D^2}$$

with

$$D = \left(\frac{1}{|X|} \sum_{x\in X} |ox|^2 \right)^{1/2},$$

where for each $x \in X$, $|ox|$ is the distance from the fixed point $o \in X$ to x. This can be obtained by comparing with the trivial uniform chain. In Theorem 2.3, take f uniformly supported on geodesic paths; that is,

$$f(\gamma) = \frac{1}{|\mathscr{G}_{xy}|} \tilde{\pi}(y)\tilde{\pi}(x) = \frac{1}{|\mathscr{G}_{xy}|\,|X|^2}$$

if γ is in $\mathscr{G}_{xy}$, the set of all geodesic paths from x to y. The point is that, for this f,

$$\sum_{\gamma \ni e} |\gamma| f(\gamma)$$

does not depend on $e \in E$. The result then follows as in Diaconis and Stroock (1991).

EXAMPLE 2.7. Let $K_{l,m} = (X, E)$ be the complete bipartite graph with $n = l + m$ vertices. To be precise, $X = \{1, \dots, l + m\}$ and $E = \{1, \dots, l\} \times \{l + 1, \dots, l + m\} \cup \{l + 1, \dots, l + m\} \times \{1, \dots, l\}$. Sinclair (1991) used the case $l = 2$, $m = n - 2$ to demonstrate the effectiveness of random paths. Here, we compare the simple walk P, π on $K_{l,m}$ with the chain $\tilde{P}(x, y) = \pi(y)$, $\tilde{\pi} = \pi$. Pairs $(x, y) \in X \times X$ are of three types:

Type 1: $(x, y) \in E$,

Type 2: $(x, y) \in \{1, \dots, l\}^2$,

Type 3: $(x, y) \in \{l+1, \dots, l+m\}^2$.

If (x, y) is of Type 1, $\mathscr{P}_{x,y} = \{(x, y)\}$ and we set $f(\gamma) = \pi(x)\pi(y)$ for these paths. If (x, y) is of Type 2, $\mathscr{P}_{x,y} = \{(x, i, y);\ i \in \{l+1, \dots, l+m\}\}$ and we set $f(\gamma) = (1/m)\pi(x)\pi(y)$ for these paths. When (x, y) is of Type 3, $\mathscr{P}_{x,y} = \{(x, i, y);\ i \in \{1, \dots, l\}\}$ and we set $f(\gamma) = (1/l)\pi(x)\pi(y)$. For this flow the constant $A(f)$ in (2.13) is

$$\begin{aligned} A(f) &= 2lm\left(\frac{lm}{(2lm)^2} + (l-1)\frac{m^2}{m(2lm)^2} + (m-1)\frac{l^2}{l(2lm)^2}\right) \\ &= \frac{3}{2} - \frac{1}{2l} - \frac{1}{2m}. \end{aligned}$$

This yields $\beta_1 \le \frac{1}{3}$ for any of these graphs. Of course, the eigenvalues of $K_{l,m}$ are known to be 1, 0 and -1 with multiplicities 1, $n-2$ and 1.

In fact, the preceding example is a special case of a generalization of Example 2.6. If the automorphism group of a graph $\mathscr{G}$ acts transitively on the set of *undirected* edges, we have $\beta_1 \le 1 - 1/D^2$ with $D^2 = \sum_{x,y} |xy|^2 \pi(x)\pi(y)$.

The complete multipartite graphs $K_{u,v,w}$ give examples where one has to use nongeodesic paths to get good bounds. An example of comparison between two nontrivial chains is given at the end of the next section.

3. The exclusion process. Let X_0 be a set with n elements. Let $E_0 \subset X_0 \times X_0$ be a symmetric set of edges such that $(X_0, E_0) = \mathscr{G}_0$ is an undirected connected graph. Before defining the exclusion process of $r \le n$ unlabelled particles on $\mathscr{G}_0$, we fix some notation. Let $d_0 = \max\{d(x);\ x \in X_0\}$ be the maximum degree in $\mathscr{G}_0$. According to (2.11) the nontrivial eigenvalues of the *simple random walk on* $\mathscr{G}_0$ satisfy

$$-1 + \frac{n}{d_0 \Delta_0^*} \le \beta_i^0 \le 1 - \frac{n}{d_0 \Delta_0}, \qquad i = 1, \dots, |X_0| - 1$$

with

$$\Delta_0 = \max_{e_0 \in E_0} \left\{ \sum_{\gamma_{xy} \ni e_0} |\gamma_{xy}| \right\}, \qquad \Delta_0^* = \max_{e_0 \in E_0} \left\{ \sum_{\gamma_{xy}^* \ni e_0} r_{xy}(e) |\gamma_{xy}^*| \right\}. \tag{3.1}$$

Our main result in this section shows that Δ_0, Δ_0^* and d_0 can also be used to bound the eigenvalues of the exclusion process of r particles hopping around on $\mathscr{G}_0$.

For $r \leq n$, the exclusion process is defined as a Markov chain with values in the r-sets of X_0. Informally, if the current state is the set A, pick an element in A with probability proportional to its degree, pick a neighboring site of this element at random and move the element to the neighboring site provided this site is unoccupied. If the site is occupied, the chain stays at A.

Formally, let $X = X_r$ be the set of the r-sets of X_0 and A_1 and A_2 be r-sets. Define

$$(3.2)\quad P(A_1, A_2) = \begin{cases} 0, & \text{if } |A_1 \cap A_2| \leq r-2, \\ 0, & \text{if } |A_1 \cap A_2| = r-1 \\ & \quad \text{and } A_1 = A \cup \{a_1\}, \\ & A_2 = A \cup \{a_2\} \text{ with } (a_1, a_2) \notin E_0, \\ 1 \Big/ \sum_{a \in A_1} d(a), & \text{if } |A_1 \cap A_2| = r-1 \\ & \quad \text{and } A_1 = A \cup \{a_1\}, \\ & A_2 = A \cup \{a_2\} \text{ with } (a_1, a_2) \in E_0, \\ \sum_{a \in A_1} d_a^*(A_1) \Big/ \sum_{a \in A_1} d(a), & \text{if } A_1 = A_2, \text{ where} \\ & \quad d_a^*(A_1) = |\{b \in A_1, (a,b) \in E_0\}|. \end{cases}$$

The chain P can be interpreted as a nearest-neighbor random walk on a graph with multiple edges [there are $\sum_{a \in A} d_a^*(A)$ loops from the r-set A to A]. It is a reversible chain with stationary distribution

$$(3.3)\qquad \pi(A) = \frac{n \sum_{a \in A} d(a)}{r\binom{n}{r}|E_0|}.$$

Hence, for $A_1 \neq A_2$ and $P(A_1, A_2) \neq 0$, we have

$$(3.4)\qquad P(A_1, A_2)\pi(A_1) = \frac{n}{r\binom{n}{r}|E_0|}.$$

Given $1 \leq r \leq n$, we define the maximum mean degree d_r over r-sets by

$$(3.5)\qquad d_r = \max_{A \in X_r} \left\{ \frac{1}{r} \sum_{a \in A} d(a) \right\} \leq d_0.$$

When $\mathscr{G}_0$ is d_0 regular [i.e., $d(a) \equiv d_0$], then P is symmetric, π is uniform on the r-sets and $d_r = d_0$. If $\mathscr{G}_0$ is not regular, a variant of the foregoing process is discussed briefly at the end of the paper. When $r = 1$, the preceding process reduces to the simple random walk on the underlying graph. When $r = n$, we get a trivial process with only one state (we will informally exclude this case). When $r = n - 1$, looking at the only unoccupied site gives a description of the process as a simple random walk with strange holding condition.

Our main results are summarized in the following theorem.

THEOREM 3.1. *Let (X_0, E_0) be a connected graph and $1 \leq r < n$ as before. The Markov chain P at* (3.2) *of the exclusion process has its eigenvalues $\beta_i(r)$,*

$1 \le i \le \binom{n}{r} - 1$, *bounded above by*

$$\beta_i(r) \le 1 - \frac{k_i(n - k_i + 1)}{r d_r \Delta_0}, \tag{3.6}$$

where $k_i = j$ *when* $\binom{n}{j-1} \le i < \binom{n}{j}$, $0 \le j \le \min\{r, n - r\}$. *In particular, the second largest eigenvalue* $\beta_1(r)$ *is bounded by*

$$\beta_1(r) \le 1 - \frac{n}{r d_r \Delta_0}.$$

Moreover, the smallest eigenvalue of P *satisfies*

$$\beta_{\min}(r) \ge -1 + \frac{n - r - 1}{d_r \Delta_0^*}. \tag{3.7}$$

Here Δ_0 *and* Δ_0^* *are given in* (3.1) *and* d_r *is the maximum mean degree over* r *sets defined at* (3.5).

REMARK 1. Because $d_r \le d_0$ where d_0 is the maximum degree in $\mathscr{G}_0$, all the foregoing estimates hold with d_r replaced by d_0. Examples 5.6 (a star) and 5.8 show that using d_r instead of d_0 can be useful.

REMARK 2. Specialize to the case of $X_0 = \{1, \dots, n\}$ with edges $E_0 = \{(i, i + 1), (i + 1, i);\ i = 1, \dots, n - 1\} \cup \{(1, 1), (n, n)\}$ (i.e., $\mathscr{G}_0$ is the n-point segment with a loop at each end). In this case, $d_0 = 2$, $\Delta_0 \le n^3/8$, $\Delta_0^* \le n^3$ (this last estimate is rough) and thus

$$\beta_1 = \beta_1(r) \le 1 - \frac{4}{rn^2}$$

and the negative eigenvalues are all bounded by

$$\beta_{\min} \ge -1 + \frac{n - r - 1}{2n^3}.$$

For this case, Fill (1991) obtained the lower bound $\beta_1 \ge 1 - 6/rn^2$. This shows our result is sharp. Fill also obtained an upper bound $\beta_1 \le 1 - (3 - o(1))/2rn^5$. He used a path bound as in (2.12). The power of using comparison with a *nontrivial* chain can be seen here. Fill introduced methods for bounding nonsymmetric exclusion by comparison with symmetric exclusion. Other examples are discussed in Section 5.

REMARK 3. Let γ_0 denote the diameter of $\mathscr{G}_0$. As a crude but universal estimate we have $\Delta_0 \le n^2\gamma_0$. This yields $\beta_1 \le 1 - 1/rn\gamma_0 d_r \le 1 - 1/rn^2 d_0$.

PROOF OF (3.6). The argument proceeds by comparison with the classical Bernoulli–Laplace model of diffusion. This is a Markov chain $\tilde{P}$ on the r sets of X_0 that can be described as follows: If the current state is the set A_1, pick an element in A_1 at random, pick an element in $A_1^c = X_0 \setminus A_1$ at random and switch the two elements. Formally, let A_1 and A_2 be r-sets. Define

$$\tilde{P}(A_1, A_2) = \begin{cases} 0, & \text{if } |A_1 \cap A_2| \le r - 2 \text{ or } A_1 = A_2, \\ 1/r(n-r), & \text{if } |A_1 \cap A_2| = r - 1. \end{cases} \tag{3.8}$$

The stationary distribution for this chain is uniform, $\tilde{\pi}(A) = 1/\binom{n}{r}$ and

$$\tilde{P}(A_1, A_2)\tilde{\pi}(A_1) = \frac{1}{r(n-r)\binom{n}{r}} \tag{3.9}$$

when $\tilde{P}(A_1, A_2) > 0$. Diaconis and Shahshahani (1987) showed that the eigenvalues of $\tilde{P}$ are

$$1 - \frac{j(n-j+1)}{r(n-r)}, \qquad 0 \le j \le r,$$

with multiplicity $\binom{n}{j} - \binom{n}{j-1}$. In other words, $\tilde{\beta}_1 = \tilde{\beta}_i = 1 - n/r(n-r)$ for $1 \le i < n$ and more generally,

$$\tilde{\beta}_i = 1 - \frac{k_i(n - k_i - 1)}{r(n-r)}, \tag{3.10}$$

where $k_i = s$ if $\binom{n}{s-1} \le i < \binom{n}{s}$ and $0 < s \le \min\{r, n-r\}$.

In order to apply the comparison technique, we now describe a path $\gamma_{A_1A_2}$ for each (A_1, A_2) such that $\tilde{P}(A_1, A_2) \ne 0$; that is, for each (A_1, A_2) such that $A_1 \cap A_2 = A$, $|A| = r - 1$. We set $A_1 = \{a_1\} \cup A$ and $A_2 = \{a_2\} \cup A$. Denote by $\gamma_{a_1a_2}$ the fixed path from a_1 to a_2 in (X_0, E_0). Say $\gamma_{a_1a_2} = (b_1, \dots, b_k)$ with

$$a_1 = b_1, b_2, \dots, b_k = a_2.$$

We can assume that $b_i \ne b_j$ if $i \ne j$ (no loops).

There are many paths from A_1 to A_2 that can be associated with $\gamma_{a_1a_2}$. In order to get a good bound, we have to choose one of them in a careful manner. We start with an informal description. First, we draw the path $\gamma_{a_1a_2}$ in the graph (X_0, E_0), and we mark the b_is that belong to A_1 by the symbol $\otimes$:

$$\begin{array}{ccccccccccc} b_1 = a_1 & & & & b_5 & b_6 & & & & & a_2 = b_{11} \\ \otimes & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \times \end{array}$$

Of course the part of A_1 that does not meet γ_{a_1, a_2} is of no importance in the description of of our path from A_1 to A_2. (It will be there and stay there all along.) Now, in the foregoing example, we start in the most obvious manner:

$$
\begin{array}{rccccccccccc}
 & a_1 & & & & & & & & & & a_2 \\
A_1 = B_1 & \otimes & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \times \\
B_2 & \times & \otimes & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \times \\
B_3 & \times & \times & \otimes & \otimes & \otimes & \times & \times & \otimes & \times & \times & \times
\end{array}
$$

Here, we realize that we are blocked (in our move toward a_2) by the particles that are on the path. Our next moves are better described by drawing them:

$$
\begin{array}{rccccccccccc}
 & a_1 & & & & & & & & & & a_2 \\
B_3 & \times & \times & \otimes & \otimes & \otimes & \times & \times & \otimes & \times & \times & \times \\
B_4 & \times & \times & \otimes & \otimes & \times & \otimes & \times & \otimes & \times & \times & \times \\
B_5 & \times & \times & \otimes & \otimes & \times & \times & \otimes & \otimes & \times & \times & \times \\
B_6 & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \otimes & \times & \times \\
B_7 & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \otimes & \times \\
B_8 & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \times & \otimes
\end{array}
$$

This ends the first part in the construction of our path: We have reached a_2, but we have left some mess behind. However, it is easy to clean up:

$$
\begin{array}{rccccccccccc}
 & a_1 & & & & & & & & & & a_2 \\
B_8 & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \times & \otimes \\
B_9 & \times & \times & \otimes & \otimes & \times & \times & \times & \otimes & \times & \times & \otimes \\
B_{10} & \times & \times & \otimes & \times & \otimes & \times & \times & \otimes & \times & \times & \otimes \\
B_{11} & \times & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \otimes
\end{array}
$$

We are done, because $B_{11} = A_2$.

Before trying to formalize this, we emphasize that the preceding construction is *mechanical*. In fact, given the path $\gamma_{a_1 a_2}$, it is enough to be given an edge (B_i, B_{i+1}) to be able to reconstruct the entire path in X. Here is an example of this fact. In the preceding example, assume we are given a_1, a_2 and the edge B_6, B_7:

$$
\begin{array}{rccccccccccc}
 & a_1 & & & & & & & & \to & & a_2 \\
B_6 & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \otimes & \times & \times \\
B_7 & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \otimes & \times
\end{array}
$$

First we see that this edge belongs to the first part of our path (moving toward a_2) because $a_2 \notin B_6$ (*not* because $a_1 \notin B_6$; see the other example that follows). In order to find A_1, we look at the particles that are on the path $\gamma_{a_1 a_2}$ and to the left of the move indicated by the given edge ($B_6 \to B_7$) (including the particle involved in this move). Each particle that does not have a left neighbor has to be moved backward until it is blocked by another particle [start from the particle involved in (B_6, B_7) and then from right to left]. Here, this gives

$$\begin{array}{cccccccccccc} & a_1 & & & & & & & & \downarrow & & a_2 \\ B_6 & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \otimes & \times & \times \\ A_1 & \otimes & \times & \times & \otimes & \otimes & \times & \times & \otimes & \times & \times & \times \end{array}$$

Alternatively, we could finish the construction of the path and thus find A_2.

We now formalize the foregoing construction (we also draw a second example):

$$\begin{array}{ccccccccccccc} & a_1 & & & & & & & & & & & a_2 \\ A_1 & \otimes & \otimes & \times & \times & \otimes & \times & \times & \otimes & \otimes & \times & \otimes & \times \\ & & i_1 & & i_2 & i_3 & & i_4 & & i_5 & i_6 & i_7 & i_8 \end{array}$$

First define i_1 to be the smallest integer $i \geq 1$ such that $b_{i+1} \notin A = A_1 \setminus \{a_1\}$ and let i_2 be the smallest integer $i \geq i_1 + 1$ such that $b_{i+1} \in A$. Then, define inductively i_{2j+1} to be the smallest integer $i \geq i_{2j} + 1$ such that $b_{i+1} \notin A$ and i_{2j+2} is the smallest integer $i \geq i_{2i+1} + 1$ such that $b_{i+1} \in A$ or $i_{2j+2} = k$ in case $b_i \notin A$ for all $i \geq i_{2j+1}$. Let ν be the integer such that $b_{i_{2\nu}} = b_k$ (i.e., $i_{2\nu} = k$). Set $i_0 = 1$ and

$$k'_\alpha = (i_{2\alpha} - i_{2\alpha-1}) + \cdots + (i_2 - i_1)$$

for $1 \leq \alpha \leq \nu$. Also, define

$$A'_1 = \left(A_1 \setminus \{b_{i_1}\}\right) = \left(A \cup \{a_1\}\right) \setminus \{b_{i_1}\})$$

and, inductively,

$$A'_\alpha = \left(A'_{\alpha-1} \cup \{b_{i_{2\alpha-2}}\}\right) \setminus \{b_{i_{2\alpha-1}}\}$$

for $1 \leq \alpha \leq \nu$. This notation will be used to describe the first part of the path (toward a_2). Namely, set

$$B_1 = A_1 = A'_1 \cup \{b_{i_1}\},$$
$$B_2 = A'_1 \cup \{b_{i_1+1}\},$$
$$B_{j+1} = A'_1 \cup \{b_{i_1+j}\},$$

for $1 \leq j \leq i_2 - i_1 = k'_1$. More generally, for $0 \leq \alpha < \nu$ and $k'_\alpha + 1 \leq j \leq k'_{\alpha+1}$, set

$$B_{j+1} = A'_{\alpha+1} \cup \{b_{i_{2\alpha+1}+j-k'_\alpha}\}$$

and consider the edges (B_j, B_{j+1}). Note that

$$B_{k'_\nu+1} = A'_\nu \cup \{b_{i_{2\nu}}\} = A'_\nu \cup \{a_2\}.$$

This ends the description of the move toward a_2.

In order to describe the "cleaning up" stage, we introduce the following notation for $1 \leq \alpha \leq \nu$: Set

$$k''_\nu = k'_\nu, \qquad k''_{\alpha-1} = \left(i_{2\alpha-1} - i_{2(\alpha-1)}\right) + k''_\alpha$$

and

$$A''_\nu = A'_\nu \cup \{a_2\},$$
$$A''_\alpha = \left(A''_{\alpha+1} \cup \{b_{i_{2\alpha+1}}\}\right) \setminus \{b_{i_{2\alpha}}\}.$$

Note that $|A'_\alpha| = r - 1$, whereas $|A''_\alpha| = r$. The cleaning up starts with the edge

$$B_{k''_\nu+1} = A''_\nu, \qquad B_{k''_\nu+2} = \left(A''_\nu \cup \{b_{i_{2\nu-1}}\}\right) \setminus \{b_{i_{2\nu-1}-1}\}.$$

More generally, for $k''_\alpha + 1 \le j \le k''_{\alpha-1}$ and $1 \le \alpha \le \nu$, set

$$B_{j+1} = \left(A''_\alpha \cup \{b_{i_{2\alpha-1}-j+k''_\alpha+1}\}\right) \setminus \{b_{i_{2\alpha-1}-j+k''_\alpha}\}$$

and consider the edges (B_j, B_{j+1}). Note that the last of the B_j is obtained for

$$\begin{aligned} j = k''_0 + 1 &= (i_1 - i_0) + (i_3 - i_2) + \cdots + \left(i_{2\nu-1} - i_{2(\nu-1)}\right) \\ &\quad + \left(i_{2\nu} - i_{2(\nu-1)}\right) + \cdots + (i_2 - i_1) + 1 \\ &= i_{2\nu} - i_0 + 1 = i_{2\nu} = k. \end{aligned}$$

Hence, the length of the path that we just described is equal to $k - 1$, which is also the length of $\gamma_{a_1a_2}$. In fact (and this will be important later on when looking at Δ^*), each edge of $\gamma_{a_1a_2}$ corresponds to exactly one edge of $\gamma_{A_1A_2}$.

Finally, we check that the foregoing path does what we want (i.e., $B_k = A_2$) by checking that *whatever is to be to the right of the particle that is moved at one given step* describes exactly the intersection of A with the right part of $\gamma_{a_1a_2}$.

Now, assume that we are given an edge $e = (C_1, C_2)$. How many of the preceding paths can pass through e? [This is the question we have to study in order to bound the constant Δ in (2.8)]. Assume that $C_1 = C \cup \{c_1\}$ and $C_2 = C \cup \{c_2\}$. First, we choose a path $\gamma_{a_1a_2}$ that contains the corresponding edge $e_0 = (c_1, c_2) \in E_0$. This fixes the endpoints a_1 and a_2 in X_0. Now, we claim that we know enough to describe completely the two ends A_1 and A_2: $A_1 = A \cup \{a_1\}$ and $A_2 = A \cup \{a_2\}$ corresponding to $\gamma_{a_1a_2}$ and the given edge (C_1, C_2). Indeed, we can first determine whether (C_1, C_2) appears in the "moving toward a_2" or in the "cleaning up" phase of this path. This only depends on whether or not $a_2 \notin C_1$.

Suppose first that $a_2 \notin C_1$. Then we are in the "moving toward a_2" phase. For instance, consider

$$\begin{array}{cccccccccccc} & a_1 & & & & & & & c_1 & \rightarrow & c_2 & a_2 \\ C_1 & \otimes & \otimes & \times & \otimes & \otimes & \times & \times & \otimes & \times & \otimes & \times \\ C_2 & \otimes & \otimes & \times & \otimes & \otimes & \times & \times & \times & \otimes & \otimes & \times \end{array}$$

In this case, starting with c_1 and proceding from right to left, we move to the left (as much as possible) the particles that do not have a left neighbor. This gives A_1:

$$\begin{array}{cccccccccccc} & a_1 & & & & & & & & & & a_2 \\ A_1 & \otimes & \otimes & \otimes & \times & \otimes & \otimes & \times & \times & \times & \otimes & \times \end{array}$$

and A_2 is then of course given by

$$\begin{array}{cccccccccccc} & a_1 & & & & & & & & & & a_2 \\ A_2 & \times & \otimes & \otimes & \times & \otimes & \otimes & \times & \times & \times & \otimes & \otimes \end{array}$$

Assume now that $a_2 \in C_1$. Then we are in the "cleaning up" phase. For instance,

$$\begin{array}{cccccccccccc} & a_1 & & & & & & & c_1 \to & c_2 & & a_2 \\ C_1 & \times & \otimes & \otimes & \times & \otimes & \times & \otimes & \otimes & \times & \times & \otimes & \otimes \\ C_2 & \times & \otimes & \otimes & \times & \otimes & \times & \otimes & \times & \otimes & \times & \otimes & \otimes \end{array}$$

Here again, we can find A_1 and A_2. To find A_2, starting from the left of c_2, we just move each particle one step to the right:

$$\begin{array}{ccccccccccccc} & a_1 & & & & & & & & & & & a_2 \\ A_2 & \times & \times & \otimes & \otimes & \times & \otimes & \times & \otimes & \otimes & \times & \otimes & \otimes \\ A_1 & \otimes & \times & \otimes & \otimes & \times & \otimes & \times & \otimes & \otimes & \times & \otimes & \times \end{array}$$

Hence, given an edge $e = (C_1, C_2)$ with $C_1 = C \cup \{c_1\}$ and $C_2 = C \cup \{c_2\}$, we established a one-to-one correspondence between the paths $\gamma_{a_1 a_2}$ in X_0 going through $e_0 = (c_1, c_2)$ and the paths γ_{A_1, A_2} in X containing the edge e. Moreover, the length of the paths is preserved in this correspondence. Hence, we certainly have

$$\Delta = \max_e \left\{ \sum_{\gamma_{A_1A_2} \ni e} |\gamma_{A_1A_2}| \right\} = \max_{e_0} \left\{ \sum_{\gamma_{a_1a_2} \ni e_0} |\gamma_{a_1a_2}| \right\} = \Delta_0. \tag{3.11}$$

From (3.4), (3.9) and (3.11) we deduce the following lemma.

LEMMA 3.1. *The comparison constant A defined in* (2.4) *with P being the exclusion process* (3.2) *and $\tilde{P}$ the Bernoulli–Laplace model of diffusion* (3.8) *satisfies*

$$A = \frac{|E_0|\Delta_0}{n(n-r)}.$$

The corresponding stationary distributions satisfy

$$\tilde{\pi} \geq \frac{|E_0|}{nd_r}\pi, \tag{3.12}$$

where d_r is the maximum mean degree over r-sets in (3.5). Hence, (2.3) implies that

$$\beta_i \leq 1 - \frac{(n-r)}{d_r\Delta_0}\left(1 - \tilde{\beta}_i\right)$$

and (3.6) follows from the values of $\tilde{\beta}_i$ given in (3.10). □

PROOF OF (3.7). In order to estimate the negative eigenvalues of P, we now construct a path of *odd length* $\gamma^*_{A_1A_2}$ from A_1 to A_2 when $\tilde{P}(A_1, A_2) > 0$. Thus, let $A = A_1 \cap A_2$, $A_1 = A \cup \{a_1\}$, $A_2 = A \cup \{a_2\}$ and consider the fixed path $\gamma^*_{a_1a_2}$ of odd length in (X_0, E_0). First, we construct $\gamma^*_{A_1A_2}$ from $\gamma^*_{a_1a_2}$ as before. Here, we face a small difficulty: $\gamma^*_{a_1a_2} = (b_1, \dots, b_k)$ might contain one "holding edge" (c, c) (we can always suppress an even number of holding edges). We have to specify when the corresponding holding edge in $\gamma^*_{A_1A_2}$ should occur. This difficulty is easily dealt with: Suppose $b_i = b_{i+1} = c$. Then, if $i = 1$, we start the path $\gamma^*_{A_1A_2}$ with the holding edge (A_1, A_1). If $i > 1$, we attach the edge (b_i, b_{i+1}) to (b_{i-1}, b_i) and whenever we perform the move (b_{i-1}, b_i) in the construction of $\gamma^*_{A_1A_2}$, we immediately follow it by the "holding edge" corresponding to (b_i, b_{i+1}). This make sense because each edge of $\gamma^*_{a_1a_2}$ yields exactly one edge of $\gamma^*_{A_1A_2}$.

Now, set $h(C) = \sum_{c \in C} d^*_c(C)$ [h stands for holding and d^*_c is defined in (3.2)]. We claim that we have

$$
\begin{aligned}
&\sum_{\gamma^*_{A_1A_2} \ni e} |\gamma^*_{A_1A_2}| \le h(C)\Delta^*_0, \quad \text{if } e = (C, C), \\
(3.13) \qquad &\sum_{\gamma^*_{A_1A_2} \ni e} r_{A_1A_2}(e)|\gamma^*_{A_1A_2}| \le \Delta^*_0, \quad \text{if } e = (C_1, C_2), C_1 \ne C_2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tilde{P}(C_1, C_2) > 0.
\end{aligned}
$$

First, consider the case when $e = (C, C)$. Let $\gamma^*_{A_1A_2}$ be a path that contains e. Of course, $\gamma^*_{A_1A_2}$ is constructed from a path $\gamma^*_{a_1a_2}$ that contains an edge $e_0 = (c, c)$ with $c \in C$. Moreover, if we fix $c \in C$ in advance, the correspondence between paths is one-to-one and preserves the length. Finally, the number of $c \in C$ that can be used to define a holding edge $e_0 = (c, c)$ is smaller than $h(C)$ because $(c, c) \in E_0$ and $c \in C$ implies $d^*_c(C) \ge 1$. This proves the first inequality.

Second, assume that $e = (C_1, C_2)$ with $C_1 \ne C_2$ and $C_1 = C \cup \{c_1\}$, $C_2 = C \cup \{c_2\}$. For this case the argument is identical to the one used in the proof of (3.6) except when $e_0 = (c_1, c_2)$ is a double edge of $\gamma^*_{a_1a_2}$ (multiple edges can always be reduced to double edges). Indeed, if e_0 is a double edge of $\gamma^*_{a_1a_2}$, either there is one path $\gamma^*_{A_1A_2}$ corresponding to $\gamma^*_{a_1a_2}$ and e is a double edge in that path or there are two paths $\gamma^*_{A_1A_2}, \gamma^*_{A'_1A'_2}$ corresponding to $\gamma^*_{a_1a_2}$ and (C, C) is a simple edge of each of these paths. In any case, we obtain

$$\sum_{\gamma^*_{A_1A_2} \ni e} r_{A_1A_2}(e)|\gamma^*_{A_1A_2}| = \sum_{\gamma^*_{a_1a_2} \ni e_0} r_{a_1a_2}(e_0)|\gamma^*_{a_1a_2}|$$

when $C_1 \ne C_2$, and this proves the second inequality in (3.13).

To finish the proof, note that (3.2) and (3.3) imply that

$$P(C, C)\pi(C) = \frac{nh(C)}{r\binom{n}{r}|E_0|}.$$

Hence, (3.4), (3.9) and (3.13) imply that the comparison constant A^* in (2.5) with P and $\tilde{P}$ as in (3.2) and (3.8) satisfies

$$A^* \leq \frac{|E_0|\Delta_0^*}{n(n-r)}.$$

Together with (2.3) and (3.12), this yields $\beta_i \geq -1 + [(n-r)/d_r\Delta_0^*](1+\tilde{\beta}_i)$ and (3.7) follows from the fact that $\tilde{\beta}_{\min} \geq -1/(n-r)$. This ends the proof of Theorem 3.1. □

REMARK. If there exists $(x, y) \in X_0^2$ such that there is no path of odd length from x to y, we set $\Delta_0^* = \infty$. This is the case if and only if the graph $(X_0, E_0) = \mathscr{G}_0$ is bipartite. Even so, it is clear that the smallest eigenvalue $\beta_{\min}$ of the exclusion process (3.1) satisfies $\beta_{\min} > -1$ as soon as $r \geq 2$. Indeed, given an r-set $A \subset X_0$, let a_1 and a_2 be two elements of A such that $|\gamma_{a_1a_2}|$ is minimum. We can assume that $\gamma_{a_1a_2}$ does not intersect A except in a_1, a_2 (if it does, replace a_2 by the first element of A on the path). Now, we can construct a loop σ_A of odd length $2|\gamma_{a_1a_2}| - 1$ by moving the particle in a_1 along $\gamma_{a_1a_2}$. Once the two particles are neighbors, we perform a holding edge and move back to the starting point.

We claim that the constant A^* in (2.6) for the exclusion process can now be bounded by

$$A^* \leq 2r\gamma_0 d_r n,$$

where γ_0 is the diameter of $\mathscr{G}_0$. Indeed, if $e = (C_1, C_2)$ is not a holding edge ($C_1 \neq C_2$),

$$\frac{1}{P(C_1, C_2)\pi(C_1)} \sum_{\sigma_A \ni e} |\sigma_A|\pi(A) \leq rd_r(2\gamma_0 - 1)|X_0|,$$

whereas if $e = (C, C)$ is a holding edge,

$$\frac{1}{P(C, C)\pi(C)} \sum_{\sigma_A \ni e} |\sigma_A|\pi(A) \leq \frac{rd_r}{h(C)}(2\gamma_0 - 1)|X_0|h(C).$$

Hence, we conclude as in (2.6) that

$$\beta_{\min} \geq -1 + \frac{1}{2r\gamma_0 d_r n}. \tag{3.14}$$

Combining (3.14) with Remark 2 following Theorem 3.1, we get the simple universal bound

$$\beta_* \leq 1 - \frac{1}{2r\gamma_0 d_r n} \leq 1 - \frac{1}{2rd_0 n^2}, \tag{3.15}$$

where $\beta_* = \max\{|\beta_{\min}|, \beta_1\}$. The order of magnitude of the bound (3.14) can sometimes be improved by further geometric considerations. For instance, consider our running example of the n-point segment with a loop at each end.

If there are r particles, we can bound the length of the foregoing loops by $2n/r$ and the number of loops using a given edge by n/r. This yields

$$\beta_{\min} \geq -1 + \frac{r}{4n^2}.$$

This is better than the bound given by (3.7). See also the examples in Section 5.

There is a variant of Theorem 3.1 using the multicommodity flow technique of Section 2C. Let P_0 denote the simple random walk on $\mathscr{G}_0$ and let $U_0 \equiv 1/n$ be the trivial uniform chain on $\mathscr{G}_0$. A (P_0, U_0)-flow is a function f_0 on simple paths in $\mathscr{G}_0$ such that

$$\sum_{\gamma \in \mathscr{P}_{0,xy}} f_0(\gamma) = \frac{1}{n^2},$$

where $\mathscr{P}_{0,xy}$ is the set of all simple paths in $\mathscr{G}_0$ from x to y. Let $A(f_0)$ be the comparison constant at (2.13) associated with such a flow. Applying Theorem 2.3 shows that the second largest eigenvalue β_1^0 of P_0 is bounded by

$$\beta_1^0 \leq 1 - \frac{|E_0|}{nd_0 A(f_0)}.$$

Consider now the exclusion process.

Theorem 3.2. *The second largest eigenvalue $\beta_1(r)$ of the Markov chain P in* (3.2) *of the exclusion process of r particles on $\mathscr{G}_0$ is bounded by*

$$\beta_1(r) \leq 1 - \frac{|E_0|}{rnd_r A(f_0)}$$

for any (P_0, U_0)-flow f_0. Here, $d_r \leq d_0$ is the maximum mean degree over r sets defined at (3.5).

Proof. Using the construction in the proof of Theorem 3.1, we establish a one-to-one correspondence between a subfamily of simple paths joining the two given r sets $A_1 = A \cup \{a_1\}$ and $A_2 = A \cup \{a_2\}$ with $(a_1, a_2) \in E_0$ and simple paths in $\mathscr{G}_0$ joining a_1 to a_2. Thus, given f_0, we obtain a $(P, \tilde{P})$-flow f for comparison between the exclusion process and the Bernoulli–Laplace model by setting

$$f(\gamma_{A_1A_2}) = \frac{n^2}{r(n-r)\binom{n}{r}} f_0(\gamma_{a_1a_2}).$$

By construction, for an edge $e = (C_1, C_2)$ with $C_1 = C \cup \{c_1\}$ and $C_2 = C \cup \{c_2\}$, $(c_1, c_2) = e_0 \in E_0$, $c_1 \neq c_2$, we have

$$\frac{1}{P(C_1, C_2)\pi(C_1)} \sum_{\gamma \ni e} |\gamma| f(\gamma) = \frac{n|E_0|}{n-r} \sum_{\gamma \ni e_0} |\gamma| f_0(\gamma) \leq \frac{n}{n-r} A(f_0).$$

Now, Theorem 2.3 yields the following lemma.

LEMMA 3.2. *The Dirichlet forms* $\mathscr{E}$ *and* $\tilde{\mathscr{E}}$ *of the exclusion process* (3.2) *and the Bernoulli–Laplace model of diffusion* (3.8) *satisfy*

$$\tilde{\mathscr{E}} \le \frac{n}{n-r} A(f_0)\mathscr{E}$$

for any (P_0, U_0)*-flow* f_0 *on* $\mathscr{G}_0$.

Theorem 3.2, as well as further bounds on the other eigenvalues, follows from this, (3.10) and (3.12) as in the proof of Theorem 3.1. □

4. Lower bound on the second largest eigenvalue of the exclusion process. This section gives a sharp lower bound for the second largest eigenvalue $\beta_1(r)$ of the exclusion process of r particles on a connected graph (X_0, E_0) with $n = |X_0|$ and $1 \le r < n$. Let β_1^0 be the second largest eigenvalue of the simple random walk on (X_0, E_0). Of course, $\beta_1^0 = \beta_1(1)$, because the random walk on the underlying graph corresponds to the (trivial exclusion) process of one particle. Recall that d_0 is the maximum degree in (X_0, E_0) and let $d_0' = \min_{x \in X_0}\{d(x)\}$ be the minimum degree.

THEOREM 4.1. *For* $1 \le r < n$, *the second largest eigenvalue of the* r *particle exclusion process* (3.2) *is bounded below by*

$$\beta_1(r) \ge 1 - \frac{d_0(1-\beta_1^0)}{d_0' r}. \tag{4.1}$$

REMARK. Consider our running example of an n-point segment with a loop at each end. Then, $\beta_1^0 = \cos(\pi/n)$, $d_0 = d_0' = 2$ and we get

$$\beta_1(r) \ge 1 - \frac{1}{r}\left(1 - \cos\frac{\pi}{n}\right) \ge 1 - \frac{\pi^2}{2rn^2},$$

which has to be compared with our upper bound $\beta_1(r) \le 1 - 4/rn^2$. Fill (1991) had the lower bound $\beta_1(r) \ge 1 - 6/rn^2$ for this case.

PROOF OF THEOREM 4.1. Let φ_0 be an eigenfunction associated with β_1^0 for the random walk on (X_0, E_0). For any r-set A of X_0, set

$$\varphi(A) = \sum_{x \in A} \varphi_0(x).$$

The variational characterization of $\beta_1 = \beta_1(r)$ gives

$$\beta_1 \ge 1 - \frac{\mathscr{E}(\varphi, \varphi)}{\operatorname{Var}(\varphi)}. \tag{4.2}$$

Here,

$$\begin{aligned}\mathscr{E}(f,f) &= \frac{1}{2}\sum_{A,B} |f(A) - f(B)|^2 P(A,B)\pi(A)\\ &= \frac{1}{2\binom{n-1}{r-1}|E_0|}\sum_{A\sim B} |f(A) - f(B)|^2,\end{aligned}$$

where $A \sim B$ means $A = A' \cup \{a\}$, $B = A' \cup \{b\}$ with $(a, b) \in E_0$ and the variance $\mathrm{Var}(f) = \mathrm{Var}_\pi(f)$ is taken with respect to the invariant measure π at (3.3). Now, for $A \sim B$ as before,

$$|\varphi(A) - \varphi(B)| = |\varphi_0(a) - \varphi_0(b)|.$$

Counting the edges (A, B) corresponding to a given edge $(a, b) \in E_0$ yields

$$\mathscr{E}(\varphi, \varphi) = \frac{n-r}{n-1}\mathscr{E}_0(\varphi_0, \varphi_0), \tag{4.3}$$

where

$$\mathscr{E}_0(f, f) = \frac{1}{2|E_0|} \sum_{(a,b)\in E_0} |f(a) - f(b)|^2$$

is the Dirichlet form on the underlying graph.

To finish the proof, we recall the following classic fact also used by Fill (1991) for his lower bounds.

LEMMA 4.1. *Let an urn contain N balls, the ith ball labelled with the real number y_i. Fix $1 \le r \le N$ and take a sample of size r from the urn without replacement. Let X be the sum of the numbers shown. Then*

$$E(X) = r\bar{Y} \quad \text{with} \quad \bar{Y} = \frac{1}{N}\sum_{i=1}^N y_i,$$

$$\mathrm{Var}(X) = \left(1 - \frac{r}{N}\right)\frac{r}{N-1}\sum_{i=1}^N (y_i - \bar{Y})^2.$$

Using Lemma 4.1 with balls labelled by the values of φ_0 shows

$$\mathrm{Var}^*(\varphi) = r\frac{n-r}{n-1}\mathrm{Var}^*(\varphi_0), \tag{4.4}$$

where the asterisk indicates that the variances are taken with respect to the uniform distribution on r-sets and the uniform distribution X_0, respectively. Thus,

$$\mathrm{Var}(\varphi) \ge \frac{d_0' n}{|E_0|}\mathrm{Var}^*(\varphi) \tag{4.5}$$

and

$$\mathrm{Var}^*(\varphi_0) \ge \frac{|E_0|}{d_0 n}\mathrm{Var}(\varphi_0). \tag{4.6}$$

Combining (4.2)–(4.6) with the fact that

$$1 - \beta_1^0 = \frac{\mathscr{E}_0(\varphi_0, \varphi_0)}{\mathrm{Var}(\varphi_0)}$$

proves the stated bound. □

Examples using Theorem 4.1 are given in next section. Sharp lower bounds are obtained for exclusion on the circle $\mathbb{Z}_n$, a finite box in $\mathbb{Z}^d$ and other graphs.

REMARK. The foregoing comparison of variances is a rather crude argument to deal with graphs that are not regular. Direct arguments can be used on specific examples, leading to improved bounds; see Examples 5.6 and 5.8. In the same spirit, d_0' can be replaced in Theorem 4.1 by the minimum mean degree over r-sets defined by $d_r' = \min_{A \in X_r}\{(1/r)\sum_{a \in A} d(a)\}$. Note however that in the above proof of Theorem 4.1 we cannot replace d_0 by the maximum mean degree d_r in (3.5).

5. Examples of simple exclusion processes. We specialize the theorems of Sections 3 and 4 to a number of graphs including the discrete circle $\mathbb{Z}_n$, the cube $\mathbb{Z}_2^d$, an $l \times m$ grid in $\mathbb{Z}^2$, the Cayley graph of the transpositions on the symmetric group S_k, a star and the complete bipartite graph $K_{l,m}$. In the following text we always consider the simple exclusion process of r particles on the given underlying graph $\mathscr{G}_0$ with n vertices and $1 \leq r < n$. In particular, β_1 and $\beta_{\min}$ are, respectively, the second largest and the smallest eigenvalue of this process.

EXAMPLE 5.1. Let $\mathscr{G}_0$ be the Cayley graph of $\mathbb{Z}_n$ with generators $\{+1, -1\}$. Here, $d_0 = 2$. Simple arguments show that $\Delta_0 = \sum_0^m k^2 = m(m+1)(2m+1)/6$, where $m = [n/2]$. Hence, (3.6) yields

$$(5.1) \qquad \beta_1 \leq 1 - \frac{12}{rn^2} + O\left(\frac{1}{rn^3}\right).$$

For the lower bound, we have $\beta_1^0 = \cos(2\pi/n)$ and thus

$$\beta_1 \geq 1 - \frac{1}{r}\left(1 - \cos\frac{2\pi}{n}\right) \geq 1 - \frac{2\pi^2}{rn^2}.$$

When $n = 2m+1$ is odd, we can use Δ_0^* to bound the smallest eigenvalue:

$$\Delta_0^* = \sum_{i, 2i+1 \leq m} (2i+1)^2 + 2\sum_{i, 2i \leq m} 2i(n+2i) \leq \frac{(n-1)(n+1)^2}{8}$$

yields

$$(5.2) \qquad \beta_{\min} \geq -1 + \frac{4(n-r-1)}{(n-1)(n+1)^2}.$$

When $n = 2m$ is even, $\Delta_0^* = \infty$. However, if $r \geq 2$ we can use an improved version of (3.14). Following the argument given there for the n-point segment, we get (for n odd or even)

$$(5.3) \qquad \beta_{\min} \geq -1 + \frac{r}{4n^2}.$$

This is better than (5.2) as soon as $r \geq 2$.

EXAMPLE 5.2. Consider the cube $\mathbb{Z}_2^d$ with is natural graph structure. Here $n = 2^d$, $d_0 = d$, $\Delta_0 = (d+1)2^{d-2}$ and $\beta_1^0 = 1 - 2/d$; see Diaconis (1988). Thus

$$1 - \frac{2}{rd} \le \beta_1 \le 1 - \frac{4}{rd(d+1)}.$$

We do not know if the extra factor of d is necessary for large r, but we believe it is not.

This graph is bipartite. For $r \ge 2$, (3.14) yields

$$\beta_{\min} \ge -1 + \frac{1 - o(1)}{2rd^2 2^d}.$$

Note, however, that if we add a trivial loop at each vertex of the cube, the chain P in (3.2) satisfies $P(A, A) \ge 1/(d+1)$ for any r set A, and (3.6) and (2.7) give

$$1 - \frac{2}{r(d+1)} \le \beta_1 \le 1 - \frac{4}{r(d+1)^2}, \qquad \beta_{\min} \ge -1 + \frac{1}{d+1}.$$

EXAMPLE 5.3. Let $\mathscr{G}_0$ be an $l \times m$ grid in $\mathbb{Z}^2$ with $lm = n$. Fix paths in $\mathscr{G}_0$ that are of minimal length, have at most one turn and always start horizontally (unless they are vertical with no turn). For this set of paths, we find

$$\Delta_0 \le \frac{lm}{4}(l+m)\max\{l, m\}.$$

Here, $d_0 = 4$, $d_0' = 2$ and, assuming $l \ge m$, (3.6) yields

$$\beta_1 \le 1 - \frac{1}{2rl^2}.$$

For the lower bound, we use comparison with a product chain as in Example 2.1 to bound the second largest eigenvalue of the grid by $\beta_1^0 \ge \cos(\pi/l)$. This and Theorem 4.1 yield

$$\beta_1 \ge 1 - \frac{2}{r}\left(1 - \cos\frac{\pi}{l}\right) \ge 1 - \frac{\pi^2}{rl^2}.$$

This graph is bipartite. Using (3.14) yields

$$\beta_{\min} \ge -1 + \frac{1}{16rl^2m} \tag{5.4}$$

for $r \ge 2$. There is an interesting argument that yields an improved bound. Let $t = t(A)$ be the minimal distance between the r elements of a given r-set $A \subset X_0$. The balls of radius $t/2 - 1$ around each of our particles are pairwise disjoint. Hence, $r(t+2)t/8 \le n$ and $t^* = \max_A t(A) \le 2\sqrt{2n/r}$. Now, the line of reasoning that gave (3.14) yields here $\beta_{\min} \geq -1 + 1/(16rt^*(t^* + 1)(t^* + 2))$. Finally, we get

$$\beta_{\min} \geq -1 + \frac{cr^{1/2}}{n^{3/2}} \tag{5.5}$$

for a universal constant c. For a square ($l = m = n^{1/2}$) and $r = n/2$ particles, (5.4) gives $\beta_{\min} \geq -1 + 1/8l^5$, whereas (5.5) yields $\beta_{\min} \geq -1 + c/l^2$. These bounds may be compared with the universal bound (3.15), which gives $\beta_* \leq 1 - 1/4l^6$ for this example. Here, using the geometry gives a big improvement.

This example extends to higher dimensions. Consider a grid in $\mathbb{Z}^d$ with size $l_1 \times \cdots \times l_d = n$ and set $l = \max l_i$. Fix paths on this grid by always moving first along the first axis, then along the second axis, etc. For this choice of paths,

$$\Delta_0 \leq \frac{ln}{4} \sum l_i \leq \frac{dl^2 n}{4},$$

$d_0 = 2d$, $d_0' = d$ and thus

$$1 - \frac{2\pi^2}{rdl^2} \leq \beta_1 \leq 1 - \frac{4}{rd^2 l^2},$$

where the lower bound is obtained as in dimension 2.

We leave to the reader the details of the estimate

$$\beta_{\min} \geq -1 + \frac{c(d) r^{1/d}}{l^{1+1/d}},$$

which generalizes (5.5). As a variant of this we mention the natural Cayley graph of the group $\mathbb{Z}_{l_1} \times \cdots \times \mathbb{Z}_{l_d}$.

EXAMPLE 5.4. Let $\mathscr{G}_0$ be the Cayley graph of the symmetric group S_k with the transpositions as a symmetric set of generators. The exclusion process on this graph is better described as a way to choose a set of r permutations in S_k without repetition. Here $n = k!$ and $d_0 = k(k-1)/2$. Using the analysis in Example 2.3 we find that $\Delta_0 \leq k((k-1)!) = k!$. Also, Diaconis and Shahshahani (1981) have shown that $\beta_1^0 = 1 - 2/(k-1)$. Thus,

$$1 - \frac{2}{rk} \leq \beta_1 \leq 1 - \frac{2}{rk(k-1)}. \tag{5.6}$$

We do not know whether or not the extra factor of k is necessary for large r in the upper bound.

This graph is bipartite and (3.14) yields

$$\beta_{\min} \geq -1 + \frac{1}{rk^2(k-1)k!}.$$

This is probably a very bad bound when $r = n/2$, but it seems difficult to improve upon.

EXAMPLE 5.5. Let G be a finite group and $\Gamma = \{g_1, \dots, g_s\}$ be a symmetric set of generators. Consider the Cayley graph $\mathscr{G}_0$ of (G, Γ). Example 3 in Section 2 above and Corollary 1 in Section 3 of Diaconis and Saloff-Coste (1993) show that $\Delta_0 \le s\gamma_0^2$, where γ_0 is the diameter of $\mathscr{G}_0$. Thus, in this case we have

$$\beta_1 \le 1 - \frac{1}{sr\gamma_0^2}.$$

EXAMPLE 5.6. Let $X_0 = \{1, \dots, n\}$ and $E_0 = \{(1, i), (i, 1);\ i = 2, \dots, n\}$ so that $\mathscr{G}_0$ is a star. The eigenvalues of the simple random walk on this graph are $1, 0 - 1$, with multiplicity 1, $n - 2$, ≤ 1. This graph is a tree, paths are forced and $\Delta_0 = 2n - 1$, $d_0 = n - 1$ and $d_0' = 1$. Thus

$$\beta_1 \le 1 - \frac{n}{r(2n-1)(n-1)} \le 1 - \frac{1}{2rn}.$$

This can be improved by using d_r in (3.5). Indeed, here

$$d_r = \max_{A \in X_r} \left\{ \frac{1}{r} \sum_{a \in A} d(a) \right\} \le \frac{n + r - 2}{r}.$$

Hence,

$$\beta_1 \le 1 - \frac{n}{(n + r - 2)(2n - 1)}.$$

When $r = n/2$, this gives $\beta_1 \le 1 - 1/3n$.

Direct application of the lower bound of Section 4 yields the uninformative inequality $\beta_1 \ge 1 - (n - 1)/r$. This is only due to the crude handling of the variances. Here, an eigenfunction φ_0 is obtained by setting $\varphi_0(x) = 0$ unless $x = 2$ or 3, in which case $-\varphi_0(2) = \varphi_0(3) = 1$. This has $\mathrm{Var}^*(\varphi_0) = 2/n$ and $\mathrm{Var}(\varphi_0) = 2/(2n - 1)$. Plugging this in at the end of the proof in Section 4 yields

$$\beta_1 \ge 1 - \frac{2}{r}.$$

We believe this is of the right order of magnitude.

EXAMPLE 5.7. Let $\mathscr{G}_0$ be a graph with automorphism group acting transitively on the set of oriented edges. Let R_0 and D_0^2 be the mean distance and the mean square distance,

$$R_0 = \frac{1}{|X_0|} \sum_{x \in X_0} |ox|, \qquad D_0 = \left(\frac{1}{|X_0|} \sum_{x \in X_0} |ox|^2 \right)^{1/2},$$

where $|ox|$ is the distance between a fixed point o and x in $\mathscr{G}_0$. Theorem 3.2 and Example 2.6 yield

$$\beta_1 \le 1 - \frac{1}{rD_0^2}.$$

For a lower bound, fix a point $o \in X_0$ and consider the test function $\psi_0(x) = |ox|$. This shows that

$$\beta_1^0 \ge 1 - \frac{1}{\overline{D}_0^2},$$

where $\overline{D}_0^2 = \mathrm{Var}(\psi_0) = D_0^2 - R_0^2$, and thus

$$\beta_1 \ge 1 - \frac{1}{r\overline{D}_0^2}.$$

EXAMPLE 5.8. Let $\mathscr{G}_0 = K_{l,m}$ be the complete bipartite graph described in Example 2.7. We will use the notation introduced there and assume $l \ge m$. We want to apply Theorem 3.2 to this example. However, the flow considered in Section 2C has to be modified. Namely, we set now $f_0(\gamma) = 1/n^2$ if γ is a simple path joining x to y with (x, y) of Type 1, $f_0(\gamma) = 1/mn^2$ for Type 2 and $f_0(\gamma) = 1/ln^2$ for Type 3. This has

$$A(f_0) = \frac{2lm}{n^2}\left(1 + \frac{l-1}{m} + \frac{m-1}{l}\right) \le 2.$$

Here, $d_0 = l$, $|E_0| = 2lm$ and Theorem 3.2 yields

$$\beta_1 \le 1 - \frac{m}{rn}.$$

When $r > m$, the bound can be improved by using d_r in (3.5) instead of d_0. Here $rd_r \le l + m(r-m) \le lm$ and we get

$$\beta_1 \le 1 - \frac{1}{n(1 + (r-m)/l)} \le 1 - \frac{1}{2n}.$$

For the lower bound, Section 4 gives

$$\beta_1 \ge 1 - \frac{l}{rm},$$

which is bad when $l \gg m$. We can fix this by considering the eigenfunction φ_0 defined as follows. Picture the graph with l vertices on the left and m on the right. For $[l/2]$ of the vertices on the left, let $\varphi_0(x) = -1$. For $[l/2]$ other vertices on the left, let $\varphi_0(x) = 1$. Proceed similarly to define φ_0 on the right vertices. Then, $\mathrm{Var}^*(\varphi_0) = 2([l/2] + [m/2])/n$ and $\mathrm{Var}(\varphi_0) = 2([l/2]m + [m/2]l)/2lm$. This yields (for $m \ge 2$)

$$\beta_1 \ge 1 - \frac{2l}{r(n-2)} \ge 1 - \frac{2}{r}.$$

Note that other choices of the eigenfunction are possible, but this one leads to the best bound. For some of these choices, the variances Var and Var* are indeed of different orders of magnitude.

6. Further results and remarks.

A. *Time to reach equilibrium.* Our main motivation in bounding eigenvalues of a finite irreducible reversible Markov chain P, π is to estimate the time the chain takes to be close to equilibrium. Classically, the variation distance

$$\|P_k^x - \pi\|_{\mathrm{TV}} = \max_{A \subset X} \{|P_k^x(A) - \pi(A)|\}$$

is used to discuss this question. Here $P_k^x(y) = P_k(x, y)$ is the iterated kernel of the chain. The relation with eigenvalues comes from the estimate

$$2\|P_k^x - \pi\|_{\mathrm{TV}} \le \pi_*^{-1/2}\beta_*^k, \tag{6.1}$$

where $\pi_* = \min_x\{\pi(x)\}$ and $\beta_* = \max\{|\beta_{\min}|, \beta_1\}$. See for instance Diaconis and Stroock (1991) or Fill (1991), which also has a version of this for nonreversible chains. If we consider the continuous time process

$$H_t = e^{-t(I-P)} = e^{-t} \sum \frac{t^n P^n}{n!},$$

the inequality

$$2\|H_t^x - \pi\|_{\mathrm{TV}} \le \pi_*^{-1/2} e^{-(1-\beta_1)t} \tag{6.2}$$

holds instead. There is no mystery behind (6.1) and (6.2); they follow from the observation that $2\|P_k^x - \pi\|_{\mathrm{TV}} = \Sigma_y |P_k(x, y) - \pi(y)|$, Jensen's inequality, and the following lemma.

LEMMA 6.1. *Let $p_k(x, y) = P_k(x, y)/\pi(y)$ be the kernel of the operator P with respect to the measure π and set $h_t(x, y) = H_t(x, y)/\pi(y)$. Let φ_i, $0 \le i \le |X| - 1$, be a basis of orthonormal eigenfunctions in $l^2(\pi)$ corresponding to the sequence $1 = \beta_0 > \beta_1 \ge \cdots \ge \beta_{|X|-1}$ of the eigenvalues. We have*

$$p_k(x, y) = \sum_0^{|X|-1} \beta_i^k \varphi_i(x)\varphi_i(y), \qquad h_t(x, y) = \sum_0^{|X|-1} e^{-t(1-\beta_i)}\varphi_i(x)\varphi_i(y),$$

$$\varphi_0 \equiv 1, \qquad \sum_0^{|X|-1} \varphi_i^2(x) = \frac{1}{\pi(x)},$$

$$\|p_k^x - 1\|_2^2 = \sum_1^{|X|-1} \varphi_i^2(x)\beta_i^{2k} \le \frac{1 - \pi(x)}{\pi(x)}\beta_*^{2k},$$

$$\|h_t^x - 1\|_2^2 = \sum_1^{|X|-1} \varphi_i^2(x) e^{-2t(1-\beta_i)} \le \frac{1 - \pi(x)}{\pi(x)} e^{-2t(1-\beta_1)}.$$

PROOF. For the first line, note that $\langle p_k^x, \varphi_i \rangle = \beta_i^k \varphi_i(x)$. For the second equality, set $\delta_x(x) = \pi(x)^{-1}$ and $\delta_x(y) = 0$ otherwise. Then, $\langle \delta_x, \varphi_i \rangle = \varphi_i(x)$ and $\|\delta_x\|_2^2 = \pi(x)^{-1}$. The last inequality follows. □

In general (6.1) and (6.2) are far from optimal. The reason is not so much the use of Jensen's inequality, but because they only take into account the value of β_1 and $\beta_{\min}$. Indeed, these inequalities are easily complemented with

$$(6.3)\qquad \begin{aligned} \sum_{x,y} |P^k(x,y) - \pi(y)|\pi(x) &\le \left(\sum_{i=1}^{|X|-1} \beta_i^{2k} \right)^{1/2}, \\ \sum_{x,y} |H_t(x,y) - \pi(y)|\pi(x) &\le \left(\sum_{i=1}^{|X|-1} e^{-2(1-\beta_i)t} \right)^{1/2}. \end{aligned}$$

If P is vertex transitive [see Aldous and Diaconis (1987)], the quantity $\|P_k^x - \pi\|_{\mathrm{TV}}$ does not depend on x and can be bounded by $(\sum_{i=1}^{|X|-1} \beta_i^{2k})^{1/2}$. Diaconis (1988) and Diaconis and Saloff-Coste (1993) give many examples of sharp bounds for random walks on groups that are obtained by using all eigenvalues.

Now, there are many examples of graphs that are not vetex transitive but for which a heuristic argument indicates that $\|P_k^x - \pi\|_{\mathrm{TV}}$ does not depend much on x. For such graphs, one expects a bound of the order of $(\sum_{i=1}^{|X|-1} \beta_i^{2k})^{1/2}$ for $\|P_k^x - \pi\|_{\mathrm{TV}}$. As an easy and typical example of this, consider the $l \times m$ grid $\mathscr{G}$ with (or without) some deleted edges as in Example 2.1. In this case, using (6.2), we find that $2\|H_t^x - \pi\|_{\mathrm{TV}} \le e^{-c}$ when t is of order $l^2(\frac{1}{2}\log(lm) + c)$. If one believes the foregoing heuristic, t of order $l^2 c$ should be enough. This is indeed the case, but the proof needs a different approach; see Diaconis and Saloff-Coste (1992b).

For the exclusion process, Theorem 3.1 and the preceding inequalities yield the following theorem.

THEOREM 6.1. *Let $\mathscr{G}_0 = (X_0, E_0)$ be a connected graph, $n = |X_0|$; for $0 \le r < n$ recall d_r from* (3.5). *The chain P of the exclusion process* (3.2) *satisfies*

$$(6.4)\qquad \|H_t^x - \pi\|_{\mathrm{TV}} \le \tfrac{1}{2}e^{-c}$$

for $t \ge n^{-1} r d_r \Delta_0 / (\log\binom{n}{r} + c)$, $c > 0$. In particular, if $r = [n/2]$, (6.4) *holds for $t \ge \frac{1}{2} d_r \Delta_0 (n + 2 + c)$, $c > 0$.*

If we specialize (6.4) to the circle $\mathbb{Z}_n$ and $r = [n/2]$ (Example 5.1), we get $\|H_t^x - \pi\|_{\mathrm{TV}} \le \frac{1}{2}e^{-c}$ for $t \ge n^3(n + 2 + c)/24$. In this case, (6.1) and the estimates (5.1) and (5.3) on β_1 and $\beta_{\min}$ also yield $\|P_k^x - \pi\|_{\mathrm{TV}} \le \frac{1}{2}e^{-c}$ for $k \ge n^3(n + 2 + c)/24$ and $n \ge 7$; compare with Fill (1991).

As a different example, consider choosing k random permutations without repetition in S_k. Using our results, we find that running the exclusion process on S_k with random transpositions yields an acceptable answer after order $\frac{1}{2}k^5 \log k$ steps starting from any fixed choice. More precisely, $\|H_t^\theta - \pi\|_{\mathrm{TV}} \le \frac{1}{2}e^{-c}$ for $t = \frac{1}{2}k^3(k^2 \log k + c)$ and any $\theta \in S_k$.

As an application of comparing *all* the eigenvalues, we get an improved bound for the mean variation distance at (6.3).

THEOREM 6.2. *Let $\mathscr{G}_0 = (X_0, E_0)$ be a connected graph, $n = |X_0|$; for $0 \le r < n$, recall d_r from* (3.5). *The chain P of the exclusion process* (3.2) *satisfies*

$$\sum_{x,y} |H_t(x,y) - \pi(y)|\pi(x) \le Ae^{-c/2} \tag{6.5}$$

for $t \ge \frac{1}{2} r d_r n^{-1}(\log n + c)$, $c > 0$. Here A is a universal constant.

PROOF. Let $\tilde{\beta}_i$, $i = 0, 1, \dots, \binom{n}{r} - 1$, be the eigenvalues of the Bernoulli–Laplace model of diffusion. Diaconis and Shahshahani (1987) proved that

$$\left(\sum_{i=1}^{\binom{n}{r}-1} e^{-2(1-\tilde{\beta}_i)t} \right)^{1/2} \le Ae^{-c/2} \quad \text{for } t \ge \frac{r(n-r)}{2n}(\log n + c).$$

Now, in Section 3, we proved (see Lemma 3.1) that

$$1 - \tilde{\beta}_i \le \frac{n-r}{d_0 \Delta_0}(1 - \beta_i),$$

where the β_is are the eigenvalues of the exclusion process. Thus, we have

$$\left(\sum_{i=1}^{\binom{n}{r}-1} e^{-2(1-\beta_i)t} \right)^{1/2} \le Ae^{-c/2} \quad \text{for } t \ge \frac{r d_0 \Delta_0}{2n}(\log n + c). \qquad \square$$

For the circle $\mathbb{Z}_n$ and $r = [n/2]$ particles, this yields $\|H_t^x - \pi\|_{\mathrm{TV}} \le Ae^{-c/2}$ for $t \ge n^3(\log n + c)/48$. For the symmetric group S_k as before, we find that, on average, order $k^3 \log k \log\log k$ steps are enough to chose k permutations at random by running the exclusion process.

We believe that the improved estimate of Theorem 6.2 holds as well for the variation distance starting from any fixed state. In Diaconis and Saloff-Coste (1992c), we prove that this conjecture is correct up to logarithmic factors.

B. *Further comments.*

1. The results obtained in this paper raise the following question. What is the relation between the second largest eigenvalue β_1^0 of the nearest-neighbor walk on a graph $\mathscr{G}_0$ and the second largest eigenvalue $\beta_1(r)$ of the

exclusion process of r particles on $\mathscr{G}_0$? Of course $\beta_1^0 = \beta_1(1)$. Also, in Examples 5.1 and 5.3 we have

$$\beta_1(r) \le 1 - \frac{c}{r}\left(1 - \beta_1^0\right). \tag{6.6}$$

It is tempting to conjecture that (6.6) holds universally. Note that Section 5 yields a lower bound of this type for graphs that are nearly regular. What we have shown here is that (6.6) holds as soon as a bound of the right order of magnitude on β_1^0 follows from the "Poincaré technique" [compare (2.11) and (3.6)]. Diaconis and Saloff-Coste (1992a) give many examples of Cayley graphs that have this property. However, the cube $\mathbb{Z}_2^d$ or the symmetric group S_k with random transpositions are examples where we do not know whether (6.5) holds or not. Note that (6.6) holds for the complete graph. Also, this conjecture agrees with a heuristic argument, often used for exclusion processes, where one "approximates" exclusion by the "free" product. Indeed, for the product $1 - (1 - \beta_1^0)/r$ is exactly the second largest eigenvalue.

2. For simplicity, we restricted ourselves in Sections 3 and 4 to an exclusion process associated with a graph. The definition and our analysis can be generalized to exclusion processes associated with a reversible chain P_0 on X_0.
3. There is a class of labelled exclusion processes for which a similar attack should work. The new chain P_{lab} is defined on r-tuples without repetition in a manner similar to (3.1). The difference is that when a particle chooses an occupied site, the two particles switch places. For $r = n$, this is a random walk on the symmetric group studied in Section 4.A of Diaconis and Saloff-Coste (1993).
4. The main estimate in Theorem 3.1 can be rephrased by saying that the Dirichlet form $\mathscr{E}$ of the exclusion process (3.1) satisfies

$$\|f\|_2^2 \le \frac{rd_r\Delta_0}{n}\mathscr{E}(f,f)$$

for all f with $\Sigma_A f(A)\pi(A) = 0$. The l^2 norm is of course taken in $l^2(\pi)$, where π is given by (3.2). Using the eigenvalues of the Bernoulli–Laplace model of diffusion and Lemma 3.1 (i.e., the comparison technique of this paper), it is possible to show that $\mathscr{E}$ satisfies also the log Sobolev inequality

$$L(f) \le \frac{crd_r\Delta_0 \log n}{n}\mathscr{E}(f,f), \qquad f \in l^2,$$

where $L(f) = 2\Sigma_A |f(A)|^2 \log(|f(A)|/\|f\|_2)\pi(A)$ and c is a universal constant. Using this, one can improve Theorem 6.1. For more of this, see Diaconis and Saloff-Coste (1992c).
5. When the underlying graph $\mathscr{G}_0$ is not regular, there is another natural exclusion process of r particles on $\mathscr{G}_0$, different from (3.2). Informally, if the process is at A, pick a particle in A *at random*, and a neighbor of this

particle at random. If the neighboring site is unoccupied, the particle moves there; if it is occupied, the system stays at A. With this definition, the chain P of the process is given by $P(A_1, A_2) = 0$ unless $A_1 \neq A_2$, $A_1 = A \cup \{a_1\}$, $A_2 = A \cup \{a_2\}$ and $(a_1, a_2) \in E_0$, in which case

$$P(A_1, A_2) = \frac{1}{rd(a_1)}$$

or $A_1 = A_2 = A$ and

$$P(A, A) = \sum_{a \in A} \frac{d_A^*(a)}{rd(a)}.$$

This is a reversible chain with reversible probability

$$\pi(A) = \frac{1}{Z} \prod_{a \in A} d(a),$$

where Z is a normalizing constant. We can compare this chain with the Bernoulli–Laplace model of diffusion. The paths are the same as in Section 3. The only difference comes from the values of the transitions and the reversible measures. Here the comparison constant (2.4) is

$$A = \max_{A_1, A_2} \left\{ \frac{Z}{(n-r)\binom{n}{r} \prod_{a \in A_1 \cap A_2} d(a)} \right\} \Delta_0,$$

where the maximum is taken over the pairs (A_1, A_2) with $A_1 \neq A_2$ and nonzero transition probability. Also,

$$\tilde{\pi} \geq \min_{A} \left\{ \frac{Z}{\binom{n}{r} \prod_{a \in A} d(a)} \right\} \pi.$$

Set

$$\delta'_* = \min_{A_1, A_2} \left\{ \prod_{a \in A_1 \cap A_2} d(a) \right\}, \qquad \delta_* = \max_{A} \left\{ \prod_{a \in A} d(a) \right\},$$

where the minimum is taken over the pairs (A_1, A_2) as before. Using (3.8) and (2.3), we get the bound

$$\beta_1(r) \leq 1 - \frac{\delta'_* n}{r \delta_* \Delta_0}$$

on the second largest eigenvalue $\beta_1(r)$ of this process. Clearly δ_* and δ'_* are rather nasty quantities to bound.

For Example 5.6 (i.e., a star), the preceding argument gives

$$\beta_1(r) \leq 1 - \frac{n}{r(2n-1)(n-1)},$$

which is the same bound as for the other exclusion process (if one uses d_0 but not d_r in Theorem 3.1).For Example 5.8 (i.e., the graph $K_{l,m}$, $l + m = n$), we need to adapt Theorem 3.2 using the multicommodity flow technique. At each flow f_0 on $\mathscr{G}_0$, a flow f corresponds for the comparison between the exclusion process and the Bernoulli–Laplace chain with

$$A(f) \le \frac{n^2 Z}{(n-r)\binom{n}{r}\delta'_*|E_0|} A(f_0).$$

Thus, we get

$$\beta_1(r) \le 1 - \frac{|E_0|\delta'_*}{rn\delta_* A(f_0)}.$$

For $K_{l,m}$, this specializes to

$$\beta_1(r) \le 1 - \frac{lm\delta'_*}{rn\delta_*}.$$

When $l = m + \mu$ for a fixed μ, we have $\delta_* \le l^r$, $\delta'_* \ge (l-\mu)^{r-1}$ and thus

$$\beta_1(r) \le 1 - \frac{1}{2r}\left(1 - \frac{\mu}{l}\right)^{r+1} \le 1 - \frac{1}{2r}e^{2\mu(1+1/n)}.$$

When m is fixed, we get instead $\delta_* \le l^m m^{r-m}$, $m^{r-1} \le \delta'_*$ and thus

$$\beta_1(r) \le 1 - \frac{l}{rn(l/m)^m}.$$

When $l = 2m = 2r$, these estimates are exponentially bad whereas, for the exclusion process of Section 3, the bound is always polynomial.

Another simple example where we can get a good bound for the process of Section 3 but not for the preceding process is a finite square grid in $\mathbb{Z}^2$ with, say, one-tenth of the edges deleted according to the rule of Example 2.1.

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